Quantitative Methods for Business Decisions: Text and Cases

To Judith

Contents

Preface

This book is intended for an introductory course in quantitative methods. It has been written for students who need to know how these methods can be helpful in the decision-making process, but who have little mathematical background beyond a course in high school algebra. The emphasis, then, is on application rather than theory. Furthermore, the book has been designed to be patient, first trying to communicate with students who are apprehensive about quantitative subjects and then trying to reinforce the learning process with numerous examples, problems, and—most important—with 29 carefully designed end-of-chapter cases.

The two features of this book that should make it both flexible and timely are:

1 *Modular design.* Most chapters contain three units, or modules: (*a*) the core text material; (*b*) cases; and (*c*) appendixes.

2 *Cases.* The end-of-chapter cases are integrated with the core text material and emphasize the applications of the techniques to real-world problems.

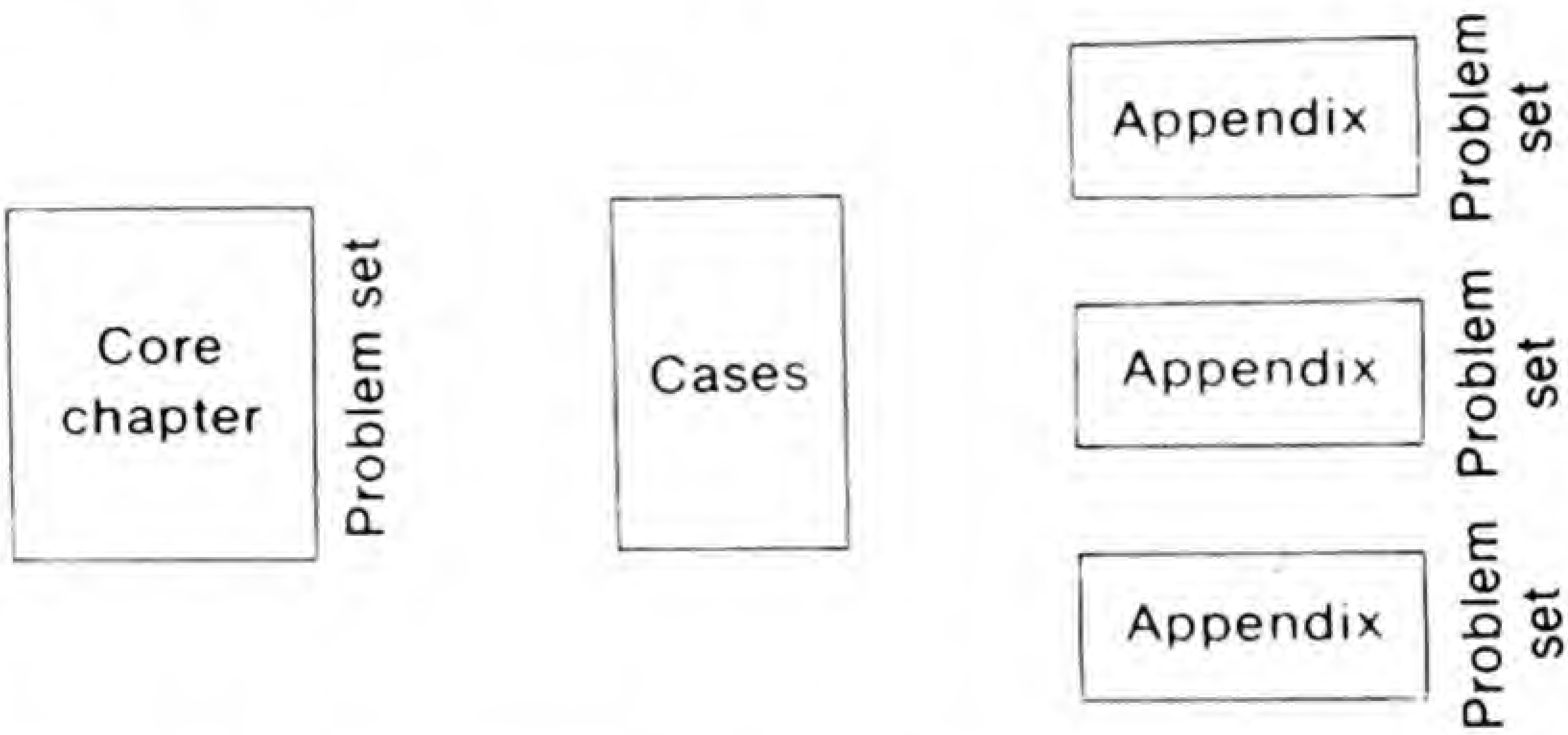
The modular design allows an unusual degree of flexibility. The core text material includes all the basic background and techniques necessary for a standard introductory course in quantitative methods. In addition, each core module has a comprehensive set of problems and exercises. The cases follow this module and have as a prerequisite only the material covered in the preceding core. They provide an elementary but interesting opportunity to try out the techniques and methods just acquired. Finally, the third module, the

appendixes, which follow the cases, can be used by those who desire a more advanced or more challenging course. The appendixes cover either similar material at a more advanced level or topics appropriate for more rigorous courses. Each of the appendixes, like the standard text core, also has a complete set of problems. The book, then, can be used to support three basic course designs:

1 A basic, introductory course using only the core material in chapter (excluding cases and appendixes)

2 A basic, introductory course using the core material plus the cases for a more applied emphasis (excluding appendixes)

3 A more advanced course using the core material, cases, and appendixes.



The 29 end-of-chapter cases, the book's second feature, illustrate the use of quantitative methods in the functional areas of marketing, production, and finance. Their settings are the private, government, and health sectors of the economy. Written specifically for this book, they differ from typically well structured textbook problems in that they generally involve some ambiguity and require that the problem confronting the decision maker be defined, that assumptions be made and supported, that a model be formulated, that a model be solved, and that problems of implementation be considered. In addition they often involve human and organizational aspects. The cases are not too difficult, nor are they too long. They were written to be meaty and realistic.

For both the student and teacher, the class sessions devoted to case discussion can be lively and exciting learning experiences. Rather than being a passive absorber of information, the student becomes an active participant. Both my experience and that of my colleagues, who have used the case method to supplement the course, suggest that the quality of the course—corroborated by student/teacher evaluations—has been improved.

For teachers unfamiliar with the case method a complete manual is available including guidelines for carrying on a case discussion.

In keeping with the book's orientation toward the application of quantitative methods in the decision-making process, chapters have been included on decision trees, computers, and the implementation of management science.

In addition the conventional topics found in most quantitative methods books have also been included.

The computer chapter, although it appears at the end of the book, may be used at any time during the course. If the students are required to solve linear-programming problems on the computer, this may be the appropriate opportunity to introduce the chapter.

I am indebted to countless people and institutions for the ideas, support, and assistance that went into the book: the General Electric Company, Boeing Company, and Hewlett-Packard Company, where I gained the working experience that provided much of the basis for the cases; my colleagues Christoph Haehling von Lanzenauer, Warren G. Briggs, and Linda G. Sprague who developed my interest in the case method; Professors Dean Plager, John Burt, Paul R. Merry, Marvin Rothstein, and E. J. Manton for their helpful suggestions; my students who tested these materials and suggested improvements; Jenifer McKinnon, Barbara Horne, and Shirley Bastianelli who typed the manuscript; Deirdre Tarzian; my mother for her love and devotion; and finally my children Tonnie Susan and Denise Fay who often make it all so worthwhile. To all of them, a very special thanks.

Barry Shore

Chapter 1

Systems and Models

INTRODUCTION

There is no question that good information can be useful during the decision-making process. It can lead to an improvement in the decision that must be made.

A simple example of this is the process of deciding what to wear in the morning. Two factors must be considered. The first is the activity that will be performed during the day and the second is the weather. To obtain some information on the weather you might call the weather bureau, check the weather report on the radio or TV, or just look outside. Finally, on the basis of this information, a decision is made. The better the information, the better will be the decision.

Organizational decision problems are considerably more complex than is the problem of deciding what to wear. Not only is the decision process, and information required by this process, more complex, but the consequences of these decisions can often be measured in the millions of dollars and can influence the performance of the organization for years into the future. If good information can lead to better decisions, the payoff to an organization for using good information can be substantial.

What is good information? It is information which is timely to the decision at hand, relevant, and accurate enough to be useful. It can come

from the formal or informal analysis of quantitative data or from the formal or informal analysis of nonquantitative data. Often data from both sources are used. Seldom is it justifiable to use one or the other exclusively.

Quantitative data are associated with those phenomena in the decision environment which can be measured. Profits, costs, expenses, interest, dividends, capacities, market share, labor-hours, and demand are all examples of these phenomena.

This book is concerned with the development of methods for analyzing these quantitative data and thereby providing information that should be useful during the decision process. The methods developed include probability analysis, bayesian analysis, decision trees, linear programming, simulation, inventory methods, and network analysis. But the emphasis is not exclusively on the development of these quantitative methods. Since it is equally as important to become familiar with how these methods are used in the decision process, case studies are included at the beginning and end of each chapter. The cases at the beginnings of the chapters are used to create a realistic setting within which the methods will be presented. The cases at the ends of the chapters will test your ability to use these methods in realistic decision situations.

The case which follows will give you the opportunity to become familiar with the ways in which quantitative methods might be used in an organization. It will also be used to uncover the relationship between systems and models, a topic that will be covered later in the chapter.

CASE STUDY: Jordan Company

The Jordan Company's rate of growth for the past 7 years had been very disappointing. During the first few years of this period the stockholders and board of directors put little pressure on President Walter Grady. He continually reassured them that this was a temporary lull and that several new products about to be introduced would return the company to its historical growth rate of 8 percent. But these products never performed well, and the period of stagnation continued.

Six months ago the board finally ran out of patience and pressured Mr. Grady to resign. Upon receiving his resignation it immediately initiated a search for his successor, and just last week named Mr. Van Porter to the post.

Mr. Porter becomes the president of the fourth largest package goods company in the United States. Jordan manufactures and distributes glues, house paints, candy, potato chips, powdered drink mixes, jellies, milk, and ice cream. The company is organized around six major divisions and distributes its products from 12 manufacturing plants scattered throughout the country. Mr. Porter's job, according to a memo issued by the board, is to "end this era

of stagnation and put this complex multiproduct company back on the road to sustained growth."

Today the first in a series of staff meetings was held which Mr. Porter had scheduled for the purpose of outlining some preliminary changes in the organization, changes which he planned to initiate as soon as possible. Attending the meeting were Bill Vance, corporate vice president of marketing, and several of his staff.

Bill Vance was nervous. It was rumored that Mr. Porter was particularly unhappy with the performance of the marketing department and would make several changes within the next month. Bill wondered if one of the changes was his job. He hoped he would have a clearer picture of his future after the meeting had taken place.

At 10 o'clock, and right on schedule, Mr. Porter walked through the doors of the conference room and prepared to address the group. After a brief statement of introduction, which did not seem to put anyone at ease, he got to the main point.

"To gain some insight into market share statistics and the profitability of our product line I spent the weekend immersed in company records, computer printouts, and financial statements. It was a frustrating experience because I never got the answers I wanted. We don't seem to have the data available in convenient form. I did suspect, however, that there are some products such as powdered and evaporated milk that have been unprofitable for 10 years. Others seem quite profitable. But these hunches aren't enough.

"We must know the facts before we can manage our product line in an intelligent way. We need relevant and timely data from the divisions, and then we need a careful analysis of these data before any recommendations can be made. In short, we need to develop and utilize an effective management information system.

"This will require a substantial change in our decision-making behavior. Gone are the days when decisions can be based on hunch, seasoned judgment, and weak facts. We must learn to manage in a more scientific way.

"Accordingly, I have recommended to the board of directors two major changes. First, that the autonomy of the six divisions be substantially reduced and that we build toward a centralized organization with more authority and responsibility here at corporate headquarters. Second, that we form a new operations analysis group whose function it will be to identify problem areas, undertake studies, collect and analyze data, and make recommendations.

"Let me illustrate how these changes can encourage better decision making. The only acceptable data I could find on market share and product profitability was for our powdered drink division. They currently hold about 15 percent of the market for all powdered drinks. But in the past they have had between 10 and 50 percent of this market. I suspected that profitability increased as the market share increased and to gain more insight into this relationship plotted profitability against market share [see Figure 1-1]. As you can see, profitability does indeed increase as market share increases, but as

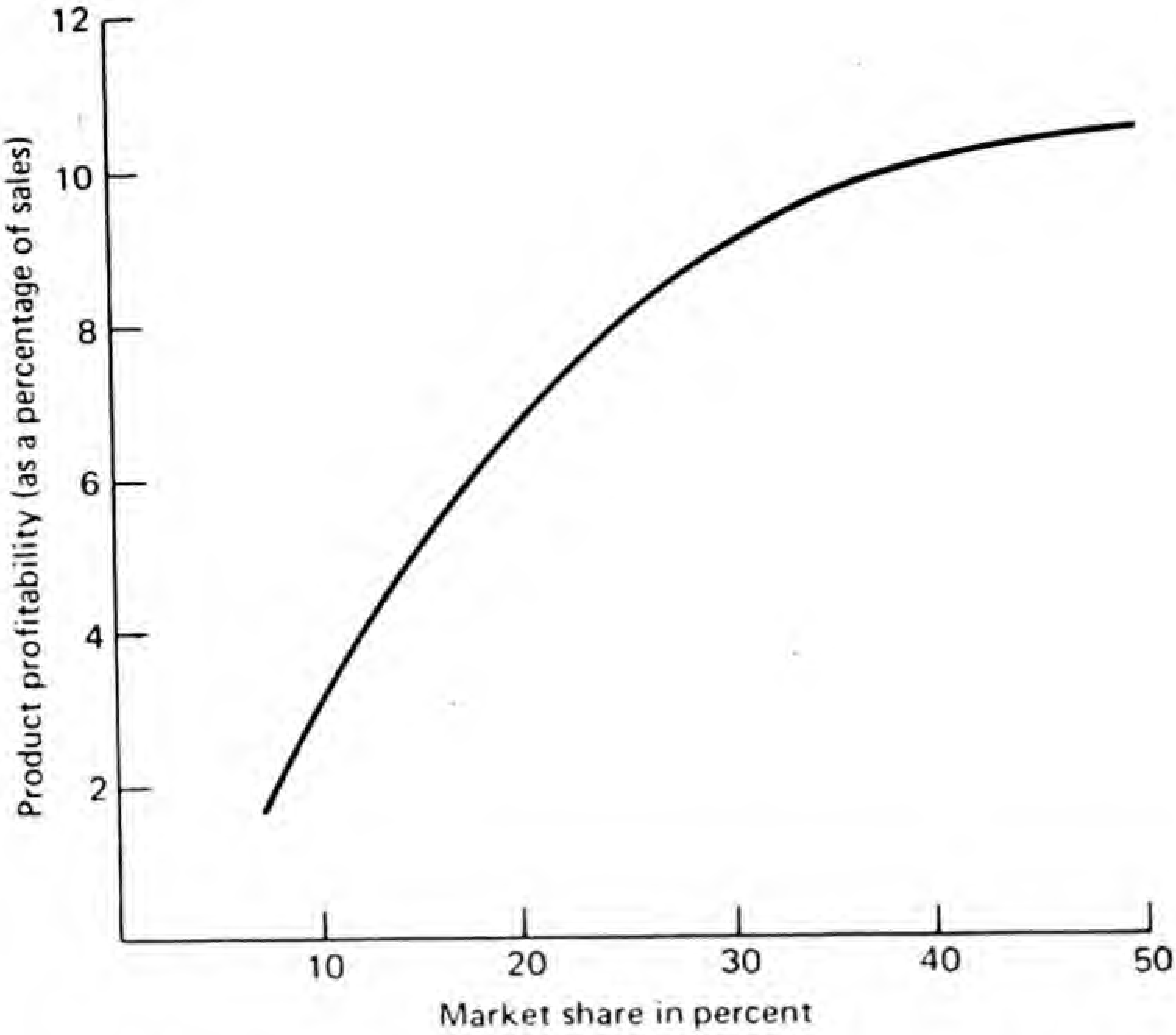


Figure 1-1 Market share and product profitability for powdered drinks.

market share reaches 40 percent any further increase does little to improve profitability. The reason behind this, I suspect, is that increases in market share beyond this level are very costly. To achieve a 1 percent increase in market share when the market share is already 40 percent costs in the vicinity of \$300,000, while it costs only \$100,000 to get this improvement if current market share is only 10 percent. From the data, then, one might seriously question the wisdom of increasing market share goals from 40 to 50 percent, but on the other hand an increase from 15 to 25 percent might make good sense.

"Good data supplied with the help of our new operations analysis group can certainly lead to better decisions. With these data we can then recommend to the divisions those products which must be dropped and set realistic market share goals for those products retained. Then, and only then, can we set advertising and promotion budgets on a product-by-product basis and hold our advertising agencies to the achievement of the market share goal within their specified budget.

"Another benefit I see from our centralization will be the control we will have over our advertising agencies. I was shocked to find that the six divisions use a total of 27 agencies. With our advertising money spread so thin we receive no close attention from any of them. Perhaps we should reduce the number of agencies to five or six. We would then become a major account for those agencies that keep our business.

"Bill, I would like you to reflect on these thoughts and report to me next week. Perhaps you can start by preparing a list of 10 products which could be dropped and collect whatever data you can find."

Bill was somewhat relieved. It was apparent that his job was not immediately in jeopardy and that his ability to adapt to this new management style would determine whether or not he would be with the Jordan Company in the future. The transition was not going to be easy!

THE ROLE OF QUANTITATIVE METHODS IN THE ORGANIZATION

Perhaps the most significant aspect of the Jordan Company case is that the company is undergoing a major change in management style. More emphasis will now be placed on facts and rigorous analysis. Quantitative methods will play a more significant role in the development of useful information.

The case points out that the use of quantitative methods can provide an organization with a wealth of information which, if used properly in the decision-making process, can lead to better decisions. Mr. Porter, for example, will be able to make a much better decision on the products to be dropped from the line, on the market share goals for those products which will remain in the line, and on the advertising budgets for those products.

Besides marketing there are countless other areas within an organization that can benefit from the use of quantitative methods. In finance, quantitative methods can be used to compare alternative capital investment strategies, help make portfolio investment decisions, and manage real estate investments. In operations, quantitative methods can be used to develop efficient production scheduling strategies, manage large construction projects, and make effective inventory control decisions. These are but a few of the areas that can benefit from this approach. If these methods are ignored, however, the firm may be overlooking a useful way of generating relevant information. The firm could then be placed at a competitive disadvantage.

But quantitative methods cannot be the only source of information upon which decisions are made. Other information must also be used. Mr. Porter, for example, might have second thoughts about dropping an unprofitable product if he found out that not only was it just six months old but that sales were rising rapidly as consumers became aware of its superior quality compared with competitive brands. Therefore, decision makers must take into account all relevant information.

The use of quantitative methods is relatively recent. Although some use of these methods was made in the 1920s, widespread use was not evident until the late 1960s. In the next section we will take a brief look at the history of these methods and explore the reasons behind their recent emergence as a major source of information in the decision-making process.

HISTORY

Quantitative methods were used as far back as the 1920s, but on a limited basis. In those years, the major telephone companies used mathematical models to determine the number of telephone circuits that were necessary to meet the demand for telephone service. Too many circuits would result in an unnecessarily large investment, while too few would result in unacceptable service to telephone subscribers. The models compared alternative circuit levels and identified the most appropriate one. These techniques are still in use today.

The 1920s also saw the first application of quantitative methods to inventory control problems. These methods helped the decision maker determine when an order for an item should be placed and how much of that item to order. Again, some of these methods, covered in Chapter 10, still remain in use today.

The use of quantitative methods remained limited until the Second World War. Then, both the British and American governments began developing and using these techniques on a broader scale. By the end of the war it was clear that quantitative methods were extremely valuable in making both strategic and tactical military decisions. Why not, then, transfer these methods to non-military decision situations in the postwar economy?

The 1950s were a decade of transference, development, and invention. Where relevant, those methods which proved so useful during the war were modified and transferred to civilian organizations. But the progress of quantitative methods did not stop there. New methods were developed by both industry and the university: the body of quantitative methods began to grow. It was not until the invention of the computer, however, that this growth rate began to accelerate at an unprecedented pace.

Quantitative methods have often required the collection and processing of large volumes of data. In the past this has meant the use of large clerical staffs. But the cost of maintaining these staffs has often overshadowed the benefits that could be attributed to these methods. Consequently, the use of quantitative methods remained limited for economic reasons.

But it was the invention of the computer that made the difference. It suddenly became possible to process huge volumes of data at a very reasonable cost. Now the use of quantitative methods made economic sense.

During the 1960s the list of successful applications began to grow. Applications were reported in the areas of marketing, accounting, operations, and finance. They were reported by manufacturing firms, banks, hospitals, government agencies, educational institutions, insurance companies, and the service industry. As these successes became known and as the cost of processing data on the computer continued to drop, the appeal of these methods attracted a wider and wider cross section of organizations.

The 1970s have seen a thorough integration of the philosophy and techniques of quantitative methods into the managerial process. Computers and

quantitative methods are no longer something new and untested. They have become an integral part of the skills required of modern managers.

SYSTEMS

A useful concept which emerged from the World War II period was the team approach. Rather than analyzing a decision problem using experts from a narrow area of an organization, the team approach requires that several individuals from related areas of an organization take part in this process. This means that a team organized to solve a marketing problem might include members from marketing, finance, and operations. The major benefit of this approach was, and still is, to ensure that a problem is viewed and solved in relation to its widest relevant consequences. In fact, this perspective is the essense of the "systems approach."

In the Jordan Company case perhaps a team approach for solving product and advertising decisions could prove to be useful. The team could consist of staff members from marketing, operations analysis, and manufacturing and might even include representatives from the advertising agencies. This team could then have the responsibility for recommending the products to be carried in the line, establishing market share goals, estimating product profitability, determining advertising budgets, and monitoring the data over a period of time to ensure the achievement of these goals.

If the Jordan Company does appoint a team, it is more likely that a systems approach to the problem will be taken. A *system* can be defined as a set of elements united by some form of interdependence and interaction between these elements and directed at achieving a goal or objective. The marketing system at Jordan would include such elements as products, budgets, consumer tastes, competition, advertising agencies, advertising media, production costs, and profits. They are all interdependent and interact with one another in certain ways. For example, if production costs increased for a certain product and competitive pressures ruled out a price increase, then product profitability would deteriorate. This could lead to lower advertising budgets, smaller market shares, and the eventual elimination of the product from the line.

Whenever one is dealing with a complex decision problem the system which surrounds the problem must be examined. What are the relevant elements? How are they interdependent? How do they interact? Complex problems cannot be narrowly defined and solved. Problems must be viewed and solved in relation to their widest relevant consequences. Often the team approach accomplishes this objective.

There is an interesting story about an inventory manager who prided himself on *never* running out of stock. Only on rare occasions were any of his shelves empty. Late in his career a team composed of representatives from production, inventory control, finance, and marketing was asked to study this inventory system which had been so "successful" for 30 years. What they

found was astonishing. His shelves were overflowing with stock. Millions of extra dollars were tied up in inventory for the purpose of preventing any possibility of an out-of-stock position. They quickly recommended a drastic reduction that would on occasion subject the organization to stockouts but would release much-needed funds for other investments. The team pointed out that the inventory manager viewed his function in very narrow terms and completely ignored the financial considerations of inventory control. Fortunately for the company, the team took a systems point of view.

A systems approach should know no organization boundaries. It should not be limited to a single functional area such as production, finance, marketing, or accounting. The bounds of the system should depend upon the particular problem under consideration. But care must be taken not to be too ambitious. It is possible to view the whole firm as a very complex *supersystem* incorporating production systems, marketing systems, financial systems, consumer systems, stockholder systems, employee systems, government systems, competitive systems, and worldwide systems. But ambition of this magnitude can be just as crippling as defining the problem from a very narrow perspective. The best strategy is to define systems which are relevant to the decision problem at hand and are of a manageable size.

We can summarize by saying that the purpose of a systems approach is to provide a larger frame of reference from which to view the decision problem. This approach should ensure that a problem is viewed from its widest relevant consequences. This should avoid the tendency to specialize and solve one part of the problem while ignoring interactions which that part has with other elements in the system.

If we are to be successful in the use of quantitative methods, decision problems must be analyzed from this point of view.

MODELS

Models are abstractions of systems, generally formulated for the purpose of solving specific problems. Models incorporate some but not all of the elements from their real-world system and often specify the relationship between these elements in an explicit way. Almost without exception these models are simpler than the systems they are designed to represent.

There are several kinds of models. Some are very informal and exist only in the mind of the decision maker. For example, many of us have models for child behavior, foreign policy, social welfare programs, and husband-wife relationships. When these topics must be discussed or decisions made, we refer to these informal models.

There are some models, however, that are not quite so informal. The engineer designing an airplane may use a wooden scale *replica* of the aircraft. This model may be flown in a wind tunnel to study the effects of wind flow on alternative design strategies. Here the model abstracts the essential physical elements from the system it is designed to represent and is used to generate

information for decision-making purposes during the design phase of the project.

In the Jordan Company case, Mr. Porter presented a *graphic* model in Figure 1-1. It depicted the relationships between two elements of the marketing system, product profitability and market share. The information generated by using a model of this kind for each product in the line could prove very helpful in making product and market share decisions.

In *mathematical* models, the relationships between elements are expressed quite explicitly in mathematical terms. Not all organizational systems, of course, can be expressed in this way. But those that can include certain investment, accounting, quality control, marketing, production, and scheduling systems, to name a few. When appropriate, mathematical models can provide extremely useful information for use in the decision-making process.

One of the major benefits of a model is that it allows the decision maker to experiment with alternative strategies. Without a model the experimentation would have to take place in the real-world system. But this is often impossible. Imagine the time and expense that would be incurred if a marketing manager had to test several new marketing strategies in the marketplace. Even if it could be done, other factors, such as changes in economic conditions, changes in competitors' strategies, and changes in consumer tastes, could mask the true consequences of these strategies. Indeed, a better approach would be to develop a model and use it to compare strategies.

Designing such models is no easy task. First it is necessary to become familiar with the real-world system, then it must be determined which elements of the system will be included in the model, then the model must be formulated, and finally it must be verified. We will now look at these steps in greater detail.

The Inclusion-Exclusion Process

Systems encountered in the real world may contain an overwhelming number of elements. Although it might seem realistic to include each and every one of them in the model, it is seldom effective to do so. Some elements must be ignored and excluded. The reason behind this is that the model must be kept to a manageable size. If too many elements are included, uncovering and expressing the interrelationships between these elements may become extremely difficult. In addition, the cost of developing large models may be prohibitive. Finally, decision makers often find large models too complex to use.

It is essential to keep in mind that the purpose of a model is to generate information for use by decision makers during the decision-making process. Therefore, the developmental stage of a model should never be limited to the technical aspects of the model itself. The role that it will play in the decision-making process must maintain a high priority. If the model is large,

complex, and difficult to use, no one will use it. For this reason simpler models with fewer elements are often more effective than complex ones.

What elements should be included? Only those elements which dominate a system should be included. Dominant elements are those that explain most but not necessarily all of a system's behavior.

In the development of a model for an inventory system, most models would include the per unit cost of purchasing an item, the cost of carrying that item in inventory, the fixed costs of placing an order, and the level of demand for that item. Experience with countless inventory systems has shown that these elements dominate most inventory systems.

But there are some inventory systems for which the elements mentioned in the last paragraph would be inadequate. In those situations, quantity discounts and out-of-stock penalties might also play a dominant role and must be included in the model. We must therefore conclude that a model for one system might be woefully inadequate as a model for another. There have yet to be developed generalized models that can be used in solving *all* inventory problems, scheduling problems, staffing problems, or investment problems. It is safer to assume, at the beginning, that each system is unique and that its dominant elements must be identified.

The model used in the Jordan Company case (Figure 1-1) includes two elements: profitability and market share. It is simple and useful. But several elements which could have made it more useful have been excluded. Certainly, profit is not strictly tied to market share. Production costs, cost of raw materials, administrative expenses, and market price might also have been included. We can see, therefore, that the model used by Mr. Porter excluded several elements. Nevertheless, the model did convey some important information. Do you think that the inclusion of other elements could have improved the usefulness of the model?

Decision makers who use these models must exercise a reasonable degree of caution. They must be aware of the elements that have been included and excluded. Above all they must recognize that the model is not a perfect representation of reality. If it was, other sources of information used in the process of reaching a decision would be unnecessary.

When Is a Mathematical Model Appropriate?

Some but not all systems can be modeled mathematically. Those that can share the characteristic that their dominant elements can be quantified. This means that they can be measured in terms of costs, profits, market share, labor-hours, efficiencies, rates of return, dividends, and so on. In addition, the relationships among these elements also can be expressed mathematically.

Formulation of the Model

Once the dominant elements have been identified and it has been determined that a mathematical model is appropriate, the formulation of the model can begin.

To illustrate this procedure, a profit model for an automotive manufacturer will be formulated. Suppose that this manufacturer produces just autos and trucks. Let the symbol A represent the number of autos produced, and let T represent the number of trucks produced. Naturally, each of these elements can be measured.

We learn from financial data that the company makes a \$100 profit on each auto and a \$150 profit on each truck. If the symbol P represents the profit level, the profit relationship between these two products can be expressed in the following way.

$$P = 100A + 150T$$

Suppose that 50 autos and 100 trucks are produced. Our model will tell us that the profit is \$20,000.

$$P = 100(50) + 150(100)$$
$$P = 20{,}000$$

We can therefore conclude that this is a mathematical model depicting the relationship between profit and levels of production. Although this model is too simple to be useful, it will be expanded later in the book to include other relevant elements and will then prove to be very useful in certain decision-making situations.

Verification of the Model

After the model has been formulated, it is necessary to verify that the model is indeed a reasonable representation of the real-world system.

Two methods of verification are used. One is to return to the model and take a fresh look at the elements included. Are they the dominant elements in the real-world system? Are the excluded elements of minor importance? Are the interrelationships expressed by the model reasonable?

Another method is to test the model by using it. Then the outcome is carefully analyzed. Do these results correspond to the outcome that would be expected from the real-world system?

Whatever method of verification is used, it is important that the model be critically examined. Only if it passes this test should it be used as part of the decision-making process.

Using the Model to Solve Decision Problems

Faced with a complex decision, the decision maker can often obtain useful information from a quantitative model. The model can be analyzed and manipulated for the purpose of gaining insight into the problem. It can be used to explore the consequences of alternative strategies. But it is not the only source of information. The final decision must depend upon several sources of information of which the model is but one.

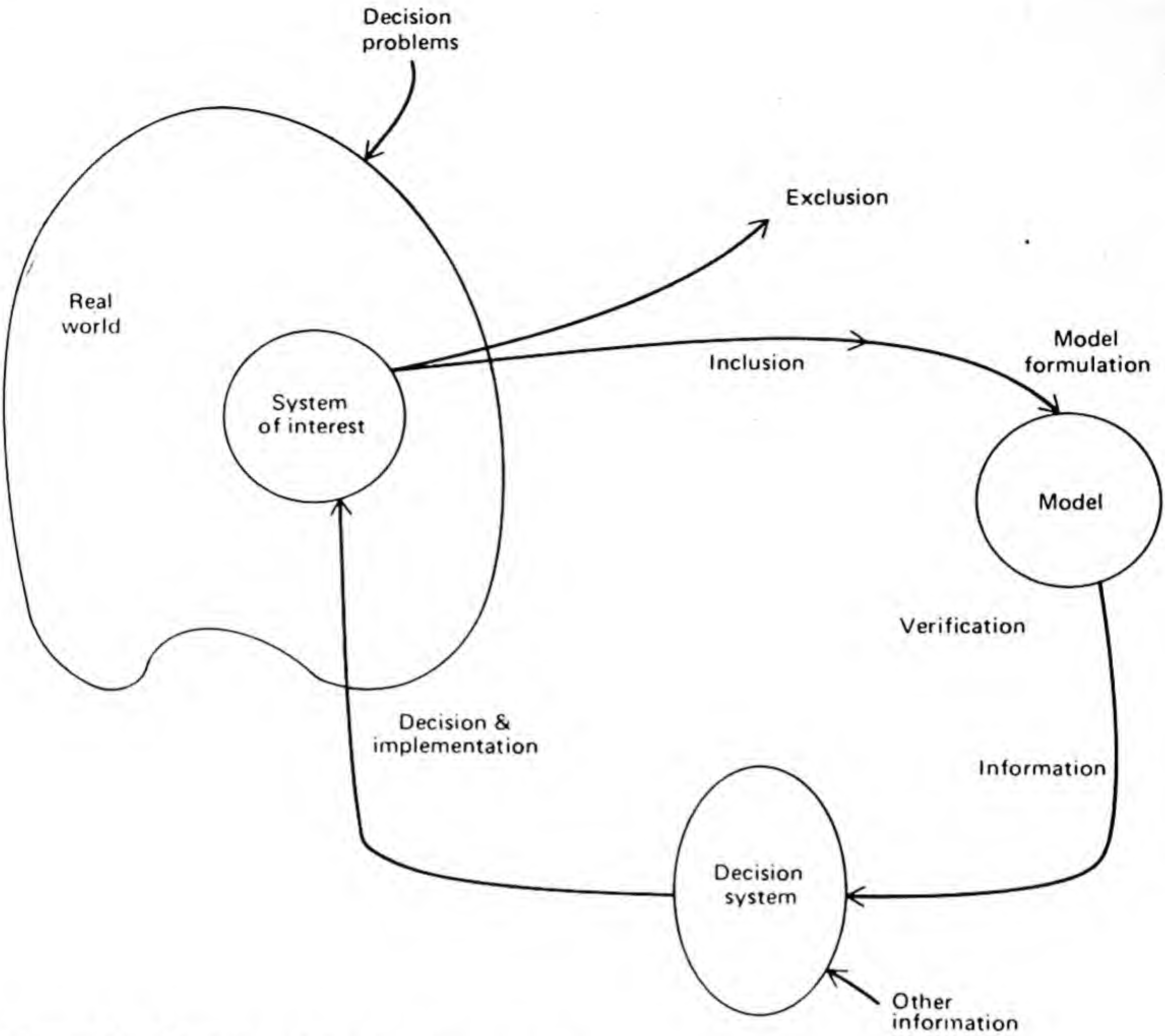


Figure 1-2 Models and the decision process.

The Modeling Process Summarized

The modeling process is summarized in Figure 1-2. The need for a model arises because a decision problem must be resolved. The first step is to isolate the system within which this problem exists and gain familiarity with the elements which comprise this system. Then it must be determined which of these elements are to be included and excluded from the model. Next, the model is formulated by establishing the relationship among the included elements. The model is then verified to ensure that it is a reasonable representation of the system of interest.

After these steps have been completed, the model can be used to generate information for the decision process. The decision maker takes this information together with information generated from other sources and reaches a decision. The last step is to implement the results of the decision.

QUESTIONS

1 What role has the computer played in the development and use of quantitative methods?
2 What is the team approach, and why will it encourage a systems point of view?
3 Identify the elements in a stereo system. Suppose that you just spent $800 on such a system. The receiver cost $400, the turntable cost $200, and the tape deck cost $150. The remainder was spent on the speakers. Was a systems view taken?
4 To be on the safe side, models should include as many elements as possible. Do you agree or disagree, and why?
5 What steps taken by Mr. Porter will lead to a systems point of view for the Jordan Company?
6 What are the consequences of not taking a systems point of view?
7 Do you think that the continuing increase in the sophistication of mathematical models will eventually eliminate the need for middle management?

CASE STUDY: Krause Company

The Krause Company is a manufacturer of mechanical and electrical parts used in large printing presses. It is located in Wichita, Kansas, and has been in business since 1953. Sales of over $15 million are projected for 1978.

In September 1977, the company obtained a license to manufacture and distribute a rotary conductor, part K5537G, which was designed and patented by a West German firm. Mr. Krause considers this product superior to any other on the market.

Rotary conductors have two functions. First they act as bearings which connect a rotary printing surface to the frame of a printing press. Second, they transmit electrical current to this rotating surface. Both the mechanical and electrical design problems associated with a complex unit such as this are quite substantial. The West German unit solves these problems in a unique way.

Conductors currently sold by Krause's competition have not satisfactorily solved these design problems. Consequently, they have an average life of only 3 months. When these units fail, the entire printing press must be stopped and a $60 replacement installed. During the 7 hours it takes to complete this job, the press is idle as are the workers who operate the machine.

Market Survey Results

Nancy Green, vice president of marketing, recently surveyed the market for rotary conductors. She spoke with printing companies who would be interested in the part as a replacement for those currently in use, and she also spoke with manufacturers of new presses. When they heard that Krause's unit had a life expectancy of 9 months, they became very enthusiastic. Green concluded from this survey that if the price could be held to $140 per unit, sales could average 2400 units per year.

The survey, however, identified a problem. The physical dimensions of part K5537G were not identical to those of all other parts currently in use. One unit could not possibly fit all existing presses. But it was possible, contended Green, for existing presses to be modified to accept a standard unit.

However, Green did not believe that a strategy of offering a single version was reasonable. In a conversation with the president she explained: "Carl, we have to offer the full line of 24 versions. Without the full line, we won't be able to capture any substantial share of the market. Our customers will not spend several hundred dollars to modify their presses just so they can buy our part. Moreover, you know our customers—when they need a part they need it in a hurry. We've got to be able to fill incoming orders within 12 hours. If we can sell them at $140 and give our usual one-day service, I figure we'll see about 10 orders coming in every month, and each production order should average about 20 identical units.

"But, Carl, we've got to offer that full line of 24 versions. A single standard part just won't fill the bill."

Production Control

Donald Heintz, manager of production control, was in Krause's office minutes after Green left.

"Look, Carl, I've heard what Green and that marketing bunch have up their sleeves, and it won't work. You approve 24 different versions, and our costs will go right through the roof.

"If I am interpreting Green's estimates correctly, approximately 10 production orders will be issued each month for an average of 20 identical units. There's no way that I can schedule all those jobs on our new automatic machinery center. I've examined the engineering drawings for these conductors, and it will take us 10 hours each time we set up that half-million-dollar baby. The cost of these setups will be $300 each. Now I admit that it will take only 1 hour to manufacture the part once the machine is set up, and per piece costs will be only $25, but a 10-hour setup for small jobs needed in a hurry is crazy! Besides, our available capacity on this machine will simply not accommodate 100 hours of setup time per month."

Krause thought for a minute and asked: "Don, what do our costs look like if we *don't* put this part on the auto center?"

"Well, if I put it through our conventional machines, the setups will drop to $2\frac{1}{2}$ hours, for a total cost of $75. But, Carl, the manufacturing time will be 3 hours and per piece costs will jump to $40. With 24 versions I can't give you the service you need with the auto center, and the costs are too damn high. Now make a standard part and I'll run lots of 200 through the auto center and we can keep them on hand in inventory. That'll spread out the $300 setup cost and keep the average cost per unit low."

QUESTIONS

1 Describe the problem facing Krause Company.
2 Identify the alternatives presented in the case.
3 In the context of this problem, what are the objectives of the marketing department? Finance? Production? The firm?
4 Discuss the difference between a systems and a nonsystems approach to describing and solving the problem.
5 Formulate a mathematical model which relates the total cost of producing a unit to its fixed and variable machining costs. Do this for both the automatic machining center and the conventional machining method.
6 Compute the per unit costs of production for each alternative using the results of question 5.
7 Identify the included and excluded elements of the model formulated in question 5.
8 Do the results of question 6 clearly point to a solution of the problem?

9 How can the results of question 6 be used to find a systems solution to the problem?
10 Have any alternative strategies been left out?
11 Are there any organizational or managerial strategies which could be employed to assure a systems definition and solution of the problem?
12 Discuss the role of quantitative methods in analyzing this case and reaching a decision.

Chapter 2

Probability Concepts

INTRODUCTION

If decision makers had access to a crystal ball and if this crystal ball were able to eliminate the uncertainty which surrounds most systems, it would be unnecessary to dwell on probability concepts at all.

But there are no crystal balls. Uncertainty cannot be eliminated. It should be acknowledged and not ignored. Indeed, any information which a decision maker can obtain about the uncertainty of a system should be carefully used in the decision-making process.

Information about uncertainty is often available to the decision maker in the form of probabilities. The purpose of this chapter is to explore some of the fundamental concepts of probability and to examine the way in which probability estimates can be made.

In the case study which follows, some of these concepts and problems of estimation are introduced in the context of a decision problem.

CASE STUDY: Star Appliance Company

One year ago the Star Appliance Company added a new TV set to their line of home entertainment products. The set was an immediate success, and sales increased rapidly. This year sales will reach 120,000 units.

Recently, the marketing department has complained that sales have been lost because the warehouse has occasionally run out of stock. When the company president became aware of this inventory problem, he asked several of his top managers to take a close look at it.

Star's current strategy is to manufacture 10,000 sets per month. On the last day of each month these sets are shipped to their warehouse and distribution center located 200 miles from the factory. During the next month this inventory is slowly depleted and is reduced to very low levels by the end of the month. Again, on the first of the following month, a new shipment of 10,000 units arrives from the factory, and inventory is replenished.

When demand during any one month runs above average, these 10,000 units are depleted ahead of schedule and the warehouse is empty during the last few days of the month. But when demand during other months is below average, there are still units in stock when the new shipment arrives.

The marketing, production, and inventory departments agreed to meet and discuss this problem. At the meeting the manager of marketing, Joe Voci, was in favor of keeping an extra 3000 sets in inventory at all times. He felt these sets would then be available to meet demand in those months when demand was above average. In this way the company would never run out of stock and no customers would be lost to their competitors.

The inventory manager, Lester Wilson, thought that this was an unreasonably large surplus to keep in stock. He asked Joe what the highest and lowest possible levels of demand were for any one month. Joe replied, "I would estimate the highest to be about 13,000 units and the lowest to be about 7000 units. On the average, our demand will continue to be about 10,000 units per month."

Lester wanted to pursue this point further. "Joe, exactly what is the chance that demand will be as high as 13,000?"

"Not very likely," was the reply.

Lester became insistent. "Exactly how likely?"

"About a 1 percent chance," answered Joe.

"And what is the likelihood that demand will be 12,000?" continued Lester.

"About 5 percent," was Joe's reply.

Lester continued asking for the probabilities at different demand levels. The results, taken from his notes, appear in Exhibit A. Lester examined his notes carefully and then spoke. "If I understand you correctly, Joe, there is a 25 percent chance that demand will exceed 10,000 units, but only a 6 percent chance that it will exceed 11,000 units. Suppose we decide to stock an additional 1000 units. This means that the chance of demand exceeding our supply is only 6 percent. Doesn't this seem like a reasonable risk to take? It seems unreasonable to increase stock by 3000 units.

"Carrying any extra stock is risky and costly," Lester went on. "Engineering is always making improvements on our products, and I don't want to be stuck with thousands of obsolete units. As far as the cost is concerned,

Exhibit A

Demand level (units)	Probability
7,000	.01
8,000	.05
9,000	.19
10,000	.50
11,000	.19
12,000	.05
13,000	.01

carrying extra units in stock is costly. We might need extra stockroom personnel, we will have to pay insurance on the extra stock, and it is very likely that we will have to lease additional warehouse space. I agree that the chances of running out of stock are too high if we carry only 10,000 units in inventory. But I disagree with the strategy of carrying an extra 3000 units. Our warehouse costs will be too high. I vote for adding 1000 units to stock."

THE USE OF PROBABILITY CONCEPTS

The Star Appliance case illustrates the role that probability concepts can play in solving decision problems. We saw how these concepts can be used to structure a problem which could have remained ambiguous. By estimating the probability associated with each level of demand, it became possible to establish an inventory level on the basis of better information.

But these probabilities do not provide all the information that is needed to make the decision. Other information, including the cost of carrying additional inventory and the risk of obsolescence, should also be considered.

It is important to recognize that even with the availability of quantitative information, the best strategy was still not obvious. But the use of this information reduced the ambiguity surrounding the problem and provided some much-needed insight.

The following sections will be devoted to the logical development of the probability concepts used in this case. First we turn to some definitions.

DEFINITIONS

Any process that leads to results for which probability concepts are applicable can be called a *random experiment*. For example, the tossing of a coin or the drawing of a poker hand is a random experiment because probability concepts can be used to express the possible results. Another example is the demand for TV sets in the Star Appliance case.

The different possible results of an experiment are called basic *outcomes*. Therefore the outcomes of a coin-tossing experiment are heads and tails; the

outcomes of demand for the Star Appliance case are 7000 units, 8000 units, 9000 units, and so on. The set of *all* possible basic outcomes in an experiment is called the *sample space.*

An *event* is a subset which consists of one or more basic outcomes of the sample space. Suppose an experiment is "the throwing of a die." The basic outcomes are 1, 2, 3, 4, 5, and 6. One possible event would be the throwing of a 1, 2, or 3. Another event would be the throwing of a 3 or 5 on any one toss of the die. These are events because they consist of one or more basic outcomes of the sample space.

BASIC STATEMENTS OF PROBABILITY

There are two fundamental statements of probability:

1 The probability of the occurrence of an event is always between 0 and 1.
2 The probabilities of all mutually exclusive and collectively exhaustive events must sum to 1.

The first statement requires that the probabilities of events should be between 0 and 1. If the probability of an event is 0, it is not possible for that event to occur. If the probability of an event is 1, the event will happen with certainty. As the probability changes from 0 toward 1, it becomes more and more likely that the event will occur.

In order to examine the second statement, the terms "mutually exclusive" and "collectively exhaustive" must be defined. Two events are said to be *mutually exclusive* if they do not contain common outcomes in the same experiment. Consider the tossing of a coin. One of two events will occur: either a head or a tail. These two events are said to be mutually exclusive since they do not contain common outcomes.

When a list of events for some experiment includes *all* possible events, the list is said to be *collectively exhaustive.* Consider the events for a coin-tossing experiment. Such a list includes:

Head
Tail

This list is collectively exhaustive because it includes all possible events.

Adding to the list the probabilities of tossing a head and tail, we obtain the following.

Event	Probability
Head	0.5
Tail	0.5
	1.0

Referring back to the two statements of probability we can verify that they have been met. The probability of each event is between 0 and 1. The events are mutually exclusive and collectively exhaustive. Furthermore, the probabilities sum to 1.

Returning to the Star Appliance Company case one can see that the two fundamental statements of probability have also been met in Exhibit A. First the probability of the occurrence of any demand level (event) is between 0 and 1. Second, each level of demand is mutually exclusive because none of them contains common outcomes. The list is also collectively exhaustive since all possible demand levels have been included in the exhibit. Finally, the probabilities associated with these mutually exclusive events sum to 1.

OBJECTIVE AND SUBJECTIVE PROBABILITIES

Probabilities can be classified as either objective or subjective. Objective probabilities are supported by relative frequency, the structure of the situation, or available historical evidence.

Consider the tossing of a fair coin. What is the probability of a head? Most would agree that it is $1/2$. In fact if a coin were tossed an infinite number of times, the *relative frequency* of heads would be $1/2$. It is this relative frequency analysis that supports the probabilities that are assigned to coin-tossing events, and we can therefore conclude that these are objective probabilities.

Consider an urn filled with three red, two green, and five white balls. If they are thoroughly mixed, one may say that the probability of selecting a certain colored ball is the following:

Event	Probability
Red	$3/10$
Green	$2/10$
White	$5/10$

These are objective probabilities since they are based upon the *structure of the situation.*

Objective probabilities are used by decision makers in such areas as auditing, quality control, and insurance. But in general there are few systems in an administrative environment that can be described with objective probabilities.

Subjective probabilities are based on personal experience or intuition. In the Star Appliance case, for example, probability estimates of demand were made from *personal experience* with the product. Objective facts upon which to base this demand forecast were unavailable. Therefore, subjective probabilities were used in the absence of objective data.

Since many management decisions must look into the future and since there is seldom objective evidence upon which to base predictions, subjective probabilities must be used. Any rational system of decision making should take into consideration all the relevant information that is available regardless of whether the source of this information is subjective or objective.

Probability concepts are the same for both subjective and objective probabilities. In the following sections we continue to explore these concepts.

A PROBLEM

The probability concepts to be covered in the remainder of this chapter will be developed around the following problem.

A large convenience-food producer has assembled a panel of 35 consumers for the purpose of evaluating several new products which the company hopes to introduce shortly. The consumers come from the Midwest, Northeast, Southwest, and Southeast areas of the United States. The group includes 17 women and 18 men. Table 2-1 categorizes the panel by area of the country and sex. From the table it can be seen that of the 5 panelists from the Midwest, none are female and 5 are male. Of the 10 panelists from the Northeast, 6 are female and 4 are male. Of the 10 panelists from the Southwest, 2 are female and 8 are male. Of the 10 panelists from the Southeast, 9 are female and 1 is male.

Suppose it is desired to select a panelist at random and ask that person several questions. The process of selecting a person would represent the experiment, and the selection of a particular person would represent the outcome of that experiment. The set of all 35 panelists would represent the sample space.

Since an event consists of one or more basic outcomes, the category "Midwest" can be described as an event. All outcomes which fall into that category will be classified as event 1. Similarly, the Northeast, Southwest, and Southeast categories will be classified as events 2, 3, and 4. These events are identified in Table 2-2. In addition, the female and male categories will be classified as events 5 and 6.

	Female	Male	
Midwest	0	5	5
Northeast	6	4	10
Southwest	2	8	10
Southeast	9	1	10
	17	18	35

Table 2-1 Categorization of 35 panelists into geographic regions and sex.

	Event 5 Female	Event 6 Male	
Event 1: Midwest	0	5	5
Event 2: Northeast	6	4	10
Event 3: Southwest	2	8	10
Event 4: Southeast	9	1	10
	17	18	35

Table 2-2 Event table.

MARGINAL PROBABILITIES

Now we are ready to select a panelist at random. Can you determine the probability that the panelist selected will be from the Midwest? Since there are a total of 35 panelists and 5 of them are from the Midwest, the probability of selecting one of them is $5/35$.

What is the probability of selecting a panelist from the Northeast? You should conclude that it is $10/35$. Can you determine the probability of selecting a panelist from the Southeast? It is $10/35$.

Since the outcome "Midwest" was classified as event 1, the probability of selecting a panelist at random who comes from the Midwest can be expressed in the following way:

$$P(E_1) = 5/35$$

where E_1 represents event 1. Continuing with the other events we have:

$$P(E_2) = 10/35$$
$$P(E_3) = 10/35$$
$$P(E_4) = 10/35$$
$$P(E_5) = 17/35$$
$$P(E_6) = 18/35$$

These probabilities are all called *marginal* or unconditional probabilities. They are given this name because they can be computed from the margin of the table.

THE INTERSECTION OF TWO EVENTS: JOINT PROBABILITY

Consider events E_2 and E_5. Their *intersection* is defined as those panelists who are *both* from the Northeast and female. In other words, the intersection of two events is a new subset which includes outcomes common to both

events. The intersection of these two events is written as $E_2 \cap E_5$ where the symbol $\cap$ is read as "and." Therefore, $E_2 \cap E_5$ can be read as the joint occurrence of event 2 *and* event 5. In the table the new subset $E_2 \cap E_5$ occurs at the intersection of the second row and first column.

The probability of this occurrence can be read directly from the table. The probability of selecting a female Northeasterner is $^6/_{35}$ because there are 6 people who are both from the Northeast and female out of 35 panelists. This probability can be expressed in the following way:

$$P(E_2 \cap E_5) = {}^6/_{35}$$

Can you determine the probability of selecting a male panelist from the Southwest? It is $P(E_3 \cap E_6) = {}^8/_{35}$

CONDITIONAL PROBABILITIES

Suppose that all female panelists are eliminated from consideration. Only male panelists remain. Given this condition, what is the likelihood of selecting a panelist from the Northeast? Since there are 4 male panelists from the Northeast and the total number of male panelists is 18, the chance of selecting one from the Northeast is $^4/_{18}$. This is called a conditional probability because it is based on the condition that only male participants remain in consideration.

Using standard notation, the example presented in the last paragraph can be represented in the following way:

$$P(E_2 | E_6) = {}^4/_{18}$$

The vertical line implies a conditional statement and is read as "given." Therefore we can read this statement as the probability of E_2 given that E_6 has occurred. That is, given that the sample space has been reduced to E_6, what is the probability of E_2?

Can you answer the following questions?

$$P(E_2 | E_5) = ?$$

$$P(E_3 | E_6) = ?$$

$$P(E_1 | E_5) = ?$$

The answers are $^6/_{17}$, $^8/_{18}$, and 0.

RULES OF ADDITION

Now that the concepts of marginal, joint, and conditional probabilities have been covered, we can turn to the rules for addition and multiplication.

The General Rule of Addition

Suppose that we would like to determine the probability of event 2 *or* event 6 occurring. Using standard notation, this is referred to as

$$P(E_2 \cup E_6)$$

where the symbol $\cup$ is read as "or." Therefore the statement can be read as the probability of the occurrence of event 2 *or* event 6.

To determine this probability, return to Table 2-2. Now we must identify those panelists who either are male *or* come from the Northeast. There are 6 panelists who are both female *and* come from the Northeast, 4 who are male and come from the Northeast, 5 who are male and come from the Midwest, 8 who are male and come from the Southwest, and 1 who is male and comes from the Southeast. Therefore we have

$$6 + 4 + 5 + 8 + 1$$

or 24 panelists who are in one or the other category. We can conclude that the probability of selecting a panelist who is either male or from the Northeast is

$$P(E_2 \cup E_6) = {}^{24}\!/_{35}$$

In a very intuitive way we have just developed the general rule of addition. Now we will state what has just been done, but in a more rigorous way. Let A and B represent any two events. Then the *general rule of addition* can be written in the following way:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

This states that the probability of event A or B occurring is equal to the probability of event A plus the probability of event B minus the probability of their intersection. Let's rewrite this rule for our specific events E_2 and E_6:

$$P(E_2 \cup E_6) = P(E_2) + P(E_6) - P(E_2 \cap E_6)$$

From Table 2-2 we have

$$P(E_2) = {}^{10}\!/_{35}$$
$$P(E_6) = {}^{18}\!/_{35}$$
$$P(E_2 \cap E_6) = {}^{4}\!/_{35}$$

We therefore have

$$P(E_2 \cap E_6) = {}^{10}\!/_{35} + {}^{18}\!/_{35} - {}^{4}\!/_{35}$$
$$P(E_2 \cap E_6) = {}^{24}\!/_{35}$$

and this coincides with the answer which we obtained from our intuitive approach. Let's compare both approaches, and you will see that they are the same.

$$\text{Intuitive approach: } P(E_2 \cup E_6) = \frac{6+4+5+8+1}{35} = \frac{24}{35}$$
$$= ({}^{6}\!/_{35} + {}^{4}\!/_{35}) + ({}^{5}\!/_{35} + {}^{8}\!/_{35} + {}^{1}\!/_{35}) = {}^{24}\!/_{35}$$

counted only once

$$= {}^{24}\!/_{35}$$

$$\text{By the general rule: } P(E_2 \cup E_6) = P(E_2) + P(E_6) - P(E_2 \cap E_6)$$
$$= {}^{10}\!/_{35} + {}^{18}\!/_{35} - {}^{4}\!/_{35}$$
$$= ({}^{6}\!/_{35} + {}^{4}\!/_{35}) + ({}^{5}\!/_{35} + {}^{4}\!/_{35} + {}^{8}\!/_{35} + {}^{1}\!/_{35}) - {}^{4}\!/_{35}$$

counted twice and
then subtracted

$$= {}^{24}\!/_{35}$$

In the intuitive approach we were careful to count the intersection of E_2 and E_6 only once. Using the general rule, if the probability of E_2 is added to the probability of E_6, the intersection $E_2 \cap E_6$ is counted twice. Therefore in the general rule we find the probability of one intersection subtracted. Otherwise we would have double-counted.

Mutually Exclusive Events

Earlier in the chapter we defined mutually exclusive events in the following way. Two events are said to be mutually exclusive if they do not contain common outcomes in the same experiment or trial. Returning to Table 2-2, we can see that events 1 and 5 are mutually exclusive because a zero is entered in that cell. Therefore it would not be possible to select a female panelist from the Midwest.

$$P(E_1 \cap E_5) = 0$$

When, in fact, two events are mutually exclusive, the general rule of addition can be simplified to the special rule of addition.

Special Rule of Addition

We have just seen that when two events are mutually exclusive, their joint probability is 0. If A and B are any two mutually exclusive events, then

$P(A \cap B) = 0$, and the general law of addition can then be simplified to the following.

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

This is called the *special rule of addition*. Returning to Table 2-2, we can compute the $P(E_1 \cup E_5)$ in the following way: $P(E_1 \cup E_5) = P(E_1) + P(E_5) = {}^{5}\!/_{35} + {}^{17}\!/_{35} = {}^{22}\!/_{35}$.

RULES OF MULTIPLICATION

We will now examine the rules of multiplication. First the general rule of multiplication will be presented. This will be followed by the concept of independence, and finally the special rule of multiplication.

General Rule of Multiplication

The general rule of multiplication can be developed by returning to the concept of conditional probability. In an earlier section we determined that the probability of event 2 *given* event 6 was the following:

$$P(E_2 \mid E_6) = {}^{4}\!/_{18}$$

Let's look behind this ratio. The numerator 4 represents the number of panelists who are both male and from the Northeast. Said another way, it represents the number of panelists who belong to the joint subset $E_2 \cap E_6$. The denominator represents the number of panelists who are male. Consequently, the conditional probability can be expressed in the following way:

$$P(E_2 \mid E_6) = \frac{P(E_2 \cap E_6)}{P(E_6)} = \frac{{}^{4}\!/_{35}}{{}^{18}\!/_{35}} = {}^{4}\!/_{18}$$

Generalizing this conditional probability statement for any two events A and B, we have:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

If we cross-multiply, we have

$$P(A \mid B)P(B) = P(A \cap B)$$

and by reversing sides this becomes

$$\boxed{P(A \cap B) = P(A|B)P(B)}$$

which is the *general rule of multiplication.*

Independence

Two events are said to be *independent* if the occurrence of one is in no way affected by the occurrence or nonoccurrence of the other. Consider the following events:

Event A = head on the second toss of a coin
Event B = head on the first toss of a coin

From our experience and intuition we know that the result of a second toss of a coin is in no way affected by the occurrence or nonoccurrence of a head on the first toss. Event A and event B are independent.

What about the following two events?

Event $A = 3$ on the second toss of a die
Event $B = 5$ on the first toss of a die

Again these two events are independent, for the outcome of the second toss in no way depends on the outcome of the first toss.

If events are not independent, they are *dependent*. Consider the following two events:

Event A = drawing an ace from a deck of 51 cards on the second draw
Event B = drawing an ace from a deck of 52 cards on the first draw

Clearly the likelihood of event A is dependent on the outcome of event B. That is, if an ace is drawn on the first draw, there are only 3 aces left in the 51-card deck and the probability of drawing a second ace is $3/51$. If on the other hand a non-ace was drawn first, the probability of drawing an ace on the second draw is $4/51$. In this case it can be concluded that the probability of event A is dependent on the outcome of event B.

Special Rule of Multiplication

If two events are independent, any given information about the one event does not affect the probability of occurrence of the other event. Therefore if A and B are independent events, then

$$P(A|B) = P(A)$$

Returning to our example, where

Event A = head on the second toss of a coin
Event B = head on the first toss of a coin

then $P(A|B) = P(A) = \frac{1}{2}$, since the probability of a head on the second toss is the same regardless of the outcome of the first toss.

Returning to the general rule of multiplication, we had the following:

$$P(A \cap B) = P(A|B)P(B)$$

If A and B are independent events, this rule can be simplified in the following way:

$$\boxed{P(A \cap B) = P(A)P(B)}$$

This is called the *special rule of multiplication.*

Returning to our example, where

Event A = head on the second toss of a coin
Event B = head on the first toss of a coin

the likelihood of a head on the first toss *and* a head on the second toss can be computed by using the special rule of multiplication.

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

SUMMARY

Often systems are composed of elements whose behavior is not known with certainty. When this is the case, probability concepts must be used in the formulation and analysis of system models.

In the next chapter these concepts are used to develop a model for solving certain decision problems under risk.

QUESTIONS

1 Define the following terms:
  a Mutually exclusive
  b Collectively exhaustive
2 Differentiate between objective and subjective probabilities.
3 Can two events be mutually exclusive and independent at the same time?
4 Differentiate between conditional and unconditional probabilities.

5 A physician has just completed a test and now feels that the likelihood of your having a specific disease is .7. Is this an unconditional or conditional probability?

6 Are the following events dependent or independent?

- a Two throws of a die
- b The outcomes of a series of medical tests
- c The selection of five cards from a deck of cards
- d Today's stock price and tomorrow's stock price
- e The level of sales this month and next month

PROBLEMS

2-1 Of 100 students in a particular course, 20 received a grade of A (Event 1), 30 received a grade of B (Event 2), 30 received a grade of C (Event 3) and 20 received a grade of D (Event 4). In the accompanying table, this group of 100 students is

Course grade	Dean's list	Event 5 Yes	Event 6 No	
Event 1	A	20	0	20
Event 2	B	15	15	30
Event 3	C	25	5	30
Event 4	D	1	19	20
		61	39	

further broken down into those who were on the dean's list (Event 5) that semester and those who were not (Event 6). Given this information, answer the following questions:

- a What is the probability that a student selected at random will be on the dean's list?
- b What is the probability that a student selected at random received a C in the course?
- c Given that the student is on the dean's list, what is the likelihood that his course grade was B?
- d What is the probability that a student received a B and is on the dean's list?
- e What is the probability that a student received an A and is not on the dean's list?
- f $P(E_1 \cap E_5)$?
- g $P(E_3 | E_6)$?
- h $P(E_1 \cup E_5)$?
- i $P(E_1 \cup E_2)$?

2-2 Given that the $P(A) = .6$ and $P(B) = .4$, find the following:

- a $P(A \cap B)$ if A and B are independent.
- b $P(A \cap B)$ if A and B are not independent.
- c $P(A \cup B)$ if A and B are mutually exclusive.

d $P(A \cup B)$ if A and B are not mutually exclusive, but are independent.
e $P(A|B)$ if A and B are independent.

2-3 Given that the $P(X) = .2$ and $P(Y) = .3$, find the following:
a $P(X \cap Y)$ if X and Y are independent.
b $P(X \cap Y)$ if X and Y are not independent.
c $P(X \cup Y)$ if X and Y are mutually exclusive.
d $P(X \cup Y)$ if X and Y are not mutually exclusive, but are independent.
e $P(X|Y)$ if X and Y are independent.

2-4 Of 1000 shoppers who have made recent purchases in a department store, 300 made small purchases (less than \$5), 500 made average purchases (between \$5 and \$15), 150 made large purchases (between \$15 and \$50), and 50 made major purchases (over \$50). In the accompanying table, these purchases are further broken down according to whether the customer had a charge card for that particular department store.

		Event 5 does have	Event 6 does not have	
Event 1	Small	100	200	300
Event 2	Average	260	240	500
Event 3	Large	100	50	150
Event 4	Major	50	0	50
		510	490	

Given this information, answer the following questions:
a What is the probability that a customer selected at random made a large purchase?
b What is the probability that a customer selected at random does not have a charge card?
c What is the probability that a customer selected at random made a large purchase with his charge card?
d Of those that do have charge cards, what percentage made average purchases?
e Given that a customer did not have a charge card, what is the likelihood that a major purchase was made?
f $P(E_2 \cap E_6) = ?$
g $P(E_4 \cup E_5) = ?$
h $P(E_2|E_5) = ?$
i $P(E_3 \cup E_6) = ?$

2-5 Given the following:

E_1 = head on the first toss
E_2 = head on the second toss
E_3 = head on the third toss
E_4 = head on the fourth toss

find the following:

$$P(E_1 \cap E_2 \cap E_3 \cap E_4)$$

In other words, you will find the likelihood of tossing four heads in a row.

2-6 Given the following:

$E_1 = 3$ on the first toss of a die
$E_2 = 3$ on the second toss of a die
$E_3 = 3$ on the third toss of a die

find the following:

$$P(E_1 \cap E_2 \cap E_3).$$

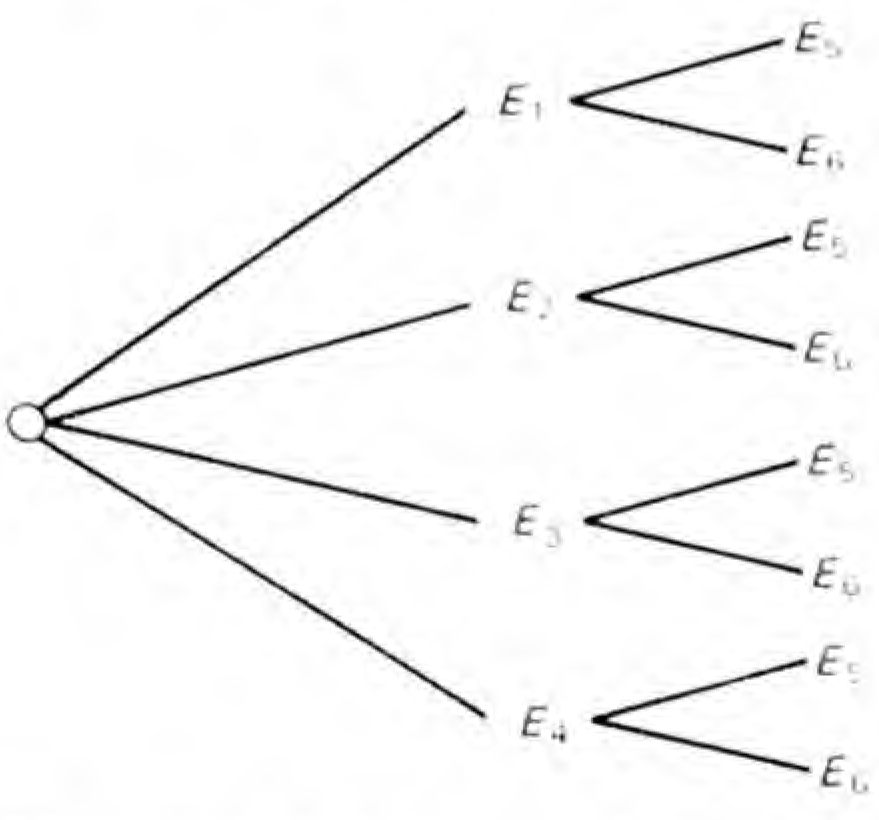
2-7 Given the following:

$E_1 =$ a diamond on the first draw from a deck of cards
$E_2 =$ a diamond on the second draw

find:

$$P(E_1 \cap E_2)$$

2-8 The problem presented in Table 2-2 of the text can be depicted as a probability tree. Identify and enter the marginal, joint, and conditional probabilities.



2-9 Draw a probability tree for problem 2-1. Identify the marginal joint and conditional probabilities.

2-10 A game is played where the player first tosses a die and then, depending upon the outcome of that toss, selects from one of three urns.
If the result of the toss is 1 or 2, the player selects from urn A; if the result is 3, 4, or 5, the player selects from urn B; and if the result is 6, the player selects from urn C.
Urn A has 90 red and 10 green marbles; urn B has 40 red and 60 green marbles; urn C has 10 red and 90 green marbles.
Draw the probability tree. Identify all the marginal and conditional probabilities. What is the probability that a player will select urn A and draw a green marble? What is the probability that a player will select a red marble?

CASE STUDY: Natural Cosmetics

The Natural Cosmetics Company of Philadelphia, Pennsylvania, produces a line of women's cosmetics including lipsticks, eye shadows, deodorants, shampoos, and facial creams. The company was founded in 1967, with first-year sales of $525,000. This year sales are expected to exceed $85 million.

Through careful screening of products and financial control of operations, the firm is able to earn a 15 percent profit on sales. This return is higher than the return earned by any other firm in the cosmetics industry.

The marketing department receives new-product ideas from one of several sources: marketing, research and development, production, retail chains, or sales representatives. Once these ideas have been received they are carefully screened before any major commitment of funds is made. So thorough is this screening that few products have ever failed in the marketplace.

Screening Process

Screening generally involves two stages. First, a complete economic analysis of the product is undertaken. If it passes this profitability test, a budget is authorized for the purpose of developing a sample product. Shortly after the sample is finished, it is tested before a panel of consumers, and if it passes this test, full-scale plans are made for production and marketing.

The record established by the consumer panel has been excellent. Ninety percent of the products which they have approved have turned out to be successful. But before the panel was used for all products, only 40 percent of the products not tested were successful.

New Products

At the present time the company is considering two new-product ideas. The first is a shampoo which will have a natural additive for combating dandruff. The second is a line of panty hose. This product, however, does represent a departure from the kinds of products that currently constitute the product line.

The market research group has spent the last 3 weeks compiling economic data on these products. It has completed the analysis for the panty hose project and is in the process of analyzing the shampoo project.

Panty Hose

The data on the panty hose project indicate that it could prove to be very profitable. Sales are projected to be 2 million annually with a profit of $290,000 before taxes.

In addition to making these "A" estimates, Natural Cosmetics also makes what they call a "B" estimate. An "A" estimate represents the most likely level of profit that the project is expected to return if it is successful. The "B"

estimate represents the most likely loss that can be incurred if the project is unsuccessful.

The "B" estimate for the panty hose project is a sales level of $600,000 with a loss of $150,000 annually.

Once "A" and "B" estimates have been made, the probabilities of achieving these estimates are determined. But determining these probabilities for the panty hose project has been difficult. The market research group has had very little experience with products in this category. Therefore, there was very little historical data that could be used. The group finally had to rely on its own judgment.

The probabilities that the product would achieve its "A" and "B" estimates were concluded to be 30 and 70 percent, respectively. However, once a sample of the product is developed and shown to the consumer panel, the probabilities associated with the "A" and "B" estimates would change to 85 and 15 percent, respectively, *if* the panel approved the product.

Shampoo

The data on the shampoo project have been collected and are summarized in Exhibit A. Included in this exhibit are both the "A" and "B" estimates.

Since this project is not unlike most of the projects considered at Natural Cosmetics, the market research group has concluded that the probabilities of achieving the "A" and "B" estimates are 40 and 60 percent, respectively. But if the panel approves the sample, these probabilities will change to 90 and 10 percent, respectively.

Exhibit A Sales and Expense Data: Shampoo Project

	A	B
Sales	1,300,000	850,000
Labor	300,000	250,000
Material	200,000	170,000
Marketing expenses associated with new product	350,000	280,000
Administrative expenses associated with new product	100,000	80,000
Decrease in profit contribution of other shampoo products	40,000	20,000
Allocated Expenses	130,000	85,000

Note 1 All figures are on an annual basis.

Note 2 Allocated expenses represent the fixed expenses of the firm that are too difficult to allocate on an individual basis to each product. Since the fixed expenses of the firm represent 10 percent of the firm's *total* sales, each *product* is charged an amount equal to 10 percent of *its* sales.

A Preliminary Recommendation

Sally Lumm, a market research analyst, has recommended that only the panty hose project be accepted. She has supported this position on the basis of several factors.

First, and most important, the panty hose project can clearly generate higher profits than the shampoo project.

Second, the probability that the panty hose project will be successful is high.

Third, she feels that the company should take this opportunity to expand into new markets. Unless new markets are developed, future growth may be limited.

QUESTIONS

1. Determine the "A" and "B" profit estimates for the shampoo project.
2. What are the marginal and conditional probabilities associated with these projects? What do they mean? At what stage in the screening process is each one relevant?
3. Assume that the budget for new-product development is limited. Since it will cost over $10,000 to develop a sample of each product for testing purposes, only one can be developed. On the basis of the "A" estimate alone, and neglecting the probabilities, which project should be selected?
4. Assume, as we did in question 3, that only one project can be selected. Identify the most attractive project on the basis of all the information given.
5. What probabilities did you use in question 4, marginal or conditional? Why? How did these probabilities affect your decision?
6. Why and under what conditions would you recommend that sample products be developed for both products?
7. Suppose that a decision was made to develop both sample products and that both of them passed the consumer panel test. If budgets limited the approval to only one new product, which one would you choose? Why?
8. Suppose that the firm had the resources to invest in both projects. Furthermore, assume that both projects passed the consumer panel. What is the likelihood that both will be successful? Are any additional assumptions about the projects necessary to compute the likelihood?
9. Under the same assumptions of problem 8, what is the likelihood that one or both will be successful?
10. Are these projects mutually exclusive?
11. Do you support Sally Lumm's position? Why?
12. What is your recommendation for action? Why?

APPENDIX A: Random Variables

A random variable is a rule or *function* which assigns numerical values to the outcome of an *experiment*. Suppose the experiment is the toss of a die. The outcomes correspond to the number of spots on the top face of the die. It is therefore possible for the outcome to take on one of the following numerical values: 1, 2, 3, 4, 5, or 6 spots. The random variable in this case is "the number of spots."

In a supermarket tomorrow's demand for bread may take on one of several numerical values. Demand might be as low as 100 or as high as 300 with all numerical values in between these extremes possible. The random variable in this case is "demand." These and other random variables are summarized in Table 2A-1.

Table 2A-1 Random Variables

Random variable (denoted by capital letter)	Values of the random variable	Description of the values of the random variable
A	1, 2, 3, 4, 5, 6	Possible outcomes from throwing a die
B	100, 101, . . . , 300	Possible demand for bread
C	0, 1, 2	Possible number of heads in two tosses of a coin
X	10, 15, 20, 25	Possible rates of return for an investment project
Z	20, 21, . . . , 1000	Possible reorder quantities for a part kept in inventory

Probability Function

We can define a probability function as a rule which assigns probabilities to each of these numerical values of the random variable. For example, a probability function can be determined which will assign probabilities to the occurrence of 0, 1, or 2 heads in two tosses of a coin. These probabilities are given in Table 2A-2. The probability function which assigned these probabilities is called the binomial probability function and will be covered later in Appendix B.

Table 2A-2 Coin-tossing Experiment with Probabilities

Value of the random variable (no. of heads)	Probability
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Expected Value of a Random Variable

The expected value of a random variable is computed by multiplying each possible value of the random variable by the probability of that value and then adding all these weighted values. For example, the expected value of the coin-tossing experiment in Table 2A-2 is

$$E(C) = \tfrac{1}{4}(0) + \tfrac{1}{2}(1) + \tfrac{1}{4}(2) = 1$$

where $E(C)$ is the expected value of the random variable C.

CONTINUOUS AND DISCRETE PROBABILITY FUNCTIONS

Probability functions can be either discrete or continuous. A probability function is said to be discrete when the random variable can take on only a limited number of selected values. For example, the random variable in Table 2A-2 can take on only the values 0, 1, and 2. The probability function which generated those probabilities would therefore be referred to as a discrete probability function. The binomial probability function is one example of a discrete probability function.

A probability function is said to be continuous when the random variable can take on any number within some given range of values. Consider the random variable "weight." It is possible that one package transported by an air freight company weighs 106.24 pounds, another weighs 1162.132 pounds, and another weighs 3221.34 pounds. Indeed it is possible for this random variable to take on any value between zero and infinity. Consequently, the probability function which would generate the probability of a certain weight is called a continuous probability function. The most useful continuous probability function is the normal probability function. It is covered in Appendix C.

APPENDIX B: Binomial Distribution

BERNOULLI'S PROCESS

A Bernoulli process is said to exist under the following conditions:

1 The outcome of each trial can be placed in one of two categories; that is, the categories or events are mutually exclusive.

2 The probability of the outcome of any one experiment or trial remains the same for each subsequent trial.

3 Each trial is independent of all trials before it.

4 There are a discrete number of trials, such as 10, 15, or 20.

In the tossing of a coin, a Bernoulli process is said to exist because:

1 The outcome of each toss (trial) can be categorized as either head or tail.

2 The probability of a head remains the same for the first and all subsequent trials.

3 The probability of a head on any trial is independent of the outcome on previous trials.

4 There will always be a discrete number of trials (it makes no sense to talk about $10\tfrac{1}{2}$ tosses of a coin).

The tossing of a coin is only one example of a Bernoulli process. Others are the generation of good or bad parts from a machine and a research questionnaire designed to uncover whether consumers intend to buy a new product.

BINOMIAL PROBABILITY DISTRIBUTION

Certain questions concerning a Bernoulli process can be answered by the binomial distribution. For example, we might like to know the likelihood of two heads in three tosses of a fair coin. First we will discover how this can be answered by drawing a probability tree and then by using the binomial distribution.

On the first toss of the coin, a head or tail can occur, on the second toss a head or tail, and on the third toss a head or tail. These possible outcomes are portrayed as a probability tree in Figure 2B-1.

From the tree it can be seen that exactly two heads can occur in the following ways:

1 HHT
2 HTH
3 THH

Since these are all independent events, the likelihood of each can be expressed in the following ways:

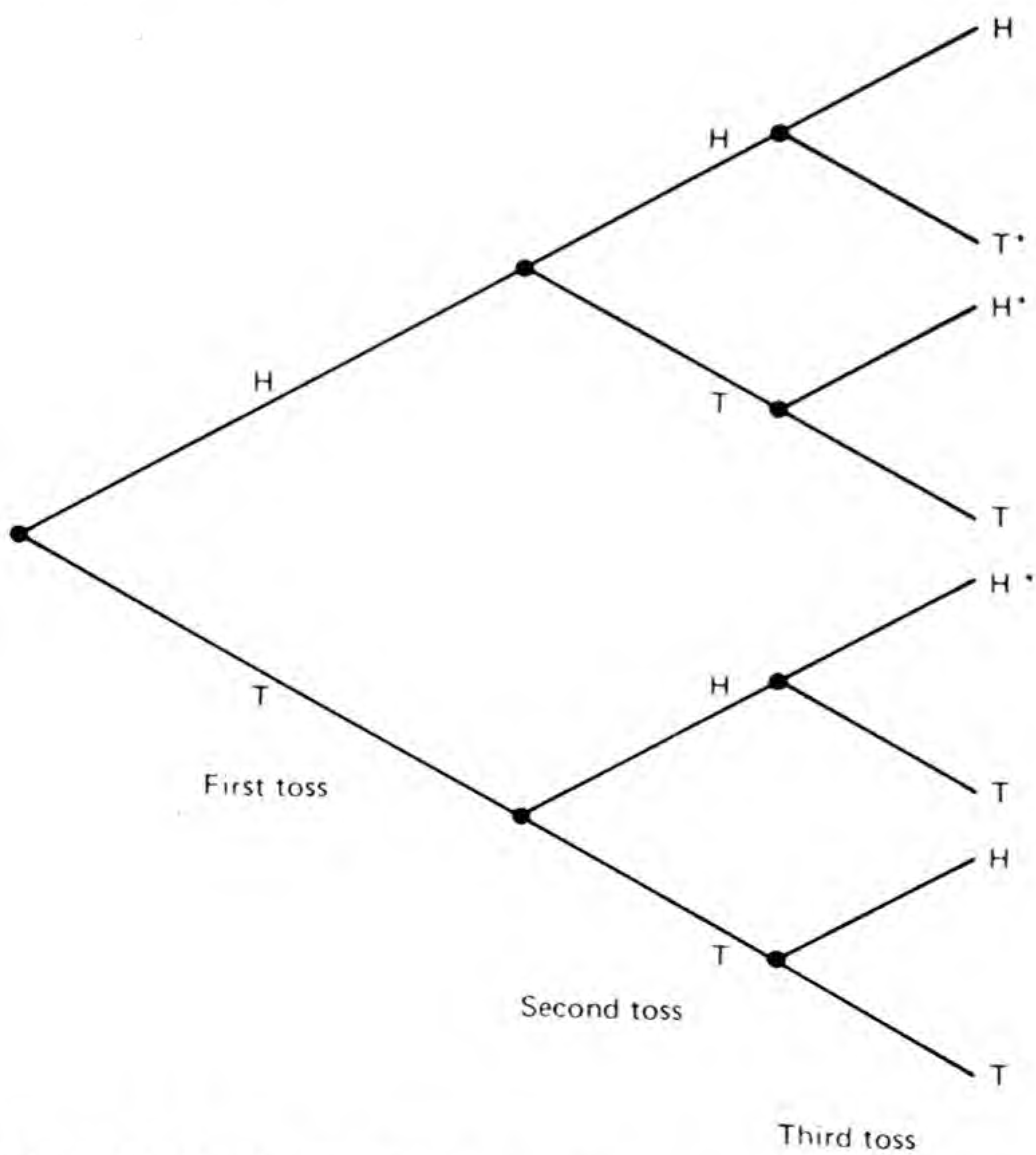


Figure 2B-1 The likelihood of two heads in three tosses of a coin.

$$P(H \cap H \cap T) = \tfrac{1}{2}(\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{1}{8}$$
$$P(H \cap T \cap H) = \tfrac{1}{2}(\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{1}{8}$$
$$P(T \cap H \cap H) = \tfrac{1}{2}(\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{1}{8}$$

All three of these outcomes are mutually exclusive; therefore the likelihood of getting exactly two heads the first way *or* the second way *or* the third way is

$$\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{3}{8}$$

We can therefore conclude that the probability of tossing exactly two heads in three tosses of a coin is $\tfrac{3}{8}$. This is called a binomial probability.

Suppose that instead of finding the likelihood of two heads in three tosses, you wanted to find the likelihood of exactly 30 heads in 50 tosses. The use of a probability tree would be impossibly cumbersome: there must be a better way! Indeed there is, and it employs the binomial function to find these binomial probabilities.

If we let n be the number of experiments or trials, r the number of successful outcomes, and p the probability of a success on any one trial, the probability of exactly r successes in n trials can be computed by using the binomial function.

$$P(r|n, p) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$

The notation ! is called a factorial. For example, if $n = 3$, then $n! = 3 \times 2 \times 1$. Consequently a factorial directs us to multiply that number by all integers below it.[1]

To return to our example, the probability of exactly two heads ($r = 2$) in three ($n = 3$) tosses where the probability of a head is $\tfrac{1}{2}$ on any one toss is computed in the following way:

$$\begin{aligned} P(2|3, \tfrac{1}{2}) &= \frac{3!}{2!(3-2)!} (\tfrac{1}{2})^2 (\tfrac{1}{2})^1 \\ &= \frac{3 \times 2 \times 1}{2 \times 1 \times 1} (\tfrac{1}{2})^2 (\tfrac{1}{2})^1 \\ &= 3(\tfrac{1}{2})^3 \\ P(2|3, \tfrac{1}{2}) &= \tfrac{3}{8} \end{aligned}$$

which is precisely the answer we found by using a probability tree.

Consider another example. What is the likelihood of tossing exactly 20 heads in 50 tosses of a fair coin? Substituting these values in the binomial formula, we have the following:

$$P(20|50, \tfrac{1}{2}) = \frac{50!}{20!30!} (\tfrac{1}{2})^{20} (\tfrac{1}{2})^{30}$$

Solving this would be quite tedious. Fortunately we have binomial tables and these can be found at the back of the book, Table 2.

The use of this table requires values of n, p, and r. First we locate the group of

[1] By definition $0! = 1$.

probabilities under $n = 50$, then we find the column in this group labeled $p = .50$, and finally we find the row in this column labeled $r = 20$. Where these intersect, we find our probability.

$$P(20 \mid 50, 1/2) = .0419$$

We can therefore conclude that there is slightly more than a 4 percent chance of tossing exactly 20 heads in 50 tosses of a coin.

PROBLEMS

2B-1 Using a probability tree, determine the likelihood of tossing two heads in four tosses of a fair coin.

2B-2 Using the binomial probability table, determine the probability of tossing two heads in four tosses of a fair coin.

2B-3 From the binomial probability table determine the probability of the following:
- **a** Exactly 10 heads in 20 tosses of a fair coin.
- **b** Exactly 15 heads in 20 tosses.
- **c** Exactly 10 tails in 20 tosses.
- **d** Compare the results of parts *a* and *c*.

2B-4 What is the probability of tossing sixteen 1s in 100 tosses of a fair die?

2B-5 A machine produces an average of 3 percent defective items. If you sample 50 items, what is the probability of observing exactly two defective items?

2B-6 If a machine produces an average of 3 percent defective items, what is the likelihood of observing two or fewer defectives ($n = 50$)?

2B-7 If a machine produces an average of 3 percent defective items, what is the likelihood of observing more than two defectives ($n = 50$)?

2B-8 The quality control manager for the Craftworks Company has received several complaints about the quality of one of his products from sales representatives in the field. To combat this problem he has decided to inaugurate a new quality control system in final inspection. He has instructed one of his employees to randomly select 100 items from a very large lot and then determine if the items are good or bad. If two or fewer are bad, the lot is accepted. If more than two are bad, the lot is rejected and returned to the machine shop.
- **a** If the actual lot percent defective were 1 percent, what would be the probability that the lot would be accepted?
- **b** If the actual lot percent defective were 2 percent, what would be the probability that the lot would be accepted?
- **c** Continue this analysis for an actual lot percent defective of 3, 4, 5, and 6 percent.
- **d** Plot your result using the following format. The X axis will be "percent defective," and the Y axis will be "probability of acceptance."
- **e** Suppose that you consider good-quality lots to be 1 percent defective. Is there a chance that lots of this quality can be rejected? What is this chance? This is referred to as a type I error or producer's risk.
- **f** Suppose that you consider bad-quality lots to be 6 percent defective. Is it likely that you will accept lots of this quality? What is the probability? This is called a type II error or consumer's risk.

APPENDIX C: Normal Probability Function

The normal probability function can be represented as a smooth, continuous, bell-shaped curve as shown in Figure 2C-1. It is completely described by its mean μ and standard deviation σ, where the mean is the expected value of the distribution and the standard deviation is a measure of its variability.[1]

The area under the curve between two values a and b represents the likelihood that the random variable will fall in this interval. For example, suppose the sales forecast for a particular product can be represented by a normal distribution with average sales of 150 units and a standard deviation of 20 units. This distribution is shown in Figure 2C-2. From it we can see that the likelihood of sales between 90 and 110 is much smaller than the likelihood of sales between 150 and 170.

The total area under the normal distribution is equal to 1.0. Therefore, the likelihood that the random variable will take on one of the values under the curve is 1.0, or certainty.

Since the normal distribution is symmetrical, the likelihood that the random variable will take on values below the mean is 50 percent, and the likelihood that it will take on values above the mean is also 50 percent.

CHARACTERISTICS OF NORMAL PROBABILITY DISTRIBUTIONS

When the standard deviation is associated with a normal distribution, it has special properties. For example, the likelihood that the random variable will fall between plus and minus 1 standard deviation of the mean is 68.26 percent. The likelihood that the random variable will fall between plus and minus 2 standard deviations is 95.44 percent, and the likelihood that the random variable will fall between plus and minus 2.5 standard deviations is 98.76 percent. These special properties are illustrated in Figure 2C-3.

[1] The standard deviation can be computed in the following way:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}}$$

where X_i = the ith observation in the population
μ = the mean of the distribution
N = the number of observations in the population

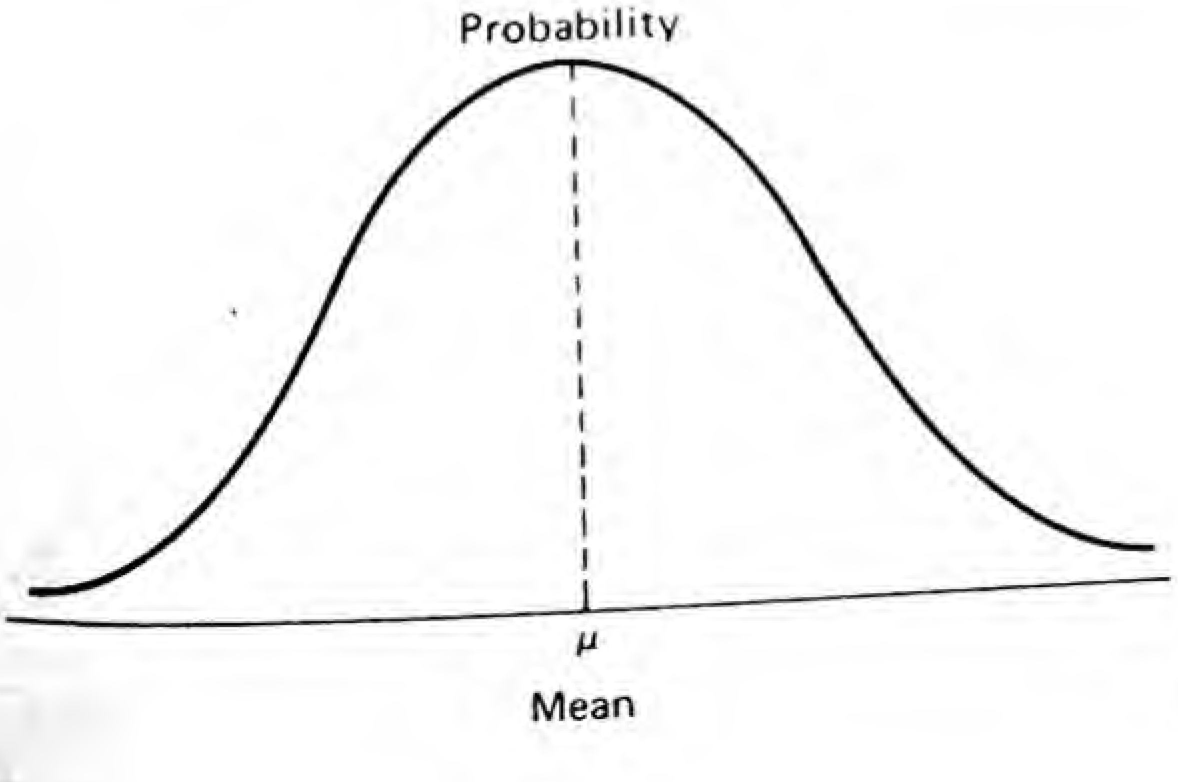


Figure 2C-1 Normal probability distribution.

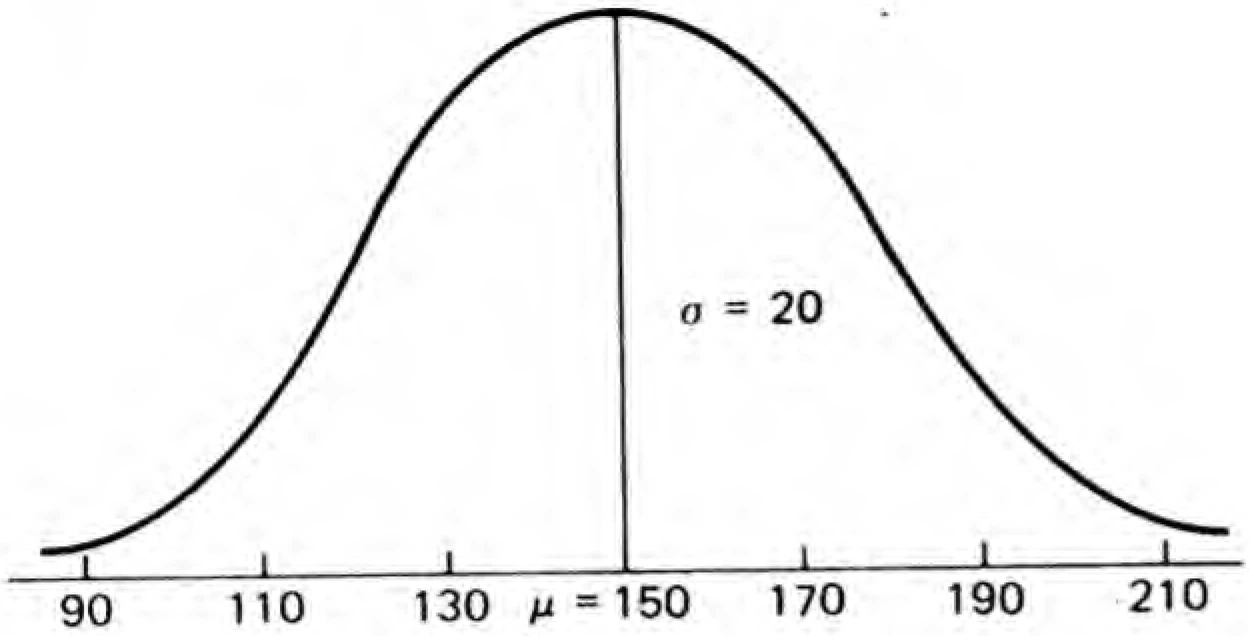


Figure 2C-2 Sales forecast represented by a normal distribution.

Applying this to the example where the average sales forecast is 150 units and the standard deviation is 20 units, we can conclude the following. There is a 68.26 percent chance that sales will fall between plus and minus 1 standard deviation of the mean.

$$\mu \pm 1(\sigma)$$
$$150 \pm 1(20)$$
$$130 \text{ to } 170$$

Therefore, there is a 68.26 percent chance that sales will fall between 130 and 170 units. In the same way we can also determine that there is a 95.44 percent chance that sales will fall between $150 \pm 2(20)$, or 110 and 190 units, and a 98.76 percent chance that sales will fall between $150 \pm 2.5(20)$, or 100 and 200 units.

USE OF THE STANDARD NORMAL TABLE

The special properties of the standard deviation can be read directly from a table. Table 1 at the back of the book includes the areas under the curve between the mean and some number of standard deviations Z from the mean for a standard normal distribution. To use this table, it is necessary to convert the distribution with which you are working into standard form. The conversion takes place in the following way:[1]

$$Z = \frac{X - \mu}{\sigma}$$

[1] The variable Z is a relative measure which permits the standardization of all normal distributions in order to use tables. The standard normal distribution into which all distributions are thereby converted has a mean of zero and a standard deviation of 1.

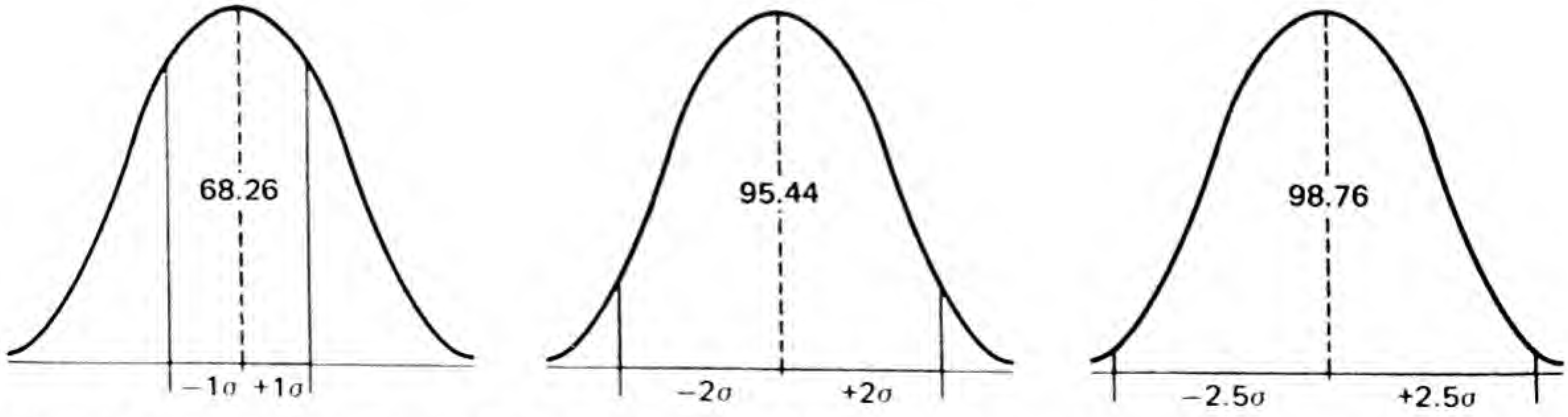


Figure 2C-3 Special properties of the standard deviation for a normal probability distribution.

where Z = converted value of X representing the number of standard deviations which the value X is away from the mean μ

X = value to be converted

μ = mean of the population

σ = standard deviation of the population

Suppose we are interested in the likelihood that sales will fall between 150 and 190 units. Since the table gives us the areas or likelihoods that the values will fall between the mean and some number of standard deviations Z of the mean, we must first compute Z.

$$Z = \frac{X - \mu}{\sigma} = \frac{190 - 150}{20} = \frac{40}{20} = 2$$

Then we look down the first column of Table 1 until we find $Z = 2$ and read from the next column that the likelihood of a value falling between the mean and 2 standard deviations from the mean is .4772 or 47.72 percent. Since the normal distribution is symmetrical, the likelihood of a value falling between plus and minus 2 standard deviations from the mean is 95.44 percent. Verify this from Figure 2C-3.

Returning to the same example, this time we will determine the likelihood that sales will fall between 160 and 185 units. This area is shaded in Figure 2C-4, and it should be clear that the area between 160 and 185 must be found. To do this, however, requires three steps; first, the area between the mean and 185 must be determined; second, the area between 150 and 160 must be determined; and third, the difference between these two must be computed.

The area between 150 and 185 is illustrated in Figure 2C-5. Performing the conversion, we have the following:

$$Z = \frac{185 - 150}{20} = \frac{35}{20} = 1.75$$

From Table 1 we find the area between the mean and $Z = 1.75$ is .4599.

Next, we determine the area between 150 and 160. This is illustrated in Figure 2C-5. Performing the conversion, we have

$$Z = \frac{160 - 150}{20} = \frac{10}{20} = .50$$

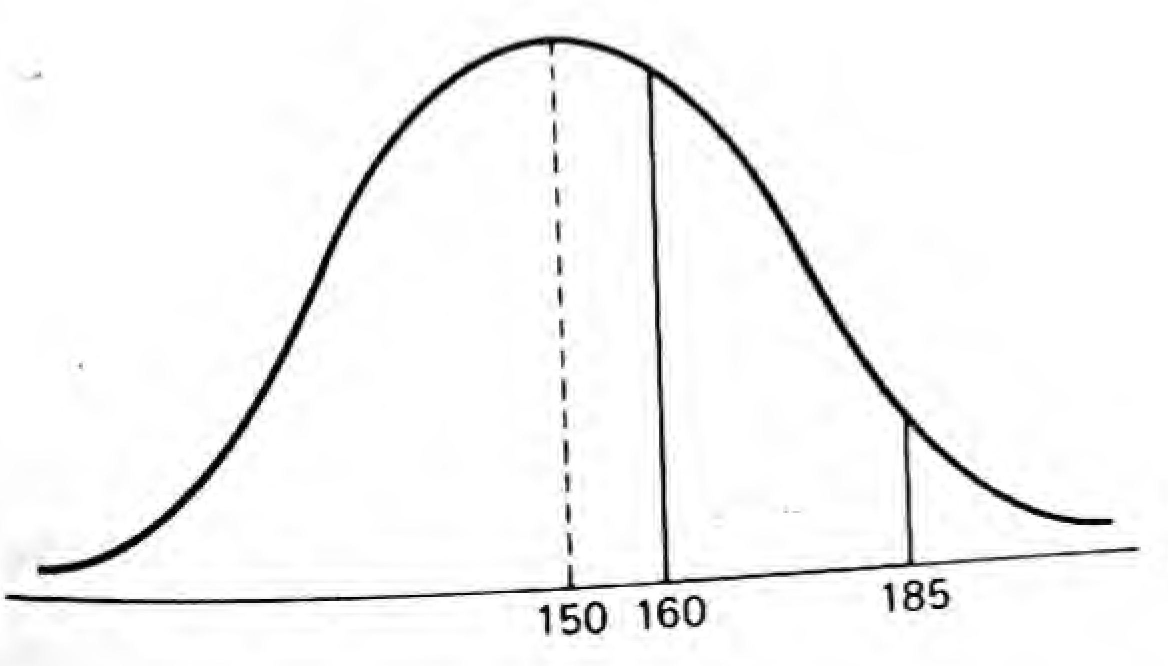


Figure 2C-4 The area between 160 and 185.

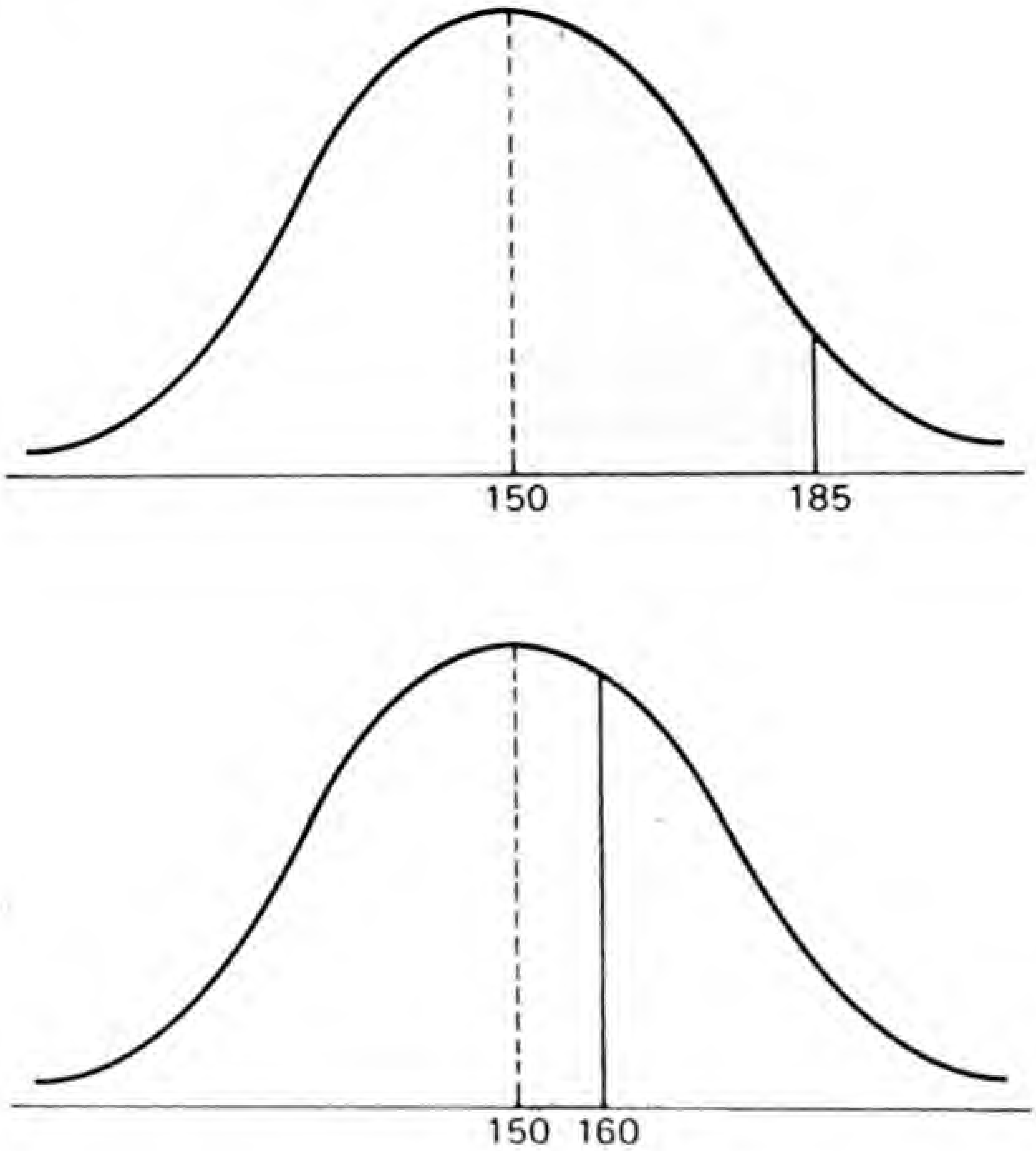


Figure 2C-5 The component parts which must be subtracted to find the area between 160 and 185.

From Table 1 we find that the area between the mean and $Z = .50$ is .1915.

Subtracting the two areas, we have the following:

$$\begin{array}{r} .4599 \\ -\ .1915 \\ \hline .2684 \end{array}$$

We can conclude that the likelihood of sales falling between 160 and 185 is 26.84 percent.

PROBLEMS

2C-1 A normal distribution has a mean of 100 and a standard deviation of 5.
- **a** Find the likelihood of values below 100.
- **b** Find the likelihood of values between plus and minus 1.5 standard deviations of the mean.
- **c** Find the likelihood of values between 90 and 115.
- **d** Find the likelihood of values between 110 and 115.

2C-2 A normal distribution has a mean of 50 and a standard deviation of 2.
- **a** Find the likelihood of values above 50.
- **b** Find the likelihood of values between 48 and 50.
- **c** Find the likelihood of values between plus and minus 1.6 standard deviations of the mean.
- **d** Find the likelihood of values between 47 and 49.

2C-3 The daily demand for a product can be described by a normal probability distribution with a mean of 1000 and a standard deviation of 50. Find the likelihood that demand will fall above 1125 units.

2C-4 The length of time it takes assembly-line workers to assemble a part can be described by a normal distribution with a mean of 105 seconds and a standard deviation of 5 seconds. What percentage of the workers require more than 112 seconds to complete the job?

2C-5 The length of time it will take to finish a large project can be described by a normal distribution with a mean of 25 days and a standard deviation of 2 days. A penalty will be imposed if the project takes longer than 26 days. What is the likelihood of incurring this penalty?

Chapter 3

The Structure of Decisions

INTRODUCTION

One decision problem is not completely unlike another. Most problems have several characteristics in common. The purpose of this chapter is to identify these characteristics and to develop a general framework for the formulation and analysis of models that can be used to generate useful information for the purpose of resolving these decision problems.

The case which follows will be used to identify these common characteristics.

CASE STUDY: City of Southbridge Parking Garage

Parking in the business district of Southbridge has always been a problem. A recent study indicated that the city needed a total of at least 1000 parking places, but current facilities accommodated only 600 cars. This shortage of spaces has increased traffic, led to congestion, and discouraged shoppers from driving into the downtown area.

Several months ago, the city planner, Ray Ouellette, started to gather some data that could be used in support of a city parking garage. He found that the federal government would finance 80 percent of the structure, the rest coming from a municipal bond that Southbridge would have to issue.

Although the federal government would finance the largest share of the structure, operating expenses such as wages and maintenance would have to be met from parking fees. These fees would have to be higher than the ones in effect at parking meters. Ray had some concern about the project's ability to generate the revenues necessary to meet these expenses. Above all he did not want this project to be a drain on the city's limited resources. It must not lose money. Any profit which it could make would, of course, be welcomed.

Profit was closely tied to the size of the garage. With the use of a piece of land the town already owned, three sizes were possible. A one-level enclosed facility would accommodate 500 autos, a two-level facility would accommodate 1000 autos, and a three-level facility, would accommodate 1500 autos.

Profit for each of these alternatives was dependent on demand for parking. If a 500-space facility were built and demand for parking was low, annual profit—after meeting all expenses—would be $10,000. A high level of demand would lead to a profit of $50,000 for the same facility.

If a 1000-space facility were built and demand for parking was low, an annual loss of 10,000 would result. A high level of demand, however, would lead to a profit of $75,000.

If a 1500-space facility were built and demand for parking was low, an annual loss of $50,000 would be incurred. A high level of demand would lead to a profit of $120,000.

Ray was uncertain as to which of these alternatives to recommend. The business community would favor the largest facility since it would ensure an adequate supply of parking places for at least 5 years. The mayor, however, more interested in the economics of the project, would favor at most the 1000-space facility.

COMMON CHARACTERISTICS

Ray Ouellette must decide among one-, two-, and three-level garages. The choice is not an easy one to make because the system, of which this garage will be a part, is complex. The elements that make up this system include automobiles, current parking facilities, parking fees, public transportation, city politics, city financing, federal financing, the business community, shoppers, and city residents.

Perhaps a model of this system might help in making the decision. But how can the model be formulated? Fortunately there are several characteristics which are common to all decision problems, and an understanding of these characteristics can facilitate the development of such a model.

Every decision problem exhibits the following five characteristics:

1 Alternative strategies
2 Outcomes
3 Environmental states
4 Decision criterion
5 Likelihoods for each environmental state

Regardless of where the problem is found—factory, retail store, hospital, bank, governmental agency, school—all will share these characteristics.

Each of these characteristics will now be explored in greater detail.

Alternative Strategies

The most obvious characteristic shared by all problems is the choice between alternatives. In the parking garage case three alternatives were possible. They included one-, two-, and three-level garages.

As another example, consider the problem faced by an institutional investor who must invest $340,000 in a group of stocks and bonds listed on the New York Stock Exchange. The number of alternative investment combinations is beyond comprehension.

Having too few alternatives from which to choose can affect the quality of the decision. As many relevant alternatives as possible should be considered. Do you think the choice should be limited to three alternatives in the parking garage case? What about the expanded use of public transportation? Might this alternative alleviate traffic problems in the downtown area? Could better public transportation completely eliminate the need for a parking garage?

Every problem has its set of alternatives. In some problems these alternatives are obvious and plentiful, but in others they may be few and difficult to uncover. It is management's job to ensure that no reasonable alternative has been ignored.

Outcomes

The choice of an alternative will eventually result in one of several possible outcomes. In the parking garage case the choice of a two-level garage capable of parking 1000 autos will eventually lead to a loss of $10,000 or a profit of $75,000. The $10,000 loss and $75,000 profit are called outcomes.

When the institutional investor chooses a particular investment group of stocks and bonds, this choice will eventually lead to an outcome measured as a rate of return on the investment.

The second characteristic common to all decision problems is that every alternative will have associated with it one or more possible outcomes.

Environmental States

The outcome of each alternative is seldom under the control of the decision maker. In the parking garage case the outcome of a two-level garage depends upon the demand for parking. This is completely beyond the control of Ray Ouellette.

In the institutional investment problem the outcome of a particular investment group depends upon the behavior of the stock and bond markets. Clearly their behavior is beyond the control of the institutional investor.

In both of the examples given above the outcome for each alternative depends upon the environmental state that occurs. In the parking garage example there are two environmental states: low and high demand. In the institutional investment example the environmental states represent the performance of the stock and bond markets.

To clarify the relationships among alternatives, outcomes, and environmental states, it is useful to construct a model which makes these interrelationships explicit.

A model for the parking garage case is presented in Table 3-1. Down the outside column are listed the three alternatives. Across the top of the model are listed the two possible environmental states: low and high demand for parking. The outcomes are entered into the cells which occur at the intersection of an alternative and an environmental state. For example, if a one-level garage is chosen and if the environmental state is low demand, the outcome will be a profit of $10,000. The model should make it clear that the outcome is a consequence of a chosen alternative and the occurrence of a particular environmental state.

Table 3-1 is often called a conditional payoff table or conditional payoff model because each outcome (or payoff) is conditional upon the selection of a particular alternative and the occurrence of an environmental state.

DECISION CRITERION

Determining how the outcomes of the alternatives are to be measured is an important step in the development of a model. What criterion will be used? Will the outcomes be measured in terms of profit, cost, rate of return, or market share?

Table 3-1 Conditional Payoff Table for Southbridge Parking Garage

	Environmental state	
Alternatives	Low demand	High demand
One-level garage	$10,000	$50,000
Two-level garage	- $10,000	$75,000
Three-level garage	- $50,000	$120,000

In the parking garage case, the criterion was profit. Each outcome was measured in terms of its annual profit contribution. But in the institutional investment case the criterion was rate of return. Each outcome was measured as the rate of return earned by that particular group of stocks and bonds.

Before a model can be formulated and the problem solved, the decision makers must agree on a criterion. If there is disagreement with this criterion, it is unlikely that there will be agreement with the decision.

But in some problems the outcomes cannot be measured by a single criterion. Several criteria must be used. For example, when the federal government evaluates different fiscal strategies, it forecasts the effect that these strategies will have on unemployment, GNP, inflation, and the balance of payments. Not one but four criteria are involved. Consequently it is impossible for the government to summarize a possible outcome with a single number. Indeed, multiple-criteria problems are very complex.

If we return to the parking garage case, we can see that on closer inspection more than one criterion might be suggested. Besides profit, the case implies that the criterion held by the business community in the downtown area might be the maximization of parking places or the maximization of shoppers' convenience. To what extent should these criteria be considered by Ray Ouellette?

How, then, can multiple-criteria problems be modeled? One way is to reduce the criteria set to its most dominant criterion. The secondary criteria are excluded from the model. This is essentially what was done in the parking garage model. But care must be exercised when this model is used to solve the real-world problem. Remember that the model is only an abstraction of reality and not reality itself. Nonetheless we will shortly see that the model can still be very useful.

Most of the decision problems which you will encounter in this book are of the single-criterion variety. In these problems it will be clear that there is one dominant criterion and that little is sacrificed by ignoring secondary criteria. Nonetheless the alert critic will always ask: What are the relevant criteria, and will the use of a single criterion oversimplify the model?

KNOWLEDGE OF THE DECISION ENVIRONMENT

Although decision makers may have no control over the environmental states, they often possess useful knowledge about them. In some decision-making situations it may be known with certainty which environmental state will occur. In another situation it may be possible to subjectively or objectively estimate the probability that each environmental state will occur, and yet in still other situations there may be absolutely no knowledge concerning the probability of these environmental states.

These three situations are classified as certainty, risk, and uncertainty. When it is known for certain which environmental state will occur, a decision problem under *certainty* is said to exist. When probability estimates can be made for the environmental states, a decision problem under *risk* is said to

exist. A decision problem under *uncertainty* exists when no statement about the probability of these environmental states can be made.

Decision Making under Certainty

When decisions are made under conditions of certainty there is but one relevant environmental state. To some extent this simplifies the analysis of the model.

Consider the following example. The production department of a large manufacturing firm is faced with a replacement decision. Four machines have been identified for possible replacement, but the budget allows the replacement of only one. The four replacement alternatives have been listed down the first column of Table 3-2.

Although these machines require a substantial investment, they are far less costly to operate and maintain than their predecessors. The difference between these costs represents the savings or benefits associated with the investment. In Table 3-2 these savings are stated as the rate of return that will be earned on the investment.

These rate-of-return figures have been carefully determined by the production department, which feels quite confident about these outcomes. Other outcomes are not likely. Consequently it is reasonable to assume that this is a decision problem under certainty and only one environmental state is relevant.

The model presented in Table 3-2 can be analyzed quite easily. Since the criterion by which alternatives are compared is rate of return, the grinding machine should be replaced. The return earned by replacing this machine is higher than that for any other alternative.

We can conclude that for decision problems under certainty the alternatives are listed and the outcomes determined. Then the alternative with the most attractive outcome is selected.

It might seem at first glance that all decision problems under certainty are simple to solve. To the contrary, some are quite complex. They are complex

Alternative strategy	Environmental state (rate of return)
Milling machine	15%
Lathe A	20%
Lathe B	17%
Grinding machine	35%

Table 3-2 Conditional payoff table for a decision-making problem under certainty.

because there may be a long list of alternatives that must be compared. Consider an inventory ordering problem. An order may be placed for 1 unit, 10,000 units, or any quantity in between. For each alternative order quantity there is a cost to the firm. Even if these costs are known with certainty it would be a lengthy job to compare the alternatives. Contributing to the complexity of this problem is the fact that firms do not have just one item in stock but thousands. Therefore this analysis must be repeated thousands of times. Fortunately inventory theory can help us in solving this problem.

Many of the problems analyzed in this book are decision problems under certainty. But they may be very complex problems because of the countless alternatives open to the decision maker. Without access to a model, they could be very difficult to compare.

Decision Making under Risk

When the decision maker is not certain as to which environmental state will occur but is able to forecast the likelihood with which these states will occur, we have a decision problem under risk.

In the Southbridge parking garage case two possible environmental states were given: low and high demand for parking. On closer examination Ray Ouellette concluded that he did have an opinion on the probability of these states. He felt that the probability that demand would be low was 20 percent, while the probability that demand would be high was 80 percent.

It is these probability estimates that make this a decision problem under risk.

Expected Monetary Values Now that Ray Ouellette has estimated the environmental state probabilities, all information necessary to analyze the model is available.

A reasonable method for analyzing the model is to weigh all the outcomes for each alternative by the probability associated with its environmental state. Then each of these products is summed. The result will be the expected monetary value for the alternative.

Turning to the first alternative in Table 3-1 we can see that if the first environmental state, low demand, occurs the outcome will be \$10,000. If the second state occurs, the outcome will be \$50,000. Furthermore we know that the probability that state 1 will occur is 20 percent and the probability that state 2 will occur is 80 percent. The expected monetary value of alternative 1 can be computed in the following way.

$$.20(\$10{,}000) + .80(\$50{,}000) = \$42{,}000$$

The expected monetary value of the first alternative is therefore \$42,000.

The expected monetary values of the second and third alternatives are computed in Table 3-3. Alternative 3, a three-level facility with parking for 1500 cars, results in the largest expected monetary value.

Alternative 1:	.20(10,000) + .80(50,000) = 42,000
Alternative 2:	.20(−10,000) + .80(75,000) = 58,000
Alternative 3:	.20(−50,000) + .80(120,000) = 86,000

Table 3-3 Computation of expected monetary values for Southbridge Parking Garage case.

The expected monetary value method is equivalent to assuming that alternative 3 is used not just once but over and over again and that environmental state 1 will occur on 20 percent of those occasions and environmental state 2 on 80 percent. The long-run average outcome will be \$86,000.

But in our case study it is not reasonable to assume that alternative 3 will be used over and over again. It will be used only once. The size of the facility is chosen, and the decision is not made again.

Does this mean that the expected monetary value method is not very useful for "one-shot" decisions? No. Even in situations such as this one it can still be shown to be the preferred strategy. If a manager makes all decisions on the basis of expected monetary value, even though every decision is different, expected value will be maximized in the long run.

We can therefore conclude that the outcome of the parking garage model implies that a three-level garage should be recommended. What remains is for Ray Ouellette to use this information in reaching a decision.

ANOTHER EXAMPLE

When an alternative is used over and over again, the use of the expected monetary method is both intuitive and convincing. We will now turn to an example.

A supermarket manager is faced with the following decision problem under risk. Each day an order must be placed for the number of cases of milk which he desires to have in stock at the beginning of the following day. Demand fluctuates from day to day so that it is never known with certainty exactly how many cases to order. If too many cases are ordered, the leftover cases must be discarded at the end of the day and at a complete loss to the supermarket. If too few are ordered, sales are lost. The manager's problem is to determine how many cases to order. His criterion is profit.

Suppose that demand records have been kept for the last 100 days. They are presented in Table 3-4. It can be seen that on 20 percent of the days demand was for 15 cases, on 30 percent of the days demand was for 16 cases, and so on. We will assume that these historical records can be used to forecast the future. Therefore Table 3-4 will be used as a probability forecast for future demand.

The supermarket manager must pay \$10 for each case which when sold

Table 3-4 Demand for Milk

Number of cases demanded	Number of days on which this number of cases was demanded	Probability of demand
15	20	.20
16	30	.30
17	40	.40
18	10	.10

yields $13 in revenue. The problem, including its alternatives, environmental states, and outcomes is presented in Table 3-5. A more detailed analysis behind the outcomes associated with "order 17" is given in Table 3-6.

The next step in the solution of this problem is to compute the expected monetary value for each alternative by weighing the possible outcomes of that alternative by their likelihood of occurrence. This is done in Table 3-7. The best strategy is to order 16 cases every day. This will result in an average daily profit over the long run of $45.40.

Order quantity \ Environmental state (demand)	15 $P(15) = .20$	16 $P(16) = .30$	17 $P(17) = .40$	18 $P(18) = .10$
15	$45	45	45	45
16	35	48	48	48
17	25	38	51	51
18	15	28	41	54

Table 3-5 Conditional payoff table for milk problem.

Order quantity	(1) Order cost	(2) Demand	(3) Revenue	(3) − (1) Profit
17	$17 \times 10 = 170$	15	$15 \times 13 = 195$	25
17	$17 \times 10 = 170$	16	$16 \times 13 = 208$	38
17	$17 \times 10 = 170$	17	$17 \times 13 = 221$	51
17	$17 \times 10 = 170$	18	$17 \times 13 = 221$	51

Note: Although demand is for 18 units, only 17 are in stock to be sold.

Table 3-6 Computation of outcomes for alternative "order 17."

Alternative	Expected monetary value
Order 15	.20(45) + .30(45) + .40(45) + .10(45) = $45.0
Order 16	.20(35) + .30(48) + .40(48) + .10(48) = $45.4
Order 17	.20(25) + .30(38) + .40(51) + .10(51) = $41.9
Order 18	.20(15) + .30(28) + .40(41) + .10(54) = $33.2

Table 3-7 Computation of expected monetary values for milk problem.

PERFECT PREDICTIONS

To gain some insight into the cost of risk, we will first determine the level of profit that could be obtained if the decision maker could forecast the environmental state with complete certainty.

Returning to our milk example, suppose there existed a forecasting service that could perfectly forecast tomorrow's demand. Naturally we would expect a fee to be charged for this service. From Table 3-5 it can be seen that if the service forecast 15 cases of demand, 15 cases would be ordered and the profit would be $45. Since this is a perfect forecast, there would be no doubts about this $45 profit. Similarly if they forecast 16 units of demand, 16 units would be ordered and profit would be $48. If 17 units were forecast 17 units would be ordered for a profit of $51, and if 18 units were forecast, 18 units would be ordered for a profit of $54. These profits are summarized in Table 3-8.

A forecast demand of 15 with a resulting profit of $45 would occur 20 percent of the time, a forecast of 16 with a resulting profit of $48 would occur 30 percent of the time, a forecast of 17 with a resulting profit of $51 would occur 40 percent of the time, and a forecast of 18 with a resulting profit of $54

Demand forecast	Order quantity	Profit	Likelihood of occurrence
15	15	$45	.20
16	16	48	.30
17	17	51	.40
18	18	54	.10

Expected profit:
.20(45) + .30(48) + .40(51) + .10(54) = $49.2

Table 3-8 Expected profits under perfect forecasting.

would occur 10 percent of the time. In the long run the expected profit from these perfect forecasts would be \$49.20. See Table 3-8.

EXPECTED VALUE OF PERFECT INFORMATION

In the preceding section we learned that a perfect forecast would result in an expected profit of \$49.20. In the absence of a perfect forecast, we concluded from Table 3-7 that 16 units should be ordered all the time at an expected profit of \$45.40. The difference between \$49.20 and \$45.40 is

$$\$49.20 - \$45.40 = \$3.80$$

which represents the most that the decision maker should be willing to pay for a perfect forecast. This \$3.80 is really the cost of risk; it is called the expected value of perfect information (EVPI).

The expected value of perfect information is an especially important concept in decision theory. Decision makers are always interested in reducing their risk. They accomplish this by collecting more information, and this might involve obtaining more data, using a more complex model, soliciting additional advice, or hiring outside consultants. But these steps toward risk reduction come at a cost. The expected value of perfect information places an upper limit to the price that the decision maker should pay to reduce the risk. It would certainly be unwise in the milk case to spend \$20 per day for a perfect forecast which would result in a \$3.80 improvement in profits.

DECISION MAKING UNDER UNCERTAINTY

In decision making under uncertainty, the decision maker faces several alternatives, and just as in decision making under risk there may be several possible environmental states. The major difference, however, is that in decision making under uncertainty the decision maker is unable to estimate the likelihood with which these environmental states will occur. Clearly some other criterion than expected value must be used. In Chapter 4 several criteria for solving these decision problems under uncertainty are explored.

SUMMARY

Early in the chapter we examined the characteristics common to all decision problems. Later these characteristics served as a guide in the formulation of some simple models. For each of these models the alternative strategies were identified and listed down the outside column of the table, the environmental states and their likelihoods were listed across the top, and the estimated outcomes were entered in the body of the table.

In subsequent chapters, you will find that an understanding of these

characteristics will be very helpful in the formulation and analysis of more complex models.

QUESTIONS

1 Identify, define, and compare the five characteristics common to all decision problems.
2 Why are some decision problems under certainty complex? Give an example.
3 Are decision problems under certainty always limited to one relevant environmental state?
4 In addition to the information supplied by the model in the parking garage case, what would you take into consideration before reaching a decision?
5 What decision would you make in the parking garage case? Why?
6 Suppose that a new demand forecast was obtained for the garage case and it indicated that the probability of low demand for parking was 50 percent. Would this affect your decision?

PROBLEMS

3-1 Given the following payoff matrix, determine the best strategy using the expected monetary value method.

	Environmental state		
Alternative	1 $P(1) = .20$	2 $P(2) = .30$	3 $P(3) = .50$
A	\$10	16	22
B	15	11	19
C	12	20	6

3-2 Compute the expected payoff with perfect information and the value of perfect information for problem 3-1.

3-3 Given the following payoff matrix, determine the best strategy using the expected monetary value method.

	Environmental state	
Alternative	1 $P(1) = .40$	2 $P(2) = .60$
A	5	12
B	10	5
C	12	6
D	6	14

3-4 Compute the expected payoff with perfect information and the value of perfect information for problem 3-3.

3-5 Given the following payoff matrix, determine the best strategy using the expected monetary value method.

Alternative \ Environmental state	1 $P(1) = .20$	2 $P(2) = .30$	3 $P(3) = .40$	4 $P(4) = .10$
A	8	12	4	10
B	10	6	14	3

3-6 Compute the expected payoff with perfect information and the value of perfect information for problem 3-5.

3-7 A large trucking company, located in city 1, must route one of its trucks through four major cities each week. The driving times in hours are given in the following table:

From city	To city 1	2	3	4	5
1	—	2	4	3	7
2	2	—	5	2	6
3	4	5	—	4	7
4	3	2	4	—	3
5	7	6	7	3	—

a What constitutes an alternative?
b Is this a decision problem under certainty, risk, or uncertainty?
c How many alternative strategies are there?
d Is there an implied criterion in the statement of the problem? What is it? Might other criteria be useful?

3-8 A sales representative must call on four customers today. They are located far enough apart so that the driving time between them is substantial, and the sales representative would like to schedule them in such a way as to minimize traveling time.

For convenience the cities are named A, B, C, and D. Assume that you have access to the driving times between city pairs, and that driving conditions do not change during the day.

Structure the problem in terms of alternatives, environmental states, and outcomes. Is this a decision problem under risk or certainty?

3-9 The Midvale hospital is reconsidering the layout of its facilities. The hospital is divided into four main sections. The intermediate-care ward currently occupies the front of the building on the lower floor, the emergency room occupies the rear of the building on the lower floor, the intensive-care unit occupies the upper rear, and the operating rooms occupy the upper front.

The layout is being reconsidered in the hope that traffic flow—both patient and personnel—can be reduced. What constitutes the alternative strategies? What criterion should be used in comparing these strategies?

3-10 Chris Houvras has spent 210 straight days at the track waiting for his favorite horse, Nip-n-Tuck, to run. Today is the day. Just before post time he rereads *The Turf* carefully and estimates that his horse has a 70 percent chance of winning, a 20 percent chance of placing, a 5 percent chance of showing, and a 5 percent chance of losing. The betting odds are such that if his horse is first, he will win $12. If his horse places (comes in second), he will win $8, and if his horse shows (comes in third), he will win $4. Placing a bet costs $2.

a Structure the problem and construct a payoff table.

b On the basis of expected monetary value method, which strategy should he choose?

c Is the expected monetary value method a reasonable way to narrow the alternatives?

d A friend of Chris's, affectionately known as "Hot Gus", has some inside information on today's race. For $3 he will tell Chris exactly what the results of the race will be. Should Chris pay the $3?

3-11 A retail chain has been approached by a large clothing manufacturer to determine whether the chain would be interested in purchasing the surplus inventory of their 1977 winter-wear line. The manufacturer has 4000 pieces and will sell only in thousand-piece lots. The cost to the retail chain will be $5 per piece, and the selling price will be $10 each. Those items unsold at the end of the sale can be sold to a discount chain at $2 each. The estimate of demand is as follows:

Units	Probability
1000	.40
2000	.30
3000	.20
4000	.10

How many pieces should the retail chain purchase?

3-12 A bookstore must place an order for this month's issue of *Down West* magazine. The issues sell for $1 each and cost $0.80 each. Unsold copies cannot be returned. The probability of demand over the next month can be expressed in the following way:

Demand	Probability
100	.10
101	.15
102	.25
103	.25
104	.15
105	.10

How many copies should the bookstore order?

3-13 Jobs 1, 2, and 3 must be processed on two machines. Each job must pass first through machine A and then through machine B. Process times for each job are given in the following table:

Job	Machine A, hours	Machine B, hours
1	2	6
2	4	2
3	3	4

a What are the alternative schedules?

b Which one minimizes the elapsed time?

3-14 A large ski shop in Denver must place its order for skis within the next week. Demand for their skis is strictly dependent on the weather. If the winter is snowy, demand is high, but if the winter is not snowy, demand is low. In snowy winters they sell about 4000 pairs of skis; in other winters they sell about 2000 pairs. The average ski costs \$50 and sells for \$80. Leftover skis must be put on sale at the end of the season at half price. These half-price sales are always successful and all the skis are sold.

Historical records show that 60 percent of the winters in the Rocky Mountains are snowy. How many pairs should the ski shop order?

The Rocky Mountain Weather Service Company can supply a perfect long-range weather forecast for \$500. The forecast is based not only on historical trends but also on the scientific extension of present-day weather patterns. Should this service be purchased?

CASE STUDY: Atwater Publications, Inc.

Atwater Publications, Inc., has been a leading publisher of trade magazines for over 50 years. The magazines it publishes are very specialized and are sold by subscription only to those in certain trades and industries. A list of their publications includes *Steel Industry*, *Electrical Equipment*, *Hotel Supplies*, *Electronics Monthly*, and *Hospital Management*.

Until recently the company has been growing at a steady rate of 10 percent per year. But in the last 2 years sales revenues have increased by only 2 percent.

In response to this decline in growth, the board of directors suggested that the company consider diversifying into other special-interest areas outside the trade field. After this suggestion was made, several proposals were received by the president, Charles Benson. The most promising was from Ann Wilson, director of marketing. She proposed a monthly women's magazine devoted to the issues, problems, and challenges facing today's younger women.

Some preliminary analysis of this proposal had already been undertaken. Based on Ms. Wilson's recommended newsstand price of $1.50 per copy, a market research study showed a 20 percent likelihood that demand for the first issue would be 50,000 copies, a 40 percent likelihood that demand would be 100,000, a 30 percent likelihood that demand would be 150,000, and a 10 percent likelihood that demand would be as high as 200,000.

The market research study went on to develop a demand forecast for those issues beyond the first one. It showed a 10 percent chance that demand would be 50,000, a 20 percent chance that demand would be 100,000, a 40 percent chance that demand would be 150,000, and a 30 percent chance that demand would be 200,000 for each issue.

With this forecast in hand Ms. Wilson met with several advertising agencies in New York City. They purchased enough advertising space in the first 12 issues to guarantee advertising revenues averaging $30,000 per issue.

With all aspects of the project going extremely well, Mr. Benson was prepared to *set the quantity* to be printed for the first issue and *estimate the quantity* for the remainder of the issues during the first year. In addition this would give him the opportunity to confirm earlier *estimates* of the project's profitability. When it was first proposed, Ms. Wilson estimated a per year *profit contribution of $1 million*.

The quantities under consideration for printing were 50,000, 100,000 150,000, and 200,000 copies. Ms. Wilson felt that 200, 000 should be printed and distributed because the first issue needed as much exposure at newsstands, supermarkets, and drugstores as possible. She thought that this display of strength would also discourage any new competition.

Mr. Benson, on the other hand, felt that only 50,000 should be printed in what he called this first "trial" run. He pointed out that it was costly to print

more than could be sold, and that a modest start would give them the time to learn more about this venture into a marketplace that was new to them.

Mr. Benson justified his conservative approach by referring to the costs of this project. It was by far the *most costly project* that Atwater Publications had ever considered. Even the present facilities were inadequate to house the new staff. Additional space would have to be leased at a cost of $2000 per month.

The additional staff necessary to support the magazine would cost $240,000 per year, and this did not include $24,000 per year in new administrative costs. Promotion costs for the magazine were budgeted at $10,000 per month. Printing costs were divided into two categories. First was a fixed cost of $10,000 per issue regardless of the quantity printed. Second was a variable cost of 20 cents for each magazine.

Atwater did not receive the full $1.50 newsstand price for each copy. Since many would be sold at subscription discounts and the rest sold to dealers at wholesale prices, the average net revenue to Atwater would be $1.

Using these figures, Mr. Benson prepared a pro forma profit and loss statement based on the printing of 50,000 magazines. The statement is shown in Exhibit A. His conclusion was that a loss of $2000 would be incurred if the first printing was limited to 50,000.

He felt that if more were printed, there would be no guarantee that they could be sold. In addition to the printing costs which would be lost for the magazines not sold, Atwater would incur an additional handling charge of 25 cents for each magazine returned. (It is an industry practice to accept all unsold magazines at the end of the month and to give full credit to the distributor.)

Exhibit A Pro Forma Profit and Loss Statement for the Printing and Distribution of 50,000 Magazines, First Issue

Net revenue		$50,000
Less: Staff salaries	$20,000	
Printing	20,000	40,000
Less: Administrative costs	2,000	
Promotion	10,000	12,000
Loss		$ 2,000

QUESTIONS

1 Identify the key issue(s) in the case.
2 What criterion should guide the choice between alternatives?
3 Explain Exhibit A. Is it complete?
4 Are the alternatives suggested by Ms. Wilson and Mr. Benson the only ones? Specify other alternatives.

5 Which alternative would you recommend for the first issue? Why?
6 Which alternative would you recommend for future issues?
7 What is your profit projection for this magazine over the first year of its life?
8 What is the most that the company should pay to improve the demand forecast for the first issue?
9 Do you feel that Atwater should continue with this project?
10 Did the company's lack of experience in this new market affect your decision-making process?
11 Describe the role that you feel quantitative methods should play in decision-making situations similar to this one.

APPENDIX A: Conditional Losses

An *opportunity loss* is that loss incurred by failing to choose the best alternative given a particular environmental state. It is possible to reach the same conclusions for a decision problem by focusing on expected opportunity losses as we did by focusing on the expected value of payoffs.

The first step in approaching a decision problem from an opportunity loss point of view is to construct a *conditional opportunity loss table*. Starting from the conditional payoff table for the milk problem, Table 3-5, the conditional opportunity loss for each outcome is computed in the following way. First, the most profitable alternative for each environmental state is identified. Second, a zero is placed in that cell. Third, the remaining outcomes in that column are subtracted from the most profitable outcome. This is done in Table 3A-1. These steps are then repeated for the remaining environmental states. The numbers which now appear in the cells represent the opportunity loss for failure to select the best alternative. For example, if demand were 15 units and 15 units were ordered, then there would have been no opportunity loss since the maximum profit of $45 was earned. If, instead, 17 units were ordered, then a profit of only $25 would have been earned. This is $20 lower than the profit that could have been earned. We call this an opportunity loss.

It seems reasonable to set as the objective the minimization of expected opportunity losses. These are computed in Table 3A-2 for each alternative. The alternative "order 16" has the lowest expected opportunity loss. This corresponds exactly to the choice of "order 16" in Table 3-7.

Order quantity \ Environmental state (demand)	15 $P(15) = .20$	16 $P(16) = .30$	17 $P(17) = .40$	18 $P(18) = .10$
15	45 − 45 = 0	48 − 45 = 3	51 − 45 = 6	54 − 45 = 9
16	45 − 35 = 10	48 − 48 = 0	51 − 48 = 3	54 − 48 = 6
17	45 − 25 = 20	48 − 38 = 10	51 − 51 = 0	54 − 51 = 3
18	45 − 15 = 30	48 − 28 = 20	51 − 41 = 10	54 − 54 = 0

Table 3A-1 Conditional opportunity loss table for milk problem.

Alternative	Expected opportunity loss
Order 15	.20(0) + .30(3) + .40(6) + .10(9) = $ 4.2
Order 16	.20(10) + .30(0) + .40(3) + .10(6) = $ 3.8
Order 17	.20(20) + .30(10) + .40(0) + .10(3) = $ 7.3
Order 18	.20(30) + .30(20) + .40(10) + .10(0) = $16.0

Table 3A-2 Expected opportunity losses.

Also note that the expected opportunity loss for the best strategy, $3.80, is the same as the expected value of perfect information. It should be, since perfect information will reduce the expected opportunity loss to zero and thereby save $3.80.

PROBLEMS

3A-1 Develop a conditional loss table for problem 3-1, identify the best alternative, and determine the value of perfect information.

3A-2 Develop a conditional loss table for problem 3-3, identify the best alternative, and determine the value of perfect information.

3A-3 Develop a conditional loss table for problem 3-5, identify the best alternative, and determine the value of perfect information.

APPENDIX B: Normal Probability Distributions and the Expected Value of Perfect Information

INTRODUCTION

A consumer products firm is considering an investment in a new breakfast cereal. The project has reached the stage where a decision must be made between two alternatives: to introduce the product or not to introduce the product.

The market research group has reported the following information. The break-even point for the product will be 5200 units. That is, the firm will earn a profit if more than 5200 units are sold but will incur a loss if fewer than 5200 units are sold. Furthermore the profit for each unit sold above the break-even point will be $3, while the per unit loss for every unit sold below the break-even point will be $3.

The market research group has also prepared a demand forecast. It suggests that future demand for the product could best be described by a normal probability distribution with an expected demand of 6000 units and a standard deviation of 600 units.

Since expected demand (6000) is greater than the break-even point (5200), the project appears to be profitable. But the company would prefer to collect some additional information before it commits itself to the alternative of undertaking the project. Obtaining this information, however, could be expensive. Consequently it would be beneficial to know the *most* that the company should spend to obtain this information. It needs the "expected value of perfect information."

The purpose of this appendix is the development of the EVPI when the probabilities associated with the environmental states can be described by a normal probability distribution.

CONDITIONAL PAYOFF

The first step in computing the EVPI is to construct a conditional payoff table. This is done in Table 3B-1 for the information given above. Notice that it is a partial

Alternative	Environmental state (demand) 5198	5199	5200	5201	5202
Introduce	−6	−3	0	3	6
Not introduce	0	0	0	0	0

Table 3B-1 Partial conditional payoff table.

conditional payoff table since the table actually continues beyond the points at which it has been terminated.

CONDITIONAL OPPORTUNITY LOSS

A conditional opportunity loss table can be constructed from Table 3B-1. See Table 3B-2. The conditional opportunity loss, if the product is introduced, increases linearly below the break-even point and is zero above that point.

This loss can be expressed mathematically in the following way.

$$\begin{aligned} C(X_b - X) &\text{ if } X \leq X_b \\ 0 &\text{ if } X > X_b \end{aligned}$$

where C = linear loss per unit
X_b = breakeven point
X = actual demand

In our example, C is equal to \$3 per unit and X_b is 5200 units. A graph of the conditional opportunity loss function is presented in Figure 3B-1 for the alternative "introduce."

EXPECTED OPPORTUNITY LOSS

To determine the expected opportunity loss associated with introducing the product, the conditional opportunity loss function is weighted by the probability distribution of demand. In Figure 3B-2 the conditional loss function is shown superimposed on the normal probability distribution of demand. Indeed, it would be impractical to take each possible value of the loss function, multiply it by its probability of occurrence, and sum these products. A more practical method is outlined in the next section.

Alternative	Environmental state (demand) 5198	5199	5200	5201	5202
Introduce	6	3	0	0	0
Not introduce	0	0	0	3	6

Table 3B-2 Partial conditional opportunity loss table.

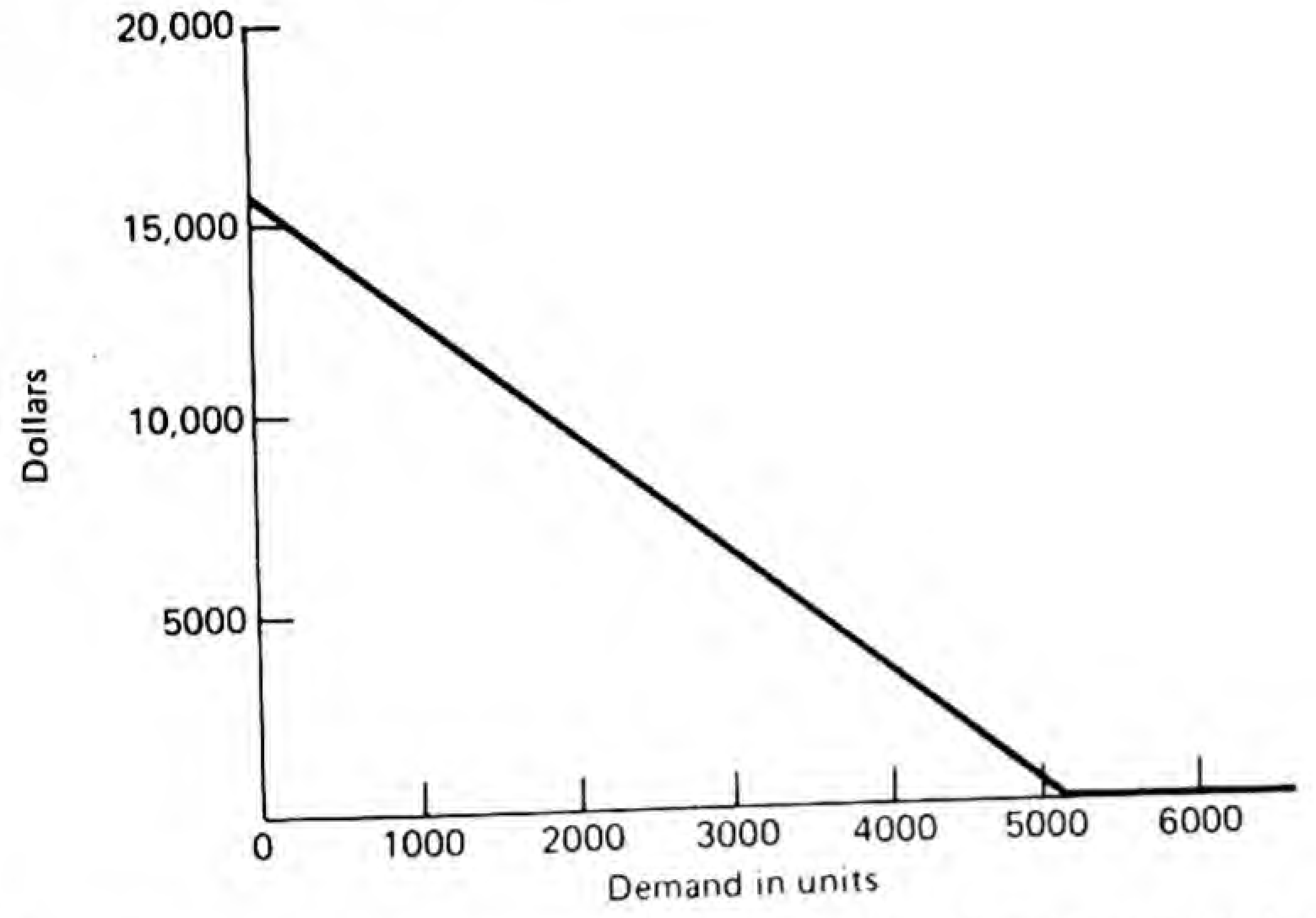


Figure 3B-1 Conditional opportunity loss function (based on the condition that the product is introduced).

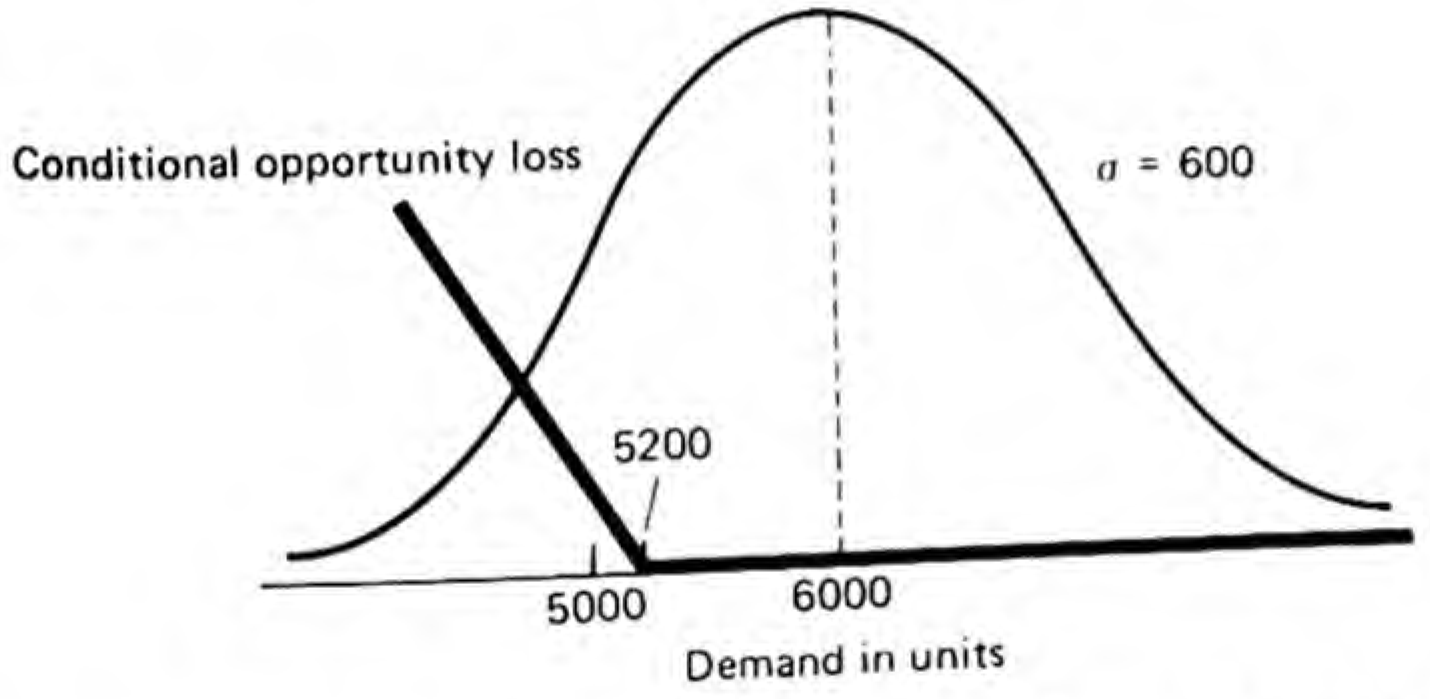


Figure 3B-2 Conditional opportunity loss function superimposed on the normal probability distribution of demand.

Computation of EOL

The expected opportunity loss (EOL) can be computed in the following way:

$$\text{EOL} = C \cdot \sigma \cdot N(D)$$

where C is the linear loss, σ is the standard deviation of the normal distribution, and $N(D)$ can be determined by first finding the value of D.

$$D = \frac{|X_b - \mu|}{\sigma}$$

where X_b = break-even point

μ = mean of the normal distribution

The value of D is therefore the number of standard deviations that the break-even

point is from the mean of the distribution. To find $N(D)$ turn to the loss function which can be found in Table 4 at the end of the book. Proceed down the first column until the value of D just computed is found. The value $N(D)$ can be read from the body of the table.

Returning to our example we now have the following:

$$C = 3$$

$$\sigma = 600$$

$$D = \frac{|X_b - \mu|}{\sigma}$$

$$= \frac{|5200-6000|}{600}$$

$$D = 1.33$$

and

$$N(D) = .04270$$

therefore

$$\text{EOL} = C \cdot \sigma \cdot N(D)$$

$$= 3(600)(.04270)$$

$$\text{EOL} = 76.86$$

The expected opportunity loss associated with the alternative of introducing the product is therefore \$76.86

It is reasonable to assume that the expected opportunity loss associated with introducing the product is lower than the expected opportunity loss of *not* introducing the product. The reason for this is that the expected level of demand is above the break-even point. It can therefore be concluded that the expected opportunity loss of the decision to introduce the product is also the expected value of perfect information.

Therefore if the firm had access to perfect information, it should be willing to spend up to \$76.86 for this information. This new information would prevent the firm from introducing the product only if it showed that demand would actually be below the break-even point.

PROBLEMS

3B-1 Suppose the standard deviation of the demand distribution for the problem given in Appendix B was 400. Recompute the expected value of perfect information. Explain why it has gone down.

3B-2 Suppose the standard deviation of the demand distribution for the problem given in Appendix B was 800. Recompute the expected value of perfect information. Explain why it has gone up.

3B-3 A firm is about to invest in a product whose break-even point is 3000 units. Below this point they will incur a loss of \$4 per unit, and above this point they

will earn \$4 per unit. Demand can be described as a normal distribution whose mean is 3500 and whose standard deviation is 500 units.

a Should the firm undertake the project?

b What is the expected value of perfect information?

c What effect would a higher certainty (lower σ) have on the expected value of perfect information?

3B-4 A product with fixed costs of \$12,000 is being considered. The product will sell for \$5 per unit. Its variable manufacturing costs are \$2 per unit. Expected demand is 5000 units, and the standard deviation of expected demand is 1000 units.

a On the basis of the expected value method, should the firm undertake the investment?

b What is the expected value of perfect information?

c What effect would even less certainty (higher σ) have on the EVPI? Explain.

3B-5 One year ago a large aircraft manufacturer contracted to manufacture several special-purpose aircraft for the government. The contract included a penalty clause. If the manufacturer was late on delivery, a penalty of \$10,000 per day would be charged to the contractor's account.

Although at the time of signing the contract the manufacturer considered it very unlikely that it would deliver the aircraft late, the present status of the project is discouraging.

At present the company has 90 days in which to meet the deadline. The project manager is not certain as to the length of time necessary to complete the project but has implied that a normal distribution with a mean of 85 and a standard deviation of 5 is a reasonable approximation.

a What is the likelihood that the project will be completed on time?

b What is the expected penalty cost at the present time?

Chapter 4

Decision Making under Uncertainty

INTRODUCTION

When the decision maker has no knowledge concerning the probability of environmental states, a decision problem under uncertainty is said to exist. The purpose of this chapter is to explore the use of models in analyzing these situations.

CASE STUDY: Crossover, Inc.

Crossover, Inc., is a large chain of hi-fi stores located on the West Coast. It has been in business for 15 years and has built a reputation for selling the finest in audio components.

Most of Crossover's stores are located in San Francisco and Los Angeles, with a few in Portland, Oregon. Recently the chain decided to expand beyond these areas. At the present time it has the opportunity to lease floor space in a new shopping mall located in Seattle, Washington.

Three alternatives are open. The first is to sign a lease for 1600 square feet. This will enable Crossover to open a branch store with a limited selection of audio components. The second alternative—one which enables

Alternative	Environmental state: High sales	Low sales
1. Branch store	50,000	10,000
2. Full-line store	150,000	−15,000
3. Neither	0	0

Table 4-1 The payoff table illustrating the consequence of opening a new store.

Crossover to open a full-line store—is to sign a lease for 8000 square feet. The third alternative is not to sign any lease and look elsewhere for space. These alternatives are incorporated in the model presented in Table 4-1.

The outcome of these alternatives depends upon the level of business that this new shopping mall will attract. If the level is high, a limited branch store will net $50,000 per year. If the level is low, it will net only $10,000 per year. A full-line store will net $150,000 per year if the level of business is high but will lose $15,000 if the level is low. If the chain decides not to open a store in this area, the payoff will be zero. These payoffs are entered in Table 4-1.

The executives of Crossover spent several hours discussing the potential business in the Seattle area. They concluded that it would be impossible for them to predict the future level of sales with the information they had available.

They would have preferred a demand forecast, but such a forecast would take at least one month to develop. Unfortunately there were several other retail firms interested in space in the mall, and a decision to lease the space had to be made within four days.

DECISIONS UNDER UNCERTAINTY

The case illustrates a situation in which there is little or no knowledge about the probabilities associated with the environmental states. In the sections which follow several methods for analyzing problems of this kind will be covered.

THE METHODS OF ANALYSIS

There are several methods that can be used when a decision problem under uncertainty must be analyzed. These methods include the following:

1. Maximin
2. Maximax
3. Laplace
4. Minimax regret

We first turn to the maximin method.

Maximin

In the maximin method, the outcome with the lowest payoff is identified for each alternative. Then the alternative associated with the best of these becomes the recommended alternative. In other words the maximum of the minimums is chosen, hence the name for this method.

> MAXIMIN
> 1 Determine the lowest outcome for each alternative.
> 2 Choose the alternative associated with the best of these.

To return to our case, the lowest payoff for each alternative is recorded alongside its row in Table 4-2. The highest of these is $10,000. According to this method, a branch store should be opened.

Reflecting on this method for a moment should make it apparent that it is the approach of a pessimist. With the worse expected for each alternative, the alternative that has been chosen is the one that will at least assure the best of the worse.

Next we turn to a selection method that would be taken by an optimist.

Maximax

The second method for choosing an alternative is to determine the best outcome for each alternative and then choose the best of these.

> MAXIMAX
> 1 Determine the best outcome for each alternative.
> 2 Select the alternative associated with the best of these.

In Table 4-3 the best outcome is identified next to each alternative. The best of these is the full-line store with a payoff of $150,000. Therefore if the

	Environmental state		
Alternative	High sales	Low sales	Lowest payoff
1. Branch store	50,000	10,000	10,000*
2. Full-line store	150,000	−15,000	−15,000
3. Neither	0	0	0

Table 4-2 Maximin method.

Alternative	Environmental state: High sales	Low sales	Best outcome
1. Branch store	50,000	10,000	50,000
2. Full-line store	150,000	−15,000	150,000*
3. Neither	0	0	0

Table 4-3 Maximax method.

decision maker uses the maximax method, a full-line store would be opened. This is clearly the optimistic approach since the choice is determined by the best outcome that could possibly occur.

Laplace

In the maximin method, the focus was on the lowest outcome for each alternative; in the maximax method, it was on the highest outcome for each alternative. The Laplace method, on the other hand, uses all the information in the payoff matrix. This is accomplished by assigning equal probabilities to each environmental state. In other words, they are all considered equally likely. In our example, since there are two environmental states, each one would be assigned the probability of $\frac{1}{2}$. Next the expected value for each alternative is computed by multiplying each of its possible outcomes by $\frac{1}{2}$ and summing. Finally the alternative with the highest expected value is chosen.

> LAPLACE
>
> 1 Assign equal probabilities to each environmental state.
> 2 Compute the expected value for each alternative by multiplying each outcome by its probability and then summing.
> 3 Select the alternative with the highest expected value.

In our example the rules can be applied in the following way:

Alternative 1: $\frac{1}{2}(50{,}000) + \frac{1}{2}(10{,}000) = \$30{,}000$

Alternative 2: $\frac{1}{2}(150{,}000) + \frac{1}{2}(-15{,}000) = \$67{,}500^*$

Alternative 3: $\frac{1}{2}(0) + \frac{1}{2}(0) = 0$

Alternative 2 has the highest expected value and is therefore chosen as the most attractive alternative under the Laplace method.

Minimax Regret

The minimax regret method focuses on the opportunity loss concept. First a conditional opportunity loss table is developed. From the data presented in Table 4-1 a conditional loss table is developed (Table 4-4). Often this is called a *regret* table. It is developed in the following way. First, the most attractive outcome for each environmental state is identified. Second, a zero is placed in those cells. Third, the remaining outcomes in each column are subtracted from the most attractive outcome. For example, the most attractive outcome for the environmental state of "high sales" is \$150,000. A zero is entered in that cell. For the cell above that one the outcome is \$50,000. Therefore the opportunity loss for that cell is the following:

$$\begin{array}{r} 150{,}000 \\ -\ 50{,}000 \\ \hline 100{,}000 \end{array}$$

This can be interpreted in the following way. *If* the environmental state were indeed high sales and *if* a branch store were opened, an opportunity loss of \$100,000 would have been incurred when compared with the best alternative that could have been selected.

After the regret table is completed the maximum opportunity loss is identified for each alternative. Finally the alternative associated with the lowest value of maximum opportunity loss is selected as the recommended alternative. See Table 4-4. The minimax regret method therefore identifies that alternative which minimizes the maximum opportunity loss that could be incurred.

MINIMAX REGRET

1 Construct a payoff table.
2 For each environmental state identify the most attractive alternative. Place a zero in those cells.
3 Compute the opportunity loss for the other alternatives.
4 Identify the maximum opportunity loss for each alternative.
5 Select the alternative associated with the lowest of these.

Alternative	Environmental state: High sales	Environmental state: Low sales	Maximum opportunity loss
1. Branch store	150,000 − 50,000 = 100,000	0	100,000
2. Full-line store	0	10,000 − (−15,000) = 25,000	25,000*
3. Neither	150,000 − 0 = 150,000	10,000 − 0 = 10,000	150,000

Table 4-4 Conditional opportunity loss table and the minimax regret method.

WHICH METHOD?

Although the decision makers in our case could have used any of these four methods, the problem is far from over because they did not reach the same conclusion. The decision makers are still left with the question "What should be done?" Well, this depends on their attitude toward risk. If they are very conservative, they should follow the conclusion of the maximin method: branch store. If they are optimistic and risk takers, they should follow the outcome of the maximax method: full-line store. If they feel that any state of nature is just as likely as any other, they can follow the Laplace method: full-line store. And if they would like to minimize their maximum opportunity loss, they can follow the minimax regret method and open a full-line store. It can be concluded that unless the decision must be very conservative, a full-line store is the most attractive alternative.

IS IGNORANCE NECESSARY?

In some situations decision makers may indeed remain ignorant of environmental state probabilities. Often this ignorance can be attributed to either a lack of time or money or both. With more time or money, additional information could be obtained and the problem converted to one of risk rather than uncertainty.

In the Crossover, Inc., case there was insufficient time to obtain new information. In other situations there may not be enough funds to obtain additional information. Regardless of the reason, there do exist situations in which decisions must be made under conditions of uncertainty.

In general, however, most decision problems are problems under risk rather than problems under uncertainty. Most decision makers find the time and money to collect some information—formally or informally—on environmental state probabilities.

SUMMARY

The models and analysis presented in this chapter have been directed at those decision problems where the decision maker is totally ignorant as to which environmental state is likely to occur. The formulation of these models proceeds in the same way as the models in Chapter 3, but several different techniques, including maximin, maximax, Laplace, and minimax regret, are used for the purposes of analysis. The choice of a particular technique depends upon the decision makers' attitudes toward risk.

QUESTIONS

1 Is decision making under uncertainty a common phenomenon?
2 What is the purpose of a payoff table?
3 Compare the maximin and maximax approaches.
4 When would a decision maker employ the Laplace method?
5 Compare the maximin and minimax regret methods.
6 Suppose that you were faced with the decision problem presented in the case. Which method would you use and why? What is your decision?

PROBLEMS

4-1 The following problem has four alternatives and three environmental states. Find the most desirable alternative under each of the following methods:
a Maximin
b Maximax
c Laplace
d Minimax regret

	Environmental state		
Alternative	A	B	C
1	5	10	6
2	11	6	12
3	14	3	7
4	4	6	14

4-2 In the following problem there are three alternatives and four environmental states. Find the most desirable alternative, using each of the following methods:
a Maximin
b Maximax
c Laplace
d Minimax regret

Alternative	Environmental state A	B	C	D
1	4	7	11	6
2	12	2	7	5
3	6	7	10	5

4-3 Given the following four alternatives and five environmental states, find the most desirable alternative under each of the following four methods:

a Maximin
b Maximax
c Laplace
d Minimax regret

Alternative	Environmental state A	B	C	D	E
1	5	7	15	6	4
2	3	10	4	7	12
3	1	16	6	3	5
4	12	12	7	6	1

4-4 The following problem has four alternatives and four environmental states. Find the most desirable alternative under each of the following methods:

a Maxi nin
b Maximax
c Laplace
d Minimax regret

Alternative	Environmental state A	B	C	D
1	20	16	25	40
2	35	37	34	36
3	17	25	35	42
4	50	40	21	15

4-5 The Beta Company is about to choose a marketing strategy for one of their new products. The vice president of marketing, Frank Mastro, feels that he must choose from three levels of promotional activity. The most he can allocate is \$100,000. He can also allocate \$60,000, but the least he can allocate is \$30,000.

The outcome of these promotional levels depends upon competitors' strategies. If \$100,000 is allocated and competitors also allocate "large" sums to their product, Beta's payoff will be \$90,000. If Beta allocates \$100,000 and competitors allocate an "average" sum, Beta's payoff will be \$170,000. And if Beta allocates \$100,000 while competitors allocate "little," Beta's payoff will be \$200,000.

If instead Beta allocates \$60,000, their payoff will be \$70,000, \$90,000, or \$140,000, depending upon whether competitors allocate large, average, or little sums to promotion.

Finally, if Beta allocates \$30,000, their payoff will be \$20,000, \$70,000, or \$125,000, depending upon whether competitors allocate large, average, or little sums to promotion.

Given these estimates the vice president of marketing is faced with making a decision. Since the product is new and he has never competed in a similar marketplace, Mr. Mastro feels that he is unable to predict the level of promotional activity chosen by his competitors.

His new product represents a very small percentage (2 percent) of estimated sales, and even if the project fails, it will not seriously jeopardize the financial viability of the firm. If, on the other hand, it is successful, it will provide a much-needed boost to Mr. Mastro's career with Beta.

What should Mr. Mastro do? Why?

4-6 The Willow Manufacturing Company will be producing a new product this month, and the production scheduling department is trying to decide how many to schedule. It can schedule either a low, medium, or high quantity of this product.

If a low quantity is scheduled and demand is low, the payoff from this strategy will be \$500. If demand is medium or high, the payoff will still be only \$500, since there will not be enough stock to take advantage of the higher level of demand.

If a medium quantity is scheduled and demand is low, profit will be only \$100, since many of the items produced will go unsold. If demand is medium, profit will be \$700, and if demand is high, profit will still be only \$700.

If the order quantity is high and demand is low, net profit will be zero. If demand is medium, profit will be \$500, and if demand is high, profit will be \$1000.

The scheduling department is quite concerned over this order quantity because the company has been losing money lately and further dramatic losses could jeopardize the viability of the firm.

Which alternative would you choose and why?

4-7 A sales representative must make four calls today. Unfortunately, the customers are between 15 and 35 miles apart. The problem is to determine the order in which the customers should be seen. Confusing the problem, however, is the fact that driving time between customers depends upon traffic. Traffic may be light, average, or heavy. Unfortunately, it is impossible to predict in advance exactly what the traffic will be like.

Structure this problem and justify the criterion you would use in its solution.

4-8 Return to problem 3-11 and reach a decision under the assumption that nothing is known about the probability of demand. Use the maximin criterion.

4-9 Return to problem 3-12 and reach a decision under the assumption that nothing is known about the probability of demand. Use the maximax criterion.

4-10 Return to problem 3-14 and reach a decision under the assumption that nothing is known about the probability of demand. Use the minimax regret criterion.

CASE STUDY: Lawrence Hospital

The Lawrence Hospital is a 400-bed hospital located in the rapidly growing region of Miami, Florida. It was built in 1960 as a 200-bed hospital and since that time has expanded to keep pace with the growing community.

In the past expansion plans were routinely approved. Financing was never a problem. But this has changed in the last few years as the public has become increasingly reluctant to accept the spiraling costs of medical care. Pressure groups, including insurance companies, politicians, and area residents, have all voiced their concern over the problem. They have insisted that the hospital hold the line on expenses while providing—at a minimum—the same quality of service.

In response to this pressure, a new hospital administrator, Fred Lathy, was hired 6 months ago. In the short period of time he has been with the hospital, he has proved a tough administrator. Several employees have been terminated, while those who have remained have been working harder. Already costs have been reduced by 12 percent. The hospital, however, is still operating at a loss.

Ever since Mr. Lathy joined the hospital staff there has been a moratorium on both hospital construction and investment in new equipment. Several proposals have been submitted, but no action has been taken.

Yesterday Mr. Lathy received a call from Winfred Knoble, Jr., chairman of the board of trustees. Mr. Knoble told Fred that several physicians had expressed their concern to him about the delay on three proposals that had been submitted during the past 6 months. Both Mr. Knoble and Mr. Lathy agreed that there were sufficient funds to accept one but not all three of these proposals. Mr. Knoble closed the conversation by asking Fred to recommend one of the proposals to the board of trustees at its next meeting in 2 weeks.

After the phone call Fred removed the proposals from his file cabinet and began to examine them carefully. The first was for the expansion of the coronary care facilities from 5 to 10 beds, the second was for the addition of a fourth operating room, and the third was for the addition of radiology facilities. The following paragraphs were taken from these proposals.

Coronary Care

The coronary care facilities at Lawrence are inadequate to meet the needs of the community, and as the community continues to grow this problem will become even more acute. Current facilities operate at an occupancy rate of 90 percent. This is higher than the occupancy rate for other units in the hospital. On many occasions patients must be accommodated in the corridor or sent to nearby hospitals. There is no question that improved facilities could save lives.

According to our estimates if aggregate demand for the facilities at

Lawrence averages 11,000 patient-days[1] per month, the excess of revenues over all costs for this project will be $120,000 per year. If demand is 9000 patient-days per month, revenues will exceed costs by $60,000, but if demand is 7000 patient-days then costs will exceed revenues by $30,000 per year.

Operating Room

Lawrence Hospital currently maintains three operating rooms. Use of these facilities is high and on several occasions elective surgery has been postponed to accommodate surgery of an emergency nature. New facilities would not only alleviate this problem but would end the reluctance of many surgeons to schedule elective surgery at Lawrence. An important by-product of this project—one not considered in the economic data—would be the increase in the demand for hospital beds in the surgical wards. Since these wards are currently operating at a net loss, this would be a welcomed addition to revenue.

If the aggregate demand for facilities at Lawrence Hospital averages 11,000 patient-days per month, revenues associated with this project will exceed costs by $190,000 per year. If demand averages 9000 patient-days per month, revenues will exceed costs by $70,000 per year, and if demand averages 7000 per month, costs will exceed revenues by $60,000 per month.

Radiology

The radiology department desires to purchase equipment used in cancer therapy. At the present time patients who require this therapy must be referred to a hospital 12 miles away.

If the aggregate demand for facilities at Lawrence Hospital averages 11,000 patient-days per month, revenues associated with this project will exceed costs by $140,000 per year. If demand averages 9000 patient-days per month, revenues will exceed costs by $90,000 per year, and if demand averages 7000, revenues will exceed costs by $60,000 per year.

The Recommendation

After reading these proposals, Mr. Lathy realized that this decision depended to some extent on his ability to estimate the future demand for hospital facilities. But several factors made this impossible. First, he had been with the hospital just 6 months and as yet had not been able to detect any trend in demand. Second, the hospital had never made demand forecasts before and there was no one with this expertise on his staff. Third, there was not enough time to make an intelligent forecast.

[1] A patient-day represents the unit of demand for hospital services. One patient using the hospital for 1 day is tabulated as one patient-day. Current demand for all hospital facilities averages 7000 patient-days per month.

The decision was therefore made on the basis of available information. Mr. Lathy recommended that the board of trustees accept the operating room proposal. He felt that it had the advantage of contributing more than the other projects toward the elimination of the hospital's financial difficulties.

QUESTIONS

1 What criterion did Mr. Lathy use in comparing the alternative proposals?
2 Construct a conditional payoff table.
3 Identify the proposal that would have been recommended had the following methods been used.
   a Maximin
   b Maximax
   c Laplace
   d Minimax regret
4 Which method did Mr. Lathy informally use to reach his conclusion? Was he justified in using this method?
5 Do you think that the choice of a criterion is an issue in this case? If so, why is it an issue? What criterion or criteria would you use?
6 Suppose that you had to make a recommendation to the board of trustees. Which proposal would you choose. Why?
7 Suppose that the probabilities associated with each of the demand levels had been estimated in the following way.

Demand	Probability
7,000	.40
9,000	.50
11,000	.10

   Is this still a decision problem under uncertainty? Find the expected monetary value of each alternative. Which alternative has the highest EMV? Which alternative would you recommend?
8 On the basis of the demand forecast given in question 7, what is the most that Lawrence Hospital should be willing to pay for a perfect demand forecast?

APPENDIX A: Game Theory

INTRODUCTION

Competitors or rivals always seem to be trying to outguess one another. Regardless of whether these rivals are competing on the battlefield of war, in a parlor game of poker, in a political campaign, or in the marketplace for goods and services, there seems to be considerable effort directed at outguessing the competition. Indeed, there is good reason for this effort. If the competitor can be outguessed, the rewards can be great.

Game theory deals with this process of competitive interaction. The purpose of this appendix is to develop models and techniques of analysis for selecting strategies in these competitive situations.

THE DEFINITION OF A GAME

Games have alternative strategies and payoffs. Each player has a set of alternatives from which to choose. The outcome which results from a given strategy is called the payoff.

A simple model of a two-person game is summarized in Table 4A-1. The left-hand column represents the strategies open to player A, and the top row represents the strategies open to player B. The entries within the table represent payoffs to player A. For example, if player A chooses strategy 1 and player B chooses strategy 2, the payoff *to* player A will be 5.

Game theory is quite different from decisions under risk, which were discussed in Chapter 3. In decisions under risk the decision maker is playing against an environment which changes its strategies in some random fashion. In game theory the decision maker is playing against an intelligent competitor who will carefully consider what strategies his opponent might employ and then calculate his own most promising strategy.

TWO-PERSON ZERO-SUM GAMES

A game is said to be a two-person zero-sum game if the gain of one player is completely at the expense of the other. Consider a poker game between two players in which the winner's gain is \$50. This \$50 gain is at the expense of the loser (what one gained the other lost). We call this a two-person zero-sum game.

Player A \ Player B	1	2	3
1	3	5	6
2	2	11	5
3	7	9	12

Table 4A-1 Payoff matrix for player A.

Coke \ Pepsi	1	2	3	4
1	.80	.30	.40	.60
2	.25	.10	.20	.15
3	.60	.20	.30	.25

Table 4A-2 Coke-Pepsi payoff matrix.

As another example consider the competition between Coke and Pepsi in the soft-drink market. For the moment, let's assume that they are the only two competitors. One objective of their marketing departments is to capture as large a portion of the market as possible. To accomplish this, they regularly choose from among countless promotional strategies.

Suppose that Coke has recently narrowed the choice down to the three strategies presented in Table 4A-2, and that Pepsi has narrowed its choice to four strategies. The body of the table represents the market share that Coke will achieve for various strategy combinations. For example, if Coke chooses strategy 2 and Pepsi chooses strategy 3, Coke's share of the market will be 20 percent.

Clearly this is a zero-sum game: what Coke gains, Pepsi loses, and what Pepsi gains, Coke loses. If Coke gains 5 percent of the market, Pepsi loses 5 percent.

Since this is a zero-sum game, it is possible to determine Pepsi's market share for various strategy combinations directly from Table 4A-2. For example, if Coke chooses strategy 2 and Pepsi chooses strategy 3, Coke's share of the market will be 20 percent. Pepsi's share will therefore be $1.00 - .20$, or 80 percent.

ASSUMPTIONS ABOUT THE PAYOFF MATRIX

There are two basic assumptions which underlie the theory of games. These include complete knowledge and utility.

Complete Knowledge

The theory of games which is about to be unfolded requires that competitors be fully aware of the strategies open to each other and their resultant payoffs. Therefore, Coke and Pepsi decision makers must have the knowledge depicted in Table 4A-2.

Utility

The second assumption is that the payoffs must represent utilities. This means that in our example a market share of 80 percent must have twice as much utility to Coke as a market share of 40 percent.

MAXIMIN AND MINIMAX STRATEGIES

According to Table 4A-2, the decision makers at Coke have three strategies open to them. Suppose for a moment that they decide to play it safe. To accomplish this, they

could identify the minimum market share which would result from each strategy. If strategy 1 were taken, the minimum would be 30 percent. If strategy 2 were taken, the minimum would be 10 percent; and if strategy 3 were taken, the minimum would be 20 percent. These minimums are recorded in Table 4A-3. The conservative strategy would then be to select the maximum of these *minimums*. Therefore, strategy 1 would be chosen, since its minimum of .30 is greater than the other two minimums. This is called the maximin method for selecting a strategy.

Now let's turn to the other player in the game: Pepsi. Suppose Pepsi also decides to play it safe. What will its strategy be? To answer this, Table 4A-3 must be reexamined. From Pepsi's point of view, lower values in the table are more desirable than larger values. This is true because a low value (.20) for Coke represents a high value ($1.00 - .20 = .80$) for Pepsi. Consequently, a safe strategy for Pepsi would be to identify the maximum value in each column and then choose the minimum of these. The maximum in the first column is .80, and the maximums of the remaining columns are .30, .40, and .60. These are entered below the body of Table 4A-3. The minimum of these is .30, and we can therefore conclude that Pepsi's strategy should be strategy 2 if indeed it wishes to play it safe. Since we chose the minimum of the maximums in determining Pepsi's strategy, a minimax method has been used.

SADDLE POINT

In Table 4A-3 it was determined that the maximin and minimax values were both .30. Furthermore, the row minimum was at the same time the column maximum. Under this condition the game is said to have a saddle point, and the *value of the game* is .30.

When a saddle point does exist, a very interesting situation occurs. Suppose that the players of this game do, indeed, choose their strategies cautiously—in other words, player 1 uses the maximin criterion and player 2 uses the minimax criterion; then they will have no incentive to change their strategies, even if each knows exactly what the opponent intends to do.

If Coke plays strategy 1, Pepsi will have no incentive to use any strategy other than strategy 2. If Pepsi changed its strategy to any other one, its market share would decrease. Correspondingly, as long as Pepsi plays strategy 2, Coke will stay with strategy 1 to preserve its market share.

It is especially important to realize that even if the players do *not* use maximin and minimax as their criterion, eventually both players will revise their strategies until

Coke \ Pepsi	1	2	3	4	Row minimum
1	.80	.30	.40	.60	.30 Maximin
2	.25	.10	.20	.15	.10
3	.60	.20	.30	.25	.20
Column maximum	.80	.30 Minimax	.40	.60	

Table 4A-3 Maximin, minimax, and saddle points.

the saddle point is reached and will thereafter have no incentive to change. As an example of this, suppose that Coke initially considered strategy 3. However, since it had full knowledge of the consequences of its own and Pepsi's strategies (assumption of game theory), it realized that if it did indeed play strategy 3, Pepsi would play strategy 2. Its counter strategy to Pepsi would then be to switch to strategy 1, and it would be back to the saddle point. Neither would have the incentive to change.

We can therefore conclude that in games where saddle points exist, player 1 will play a maximin strategy while player 2 will play a minimax strategy. Furthermore, there will be no incentive for either to change strategies.

PROBLEMS

4A-1 The following is a payoff matrix which describes the strategies available to two players and the payoff to player 1.

Player 1 \ Player 2	1	2	3
1	7	3	6
2	5	4	16
3	17	2	9

a Does a saddle point exist?
b What strategies should players 1 and 2 take?
c What is the value of the game?
d Would it be advantageous for player 1 to change strategies? Explain why.

4A-2 The following is a payoff matrix describing the strategies available to two players and the payoff to player 1.

Player 1 \ Player 2	1	2	3
1	3	16	4
2	5	11	4
3	15	21	5

a Does a saddle point exist?
b What strategies should players 1 and 2 take?
c What is the value of the game?
d Would it be advantageous for player 1 to change strategies? Explain why.

4A-3 The following is a payoff matrix which describes the strategies open to two players and the payoff to player 1.

Player 1 \ Player 2	1	2	3	4
1	14	16	12	20
2	11	15	10	31
3	21	4	5	8

a Does a saddle point exist?
b What strategies should players 1 and 2 take?
c What is the value of the game?
d Would it be advantageous for player 1 to change strategies? Explain why.

4A-4 The following is a payoff matrix describing the strategies open to two players and the payoff to player 2.

Player 1 \ Player 2	1	2	3
1	14	7	21
2	6	10	8
3	5	4	3

a Does a saddle point exist?
b What strategies should players 1 and 2 take?
c What is the value of the game?
d Would it be advantageous for player 1 to change strategies? Explain why.

APPENDIX B: Games with No Saddle Point

INTRODUCTION

We discovered in the previous appendix that whenever a game had a saddle point, it was in the best interest of each player to choose a strategy according to the maximin or minimax criterion. If the game were to be played over and over again, this strategy would be repeatedly employed. Whenever a single strategy is relied upon, we say that the player is using a *pure strategy*.

Not all games, however, have saddle points. If, indeed, none exists, we will shortly see that a pure strategy should not be used. Instead, a combination of strategies or *mixed strategies* should be employed.

Consider the two-person zero-sum game presented in Table 4B-1. Each player can

Player I \ Player II	1	2	Row minimum
1	12	30	12 — Maximin
2	15	11	11
Column maximum	15 — Minimax	30	

Table 4B-1 Example of a two-person zero-sum game with no saddle point.

choose between two strategies whose payoffs to player I are measured in dollars. For example, if both players choose strategy 1, player I gains \$12 and player II loses \$12.

Suppose that the maximin and minimax criteria are used to identify the best strategies. According to Table 4B-1 the maximin value is 12 and the minimax value is 15: the row minimum, however, does *not* correspond to the column maximum. We can conclude that there is no saddle point, and we would not expect to find the use of strategy 1 by player I and the use of strategy 1 by player II to be the most effective. Suppose for a moment that they *did* decide to use these pure strategies. Let's examine the consequence of this action.

If player I selects strategy 1 (maximin), player II will respond by also choosing strategy 1. However, player I in the interest of increasing his payoff, will then switch to strategy 2. Subsequently, player II will switch to strategy 2 to protect himself. When player II switches to strategy 2, player I will then switch to strategy 1. Player II will then respond by switching to strategy 1, and we are right back where we started. This unstable cycle will go on and on. There will be no set of strategies out of which these players will not switch. In other words, this game has no saddle point.

Clearly, there is no simple pure strategy that will work. It is possible, however, to develop a mixed strategy such that both players will reach an equilibrium and will have no incentive for changing their mixed strategies. A *mixed* strategy is some *combination* of pure strategies. For example, a mixed strategy for player I might be to use strategy 1 in one-third of the games and strategy 2 in two-thirds of the games. Next we will develop a method for determining these mixed strategies.

A GRAPHICAL METHOD FOR 2 × 2 GAMES

The game presented in Table 4B-1 is referred to as a 2×2 two-person zero-sum game because there are two strategies open to player I and two strategies open to player II. If there had been three strategies open to player I and five strategies open to player II, it would have been a 3×5 two-person zero-sum game.

The 2×2 game presented in Table 4B-1 can be solved by graphical analysis, where the vertical axis of the graph represents the payoff to player I. The horizontal axis represents the probability of player I choosing strategy 2, $P(\text{I},2)$. This axis starts at 0 and ends at 1.0. Therefore, at the right end of the axis it is certain that strategy 2 is chosen, $P(\text{I},2) = 1.0$, while at the left end of the axis it is certain that strategy 2 is not chosen, $P(\text{I},2) = .0$.

From the payoff matrix in Table 4B-1, we see that if player I chooses strategy 1, his payoff will be 12, provided that player II chooses strategy 1. This point is plotted in Figure 4B-1:

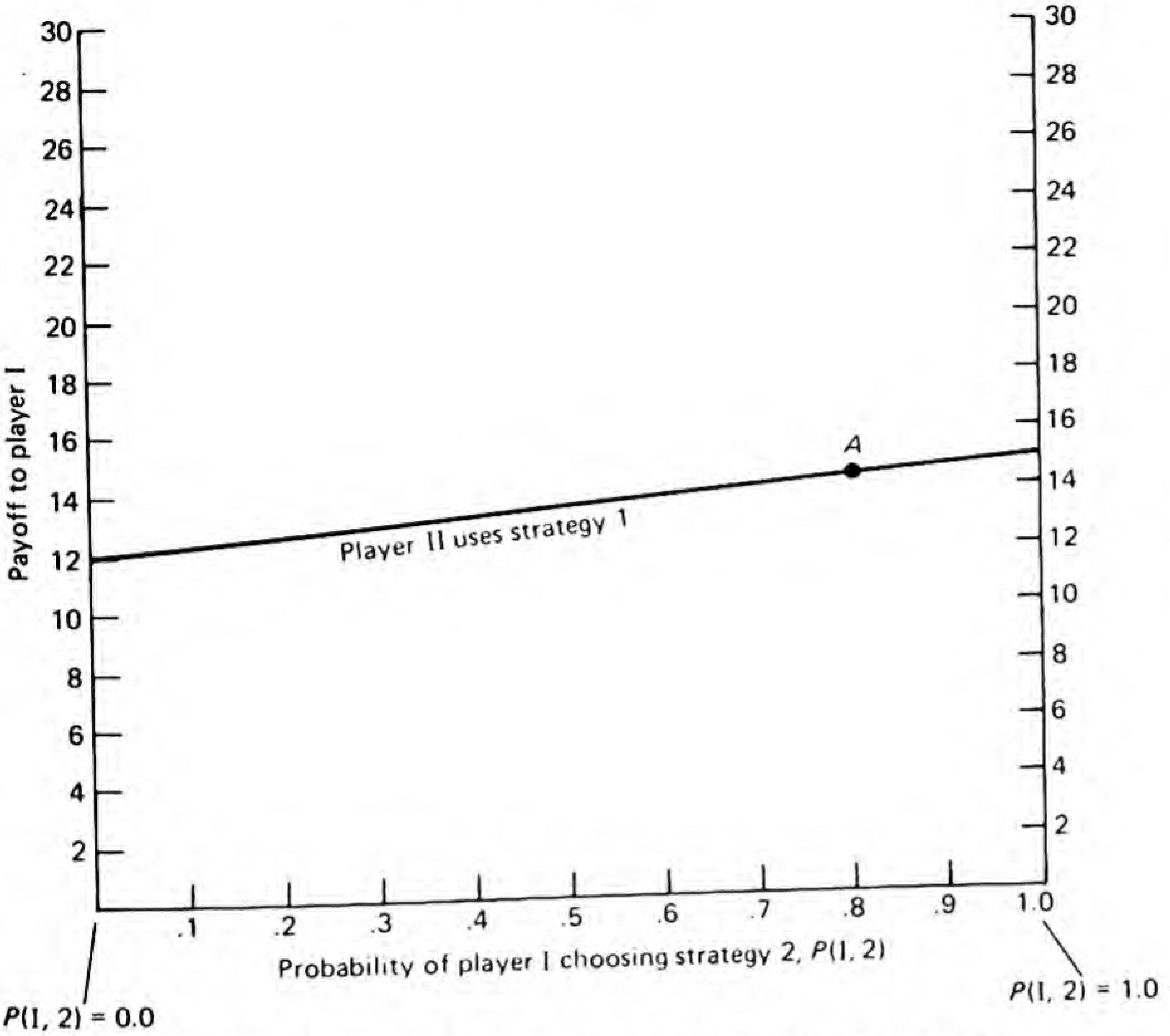


Figure 4B-1 Payoff to player I given that player II uses strategy 1.

Payoff to player I = 12
$P(\text{I},2) = .0$

On the other hand, if player I chooses strategy 2 and player II continues to choose strategy 1, the point in Figure 4B-1 will be the following:

Payoff to player I = 15
$P(\text{I},2) = 1.0$

These two points are then connected by a straight line. Points along the line represent the payoff to player I for employing a combination of strategy 1 and strategy 2 *given* that player II chooses strategy 1. For example, point A implies that if player II chooses strategy 1 exclusively, and if player I chooses strategy 2 eighty percent of the time and strategy 1 twenty percent of the time, the average payoff will be 14.4.

Our next step is to construct a line which depicts payoffs to player I when player II chooses strategy 2 exclusively. The two end points of the line are determined in the following way:

Payoff to player I = 30
$P(\text{I},2) = 0$

Payoff to player I = 11

$$P(\mathrm{I},2) = 1.0$$

These points are identified in Figure 4B-2 and are connected by a straight line. Points along this line represent payoffs for various mixed strategies when player II stays exclusively with strategy 2.

To this line is now added the line determined in Figure 4B-1. With both lines presented in Figure 4B-3 we can now proceed to select player I's best strategy.

According to the maximin principle, the minimum payoff for all possible strategies open to player I must be identified. But player I not only has the pure strategies of strategy 1 and strategy 2 available but has the option of playing any combination of these as well. The minimum payoff for these combinations is shown as the crosshatched line in Figure 4B-4. The final step is to select the maximum of these minimum points. Point M is that point.

According to the maximin principle, then, player I should choose the mixed strategy depicted by point M.[1] This strategy requires that strategy 2 be played $^{18}/_{22}$ or 82 percent of the time and strategy 1 be played $^{4}/_{22}$ or 18 percent of the time. The value of the game will therefore be 14.4545.

From Figure 4B-4 it should be clear that this mixed strategy is indeed an equilibrium strategy in which player I will have no incentive to change.

The strategy that should be taken by player II can be determined in the same way. From Figure 4B-5 it can be seen that according to the minimax criterion, point M represents the second player's best mixed strategy.[2] Accordingly, player II should employ the second strategy $^{3}/_{22}$ of the time and the first strategy $^{19}/_{22}$ of the time. The value of the game to him will be 14.4545.

From Figure 4B-5 it can be seen that if player II does play this mixed strategy, there will be no incentive for him to change.

We can therefore conclude that if both players employ the mixed strategies which

[1] The solution can be computed in the following way:

$$30[1 - P(\mathrm{I},2)] + 11P(\mathrm{I},2) = 12[1 - P(\mathrm{I},2)] + 15P(\mathrm{I},2)$$

Solving for $P(\mathrm{I},2)$,

$$P(\mathrm{I},2) = {}^{18}/_{22} \qquad \text{and} \qquad P(\mathrm{I},1) = {}^{4}/_{22}$$

The value of the game is computed in the following way:

$$30(1 - {}^{18}/_{22}) + 11({}^{18}/_{22}) = 14.4545$$

[2] The solution can be computed in the following way:

$$15[1 - P(\mathrm{II},2)] + 11P(\mathrm{II},2) = 12[1 - P(\mathrm{II},2)] + 30P(\mathrm{II},2)$$

Solving for $P(\mathrm{II},2)$,

$$P(\mathrm{II},2) = {}^{3}/_{22} \qquad \text{and} \qquad P(\mathrm{II},1) = {}^{19}/_{22}$$

The value of the game is computed in the following way:

$$15(1 - {}^{3}/_{22}) + 11({}^{3}/_{22}) = 14.4545$$

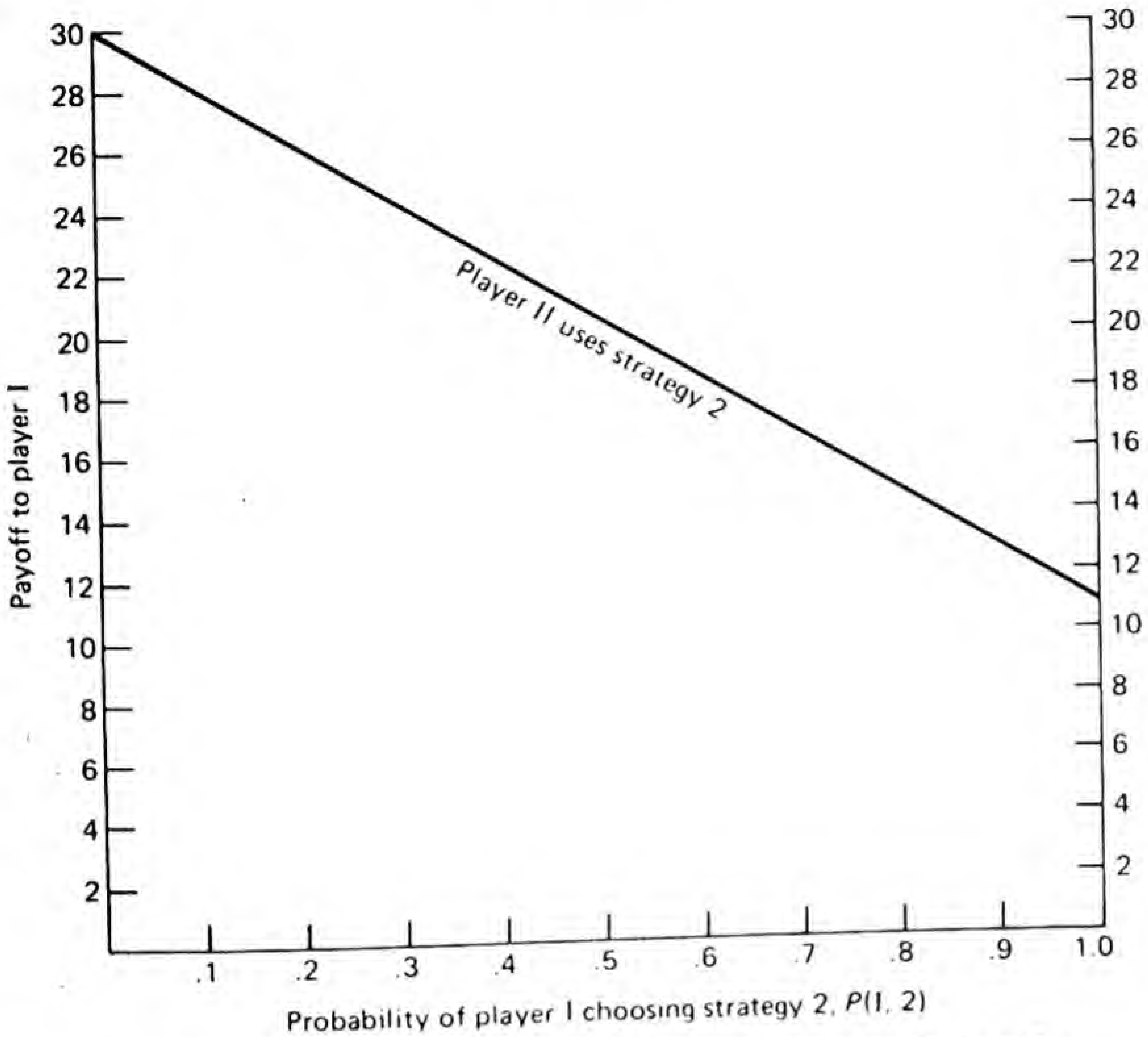


Figure 4B-2 Payoff to player I given that player II uses strategy 2.

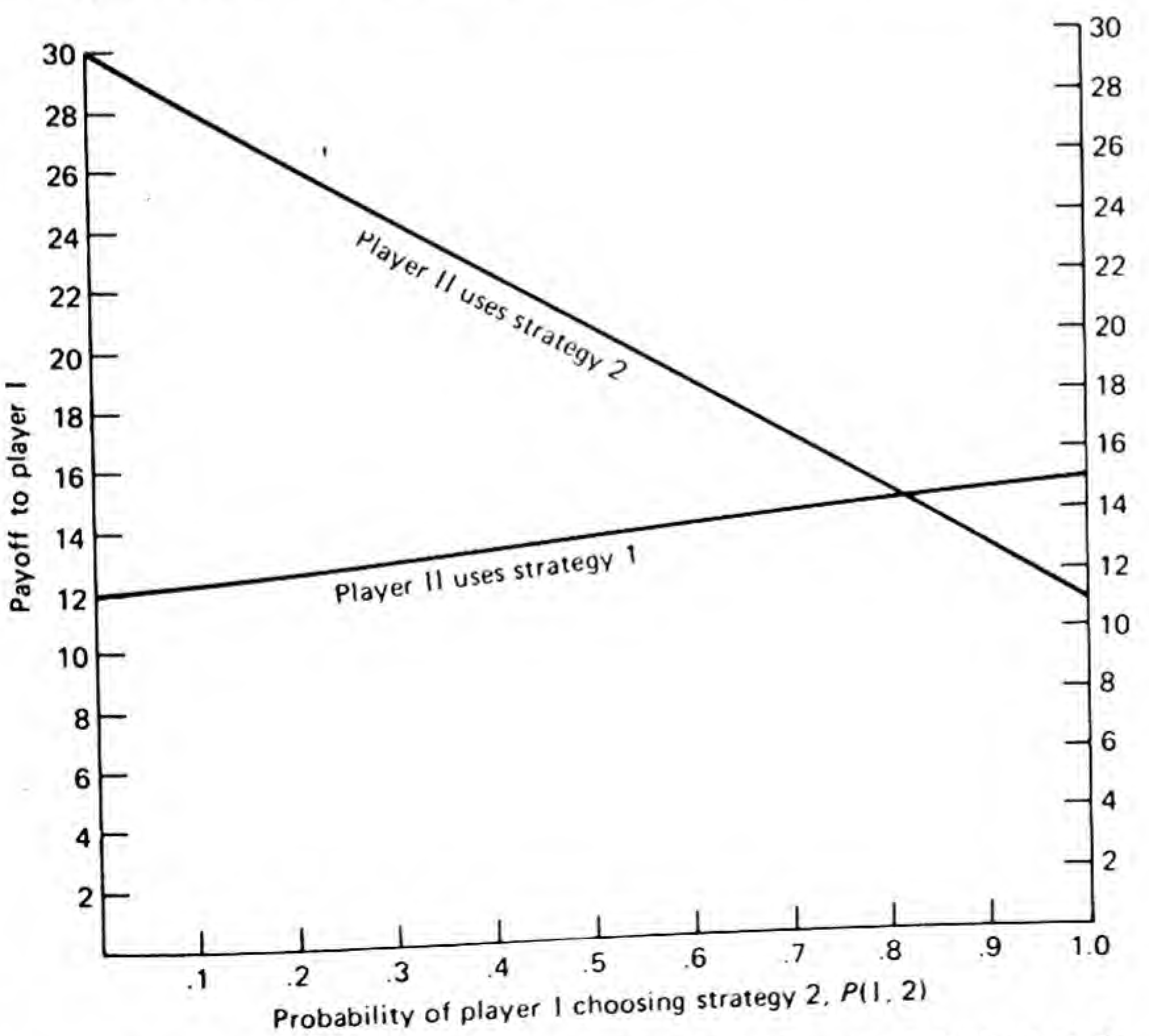


Figure 4B-3 Payoff to player I given that player II uses strategy 1 or 2.

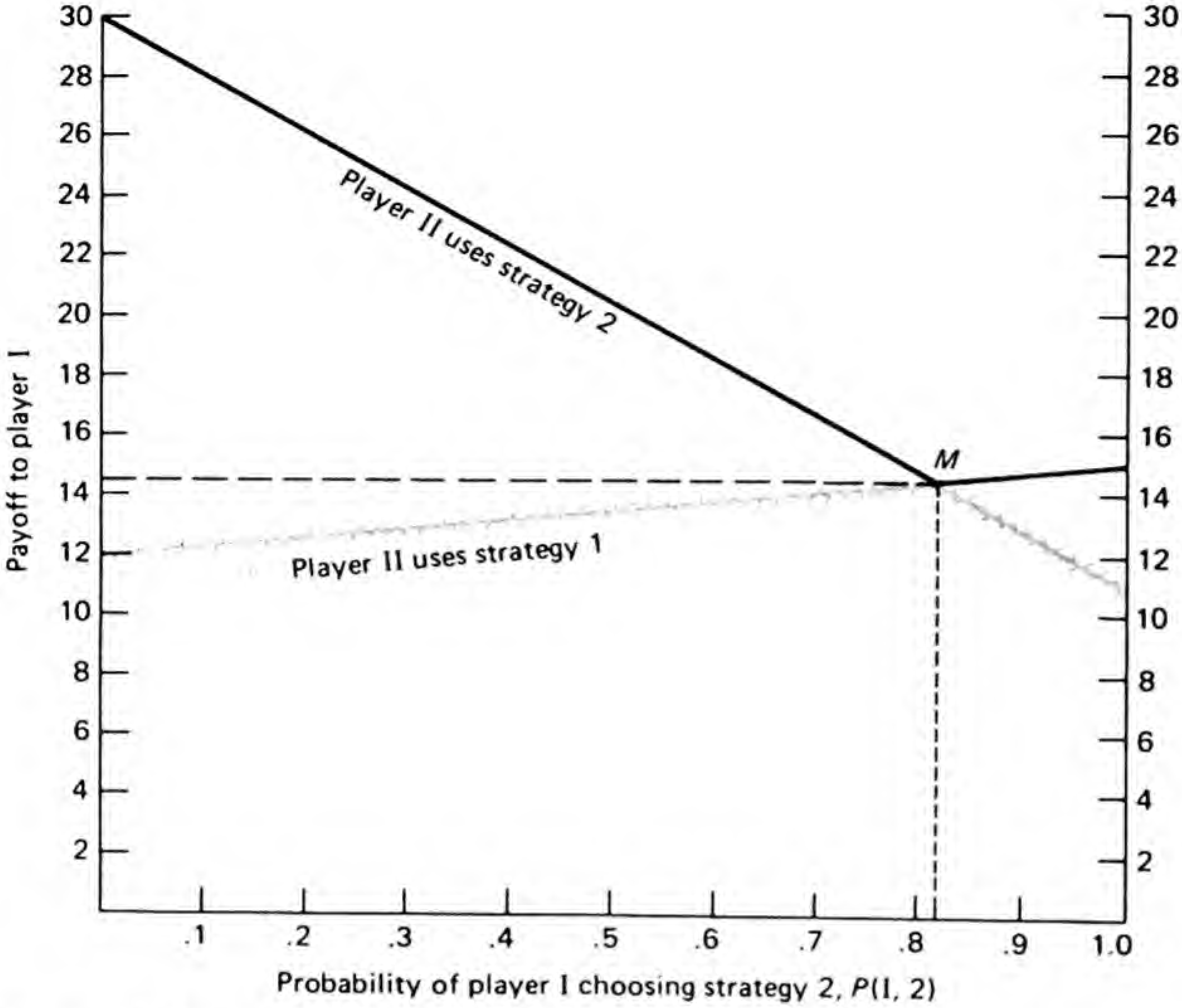


Figure 4B-4 Solution to player I's mixed strategy.

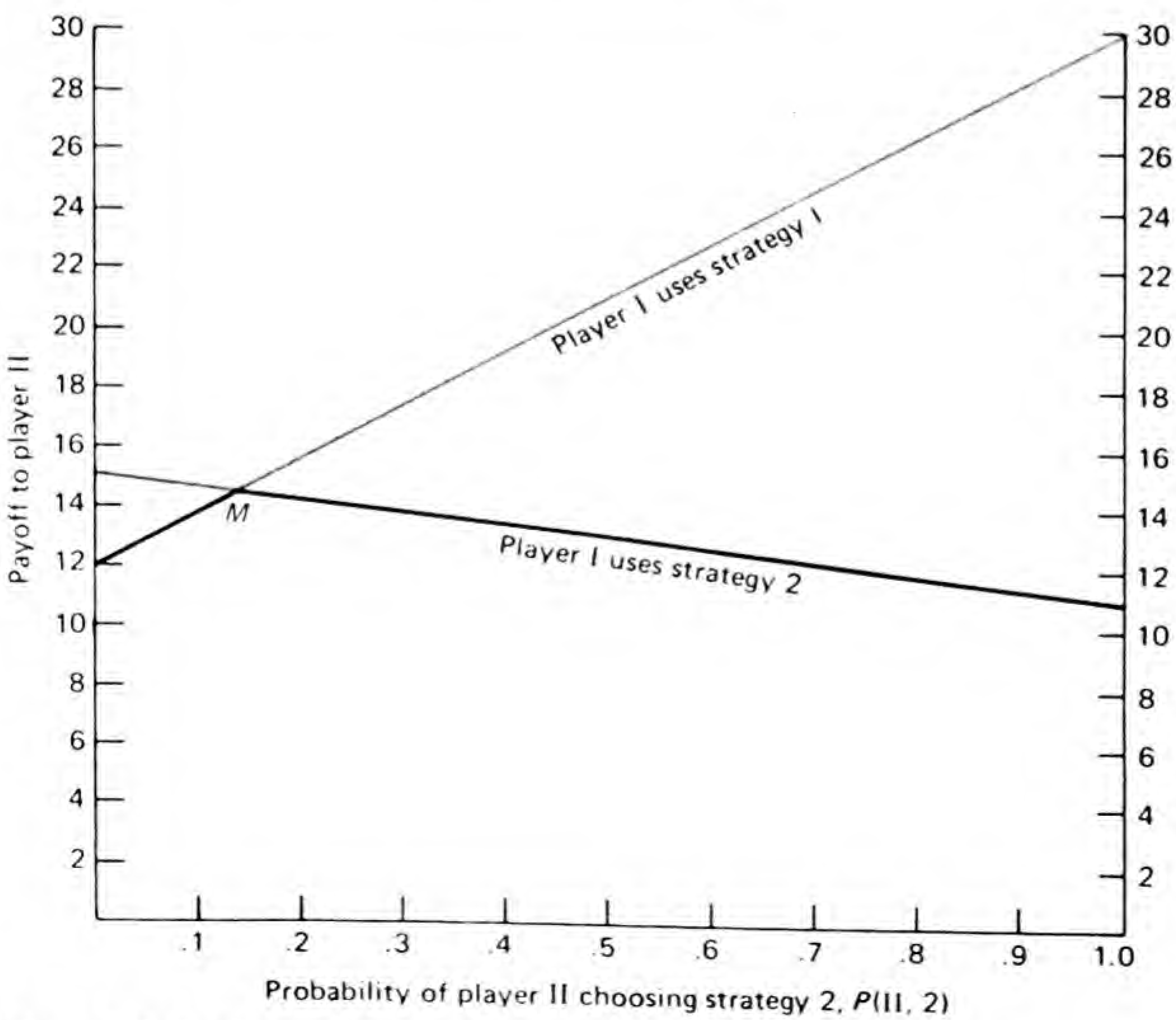


Figure 4B-5 Solution to player II's mixed strategy.

we have just developed, neither will have the incentive to change strategies and the value of the game will be 14.4545 to both of them.

THE USE OF MIXED STRATEGIES

The solution to the game presented in Figure 4B-1 requires that a mixed strategy be used. Player I, according to the solution, was required to employ the first strategy 82 percent of the time and the second strategy 18 percent of the time. To do this, the player must use some random device, such as random number tables, to ensure that there is no predictable pattern in the choice of his strategies. Clearly, it would not do to employ strategy 1 during the first 82 of 100 occurrences and then switch to strategy 2 for the last 18. The competitor, player II, would quickly become aware of this pattern and exploit this information. Strategies 1 and 2 indeed must appear 82 and 18 percent of the time, but in a *random* fashion.

PROBLEMS

4B-1 The following is a payoff matrix which describes the strategies open to two players and the payoff to player 1.

Player 1 \ Player 2	A	B
A	20	10
B	6	12

Determine the strategy that player 1 should take. What is the value of the game?

4B-2 Return to problem 4B-1 and determine the strategy that player 2 should take. What is the value of the game?

4B-3 The following is a payoff matrix which describes the strategies open to two players and the payoff to player 1.

Player 1 \ Player 2	A	B
A	20	60
B	40	10

Determine the strategy that player 1 should take.

4B-4 Return to problem 4B-3 and determine the strategy that player 2 should take.

Chapter 5

Bayes' Rule: The Revision of Probabilities in the Light of New Information

INTRODUCTION

In any decision-making situation there is always some initial amount of information—either subjective or objective—to which the decision maker has access. If this information is adequate, a decision may be made. Even if the information is inadequate, but time and money prevent the collection of additional information, a decision may still be made. But if additional information can be collected, it is very likely that this new information together with the initial information may lead to a better decision: one with less risk.

This chapter is concerned with the way in which initial information is revised once new information is made available. The case study which follows will be used to introduce these concepts in greater detail.

CASE STUDY: Howard Company

The Howard Company is a large machine shop which manufactures parts used by the defense and aircraft industries. Most of its orders are for small quantities of expensive parts. Seldom are two orders alike.

To accommodate this variety in jobs Howard operates 1000 machines grouped into 120 functional work centers. One work center is completely devoted to drilling, another to grinding, another to painting, and so on.

A primary concern of the Howard Company is the quality of its manufactured parts. Each part must be machined to the standards specified by the customer and no part can be shipped unless these standards have been met.

To ensure compliance with these standards the Howard Company has established a quality control group which reports directly to the president of the company. The group has the responsibility for checking jobs at various stages in the manufacturing process and the authority to stop this work if the standards specified by the customer have not been met.

One stage at which each job is checked is the machine setup stage. The setup or adjustment of a machine is checked by a member of the quality control department before the machine is permitted to produce the customer's order. In this way any errors can be corrected before hundreds or thousands of unacceptable parts are produced.

At the present time Frank O'Connor of the quality control department has been called to one of the grinding machines for the purpose of approving a setup. Frank is quite familiar with the machine and its operator. From this past experience he knows that the probability that the machine is properly set up is 80 percent. But Frank is reluctant to approve the setup on this information alone. Even though it will take one hour for the grinder to produce one part, he prefers to wait for this part, take some measurements, and on the basis of this new information revise his prior probability that the machine is properly set up.

If Frank decides that the machine is adjusted properly, his approval will be given to proceed with the customer's order of 45 pieces.

THE USE OF ADDITIONAL INFORMATION

In the Howard Company case Frank O'Connor had some preliminary information. He felt that the probability that the machine had been properly set up was 80 percent. This preliminary information is called a *prior* probability.

Once a sample is taken and some additional information is collected, this prior probability can be revised. But how can this revision be accomplished? It can be accomplished by using Bayes' rule.

Bayes' rule will be developed in the next section using a simple but mechanical example. After the rule has been developed we will return to the Howard Company case and use it to revise Frank's prior probabilities.

THE DEVELOPMENT OF BAYES' RULE

A Game

The rules for playing a certain game of chance require that the player first throw a die. If the outcome is 1, a marble must be selected from urn A; if the

outcome is 2 or 3, a marble must be selected from urn B; and if the outcome is a 4, 5, or 6, a marble must be selected from urn C.

Prior and Conditional Probabilities

From the description given above we can determine that the probability that urn A will be selected is $1/6$; the probability that urn B will be selected is $2/6$; and the probability that urn C will be selected is $3/6$. These outcomes and probabilities are shown in Figure 5-1. The probabilities are given a special name: they are called *prior probabilities.*

After the die is thrown and an urn selected, a marble must be picked. Suppose that urn A has 10 marbles: 6 red and 4 green. Therefore the likelihood of drawing a red marble from urn A is $6/10$. When this information is presented on the probability tree, however, it should be indicated that this is a conditional probability. The likelihood of drawing a red marble depends upon (or is conditional upon) the urn from which that marble was drawn. We can then write this probability in the following way:

$$P(R|A) = 6/10$$

Correspondingly the probability of drawing a green marble is

$$P(G|A) = 4/10$$

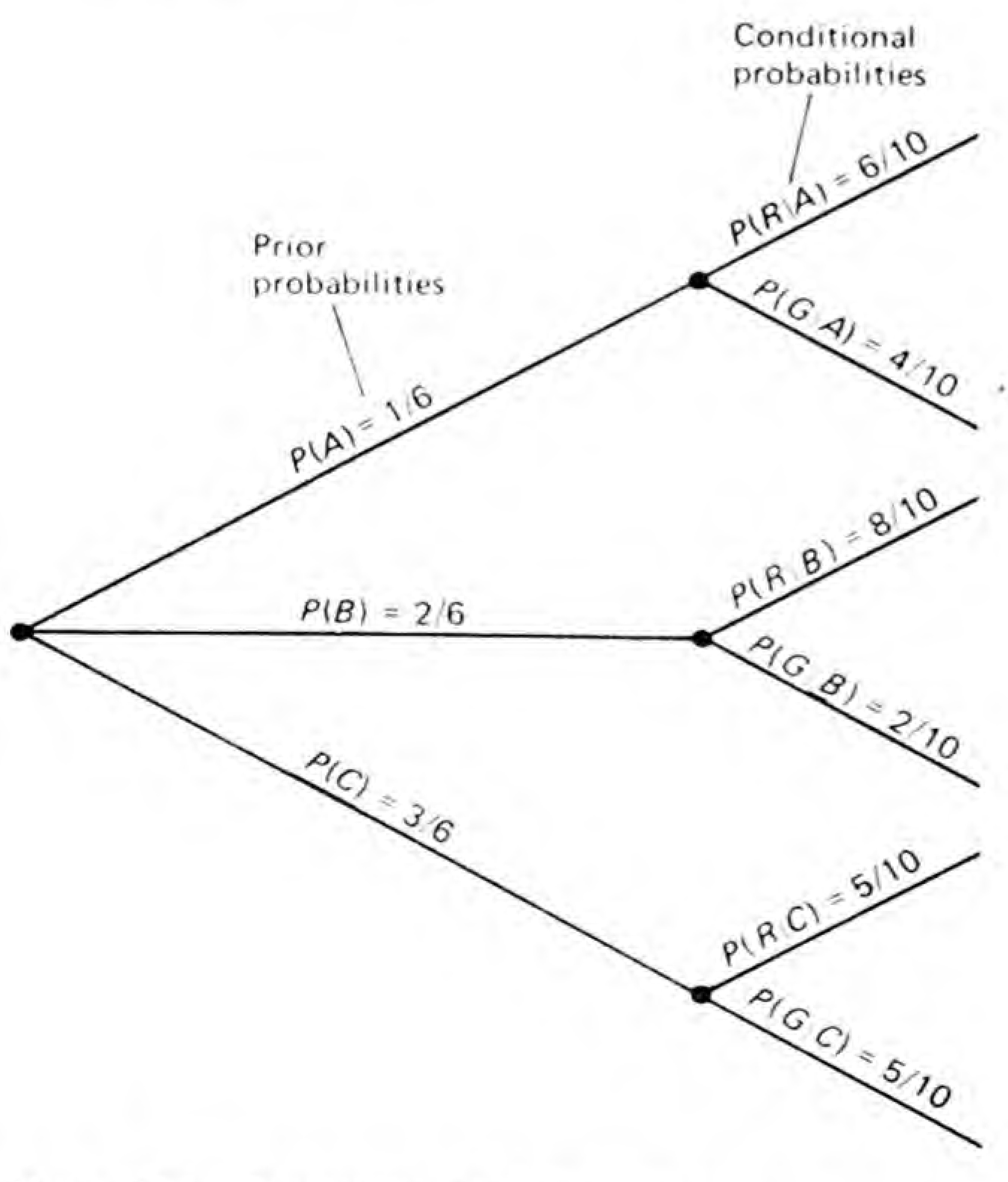


Figure 5-1 Probability tree for urn game.

In urn B there are 8 red marbles and 2 green marbles. Therefore, these conditional probabilities can be expressed in the following way:

$$P(R|B) = 8/10$$
$$P(G|B) = 2/10$$

In urn C there are 5 red and 5 green marbles. These conditional probabilities can be expressed in the following way:

$$P(R|C) = 5/10$$
$$P(G|C) = 5/10$$

All these conditional probabilities are shown in Figure 5-1.

Joint Probabilities

From these probabilities it is possible to determine the joint probability of choosing urn A and drawing a red marble, choosing urn B and drawing a red marble, and choosing urn C and drawing a red marble. By referring to the rule of multiplication which we developed in Chapter 2, we can compute these joint probabilities in the following way:

$$P(A \cap R) = P(A) \cdot P(R|A)$$
$$P(A \cap R) = 1/6 \cdot 6/10 = 6/60$$

Continuing with the other joint probabilities, we have

$$P(B \cap R) = P(B) \cdot P(R|B) = 2/6 \cdot 8/10 = 16/60$$
$$P(C \cap R) = P(C) \cdot P(R|C) = 3/6 \cdot 5/10 = 15/60$$

They are repeated in Figure 5-2.

Summing the Joint Probabilities

Suppose you want to know the likelihood of choosing a red marble. This outcome can occur in one of three ways. You might choose urn A and pick a red marble, *or* choose urn B and pick a red marble, *or* choose urn C and pick a red marble. In other words, a red marble can occur in any one of the following three ways:

$$(A \cap R) \cup (B \cap R) \cup (C \cap R)$$

The probability of this can be written in the following way:

$$P(R) = P(A \cap R) \cup P(B \cap R) \cup P(C \cap R)$$

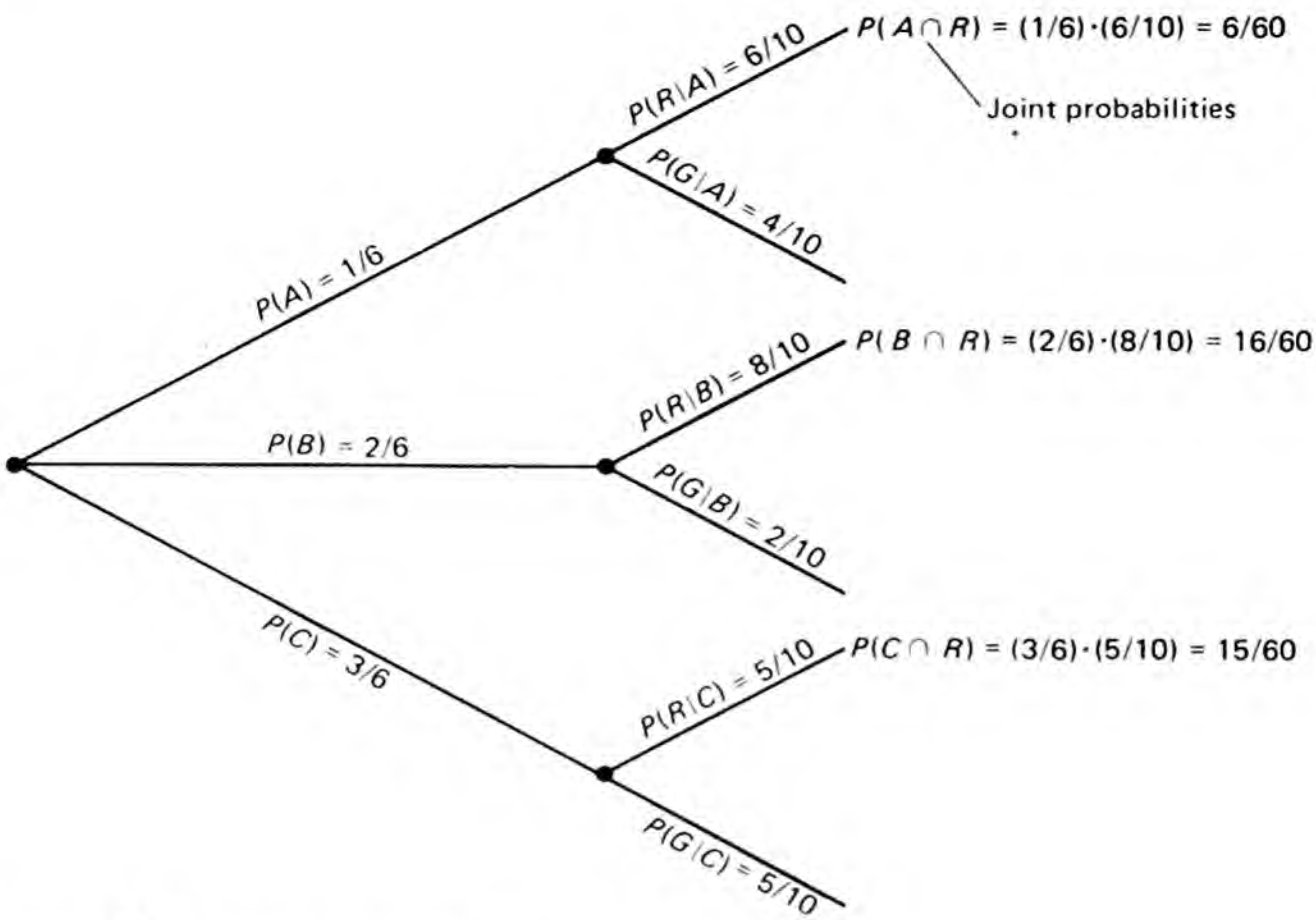


Figure 5-2 Joint probabilities.

Since these three joint events are mutually exclusive (if one happens, it rules out the others), we have no intersection between them and we need only to add these joint probabilities.

$$P(R) = {}^{6}/_{60} + {}^{16}/_{60} + {}^{15}/_{60} = {}^{37}/_{60}$$

Therefore, the likelihood that a red marble will be chosen by the game rules is ${}^{37}/_{60}$.

Problem

Find the probability of drawing a green marble.

Solution

$$P(A \cap G) = P(A) \cdot P(G \mid A) = {}^{1}/_{6} \cdot {}^{4}/_{10} = {}^{4}/_{60}$$
$$P(B \cap G) = P(B) \cdot P(G \mid B) = {}^{2}/_{6} \cdot {}^{2}/_{10} = {}^{4}/_{60}$$
$$P(C \cap G) = P(C) \cdot P(G \mid C) = {}^{3}/_{6} \cdot {}^{5}/_{10} = {}^{15}/_{60}$$
$$P(G) = P(A \cap G) \cup P(B \cap G) \cup P(C \cap G) = {}^{4}/_{60} + {}^{4}/_{60} + {}^{15}/_{60}$$
$$P(G) = {}^{23}/_{60}$$

Note that $P(G) + P(R) = {}^{23}/_{60} + {}^{37}/_{60} = 1$.

Revising the Prior Probabilities

Now that we have examined all the probabilities associated with this game, let's assume that someone has just played it. You are informed, in fact, that a red marble was drawn. This is your new information.

Before you obtained this new piece of information, you knew the prior probabilities of choosing urn A, urn B, or urn C. Now with this new information it will be possible to revise these probabilities. For example, you will be able to determine the probability that urn A was chosen *given* that a red marble has been drawn. This can be expressed in the following way:

$$P(A|R)$$

It is referred to as a *revised prior* or *posterior probability*.

Bayes' Rule

To compute the revised prior probability, we must first examine all possible ways in which a red marble could have been drawn. From Figure 5-3 we can see that a red marble could have been drawn in one of the following three ways: urn A and then a red marble, urn B and then a red marble, and urn C and then a red marble. In this example we are especially interested in one of these ways: urn A and a red marble. Therefore, if a red marble was drawn, the likelihood that the correct sequence was urn A and a red marble can be computed in the following way:

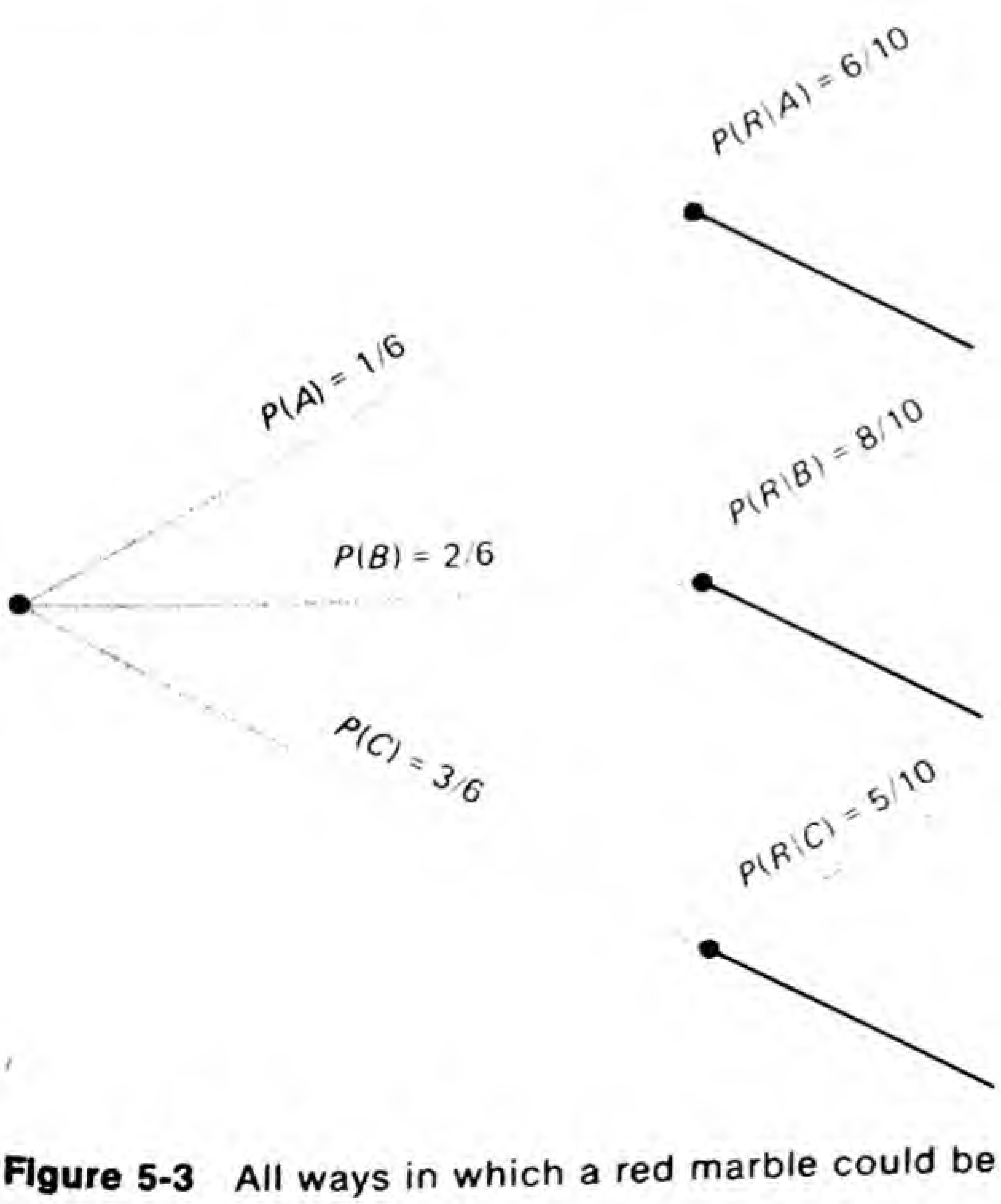


Figure 5-3 All ways in which a red marble could be drawn.

$$P(A|R) = \frac{P(A \cap R)}{P(A \cap R) \cup P(B \cap R) \cup P(C \cap R)}$$

The denominator is already familiar to us, and we could simplify the expression.

$$P(A|R) = \frac{P(A \cap R)}{P(R)}$$

This formulation is referred to as Bayes' rule and is used to revise probabilities in the light of new information.

Returning to our example, we have

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{6/60}{37/60} = 6/37$$

and we can therefore conclude that, based on the additional information that a red marble was drawn, the likelihood that it came from urn A is $6/37$.

Originally we determined that the likelihood of choosing urn A was

$$P(A) = 1/6 = .167$$

and now, after collecting some information, we revise that probability and say that the likelihood that urn A was chosen is

$$P(A|R) = 6/37 = .162$$

This is a revised probability, and we say that it was computed using Bayes' rule.

Problem

Given that a red marble was chosen, what is the likelihood that it came from urn B?

Solution

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{16/60}{37/60} = 16/37 = .432$$

Note the dramatic increase in the posterior probability $P(B|R) = .432$ from the prior probability $P(B) = 2/6 = .333$. This illustrates the impact that additional information can have on prior probabilities.

Problem

What is the likelihood that urn B was chosen given that a green marble was drawn?

Solution

$$P(B|G) = \frac{P(B \cap G)}{P(G)} = \frac{4/60}{23/60} = 4/23$$

Before moving on to more examples, a formal presentation of Bayes' rule will be made.

BAYES' RULE: A FORMAL VIEW

Given two events A and B, Bayes' rule states the following:

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$
$$= \frac{P(A \cap B)}{P(A)}$$

In a decision-making setting, event A represents the new information—usually obtained from sampling—and $P(B|A)$ represents the revised prior probability based on this new information.

THE USE OF BAYES' RULE IN THE HOWARD CASE

Returning to the Howard Company case we have already learned that in the past the grinding machine has been correctly (C) set up 80 percent of the time. Therefore, this becomes the prior probability

$$P(C) = .80$$

Correspondingly, the likelihood that the machine is incorrectly (I) set up is .20.

$$P(I) = .20$$

These are entered on the tree diagram in Figure 5-4. Upon further questioning Frank has informed us that when the machine is correctly set up, the likelihood that it will produce a good part is .90.

$$P(G|C) = .90$$

The likelihood that a correctly set-up machine will produce a bad part is therefore .10.

$$P(B|C) = .10$$

These are also entered in Figure 5-4.

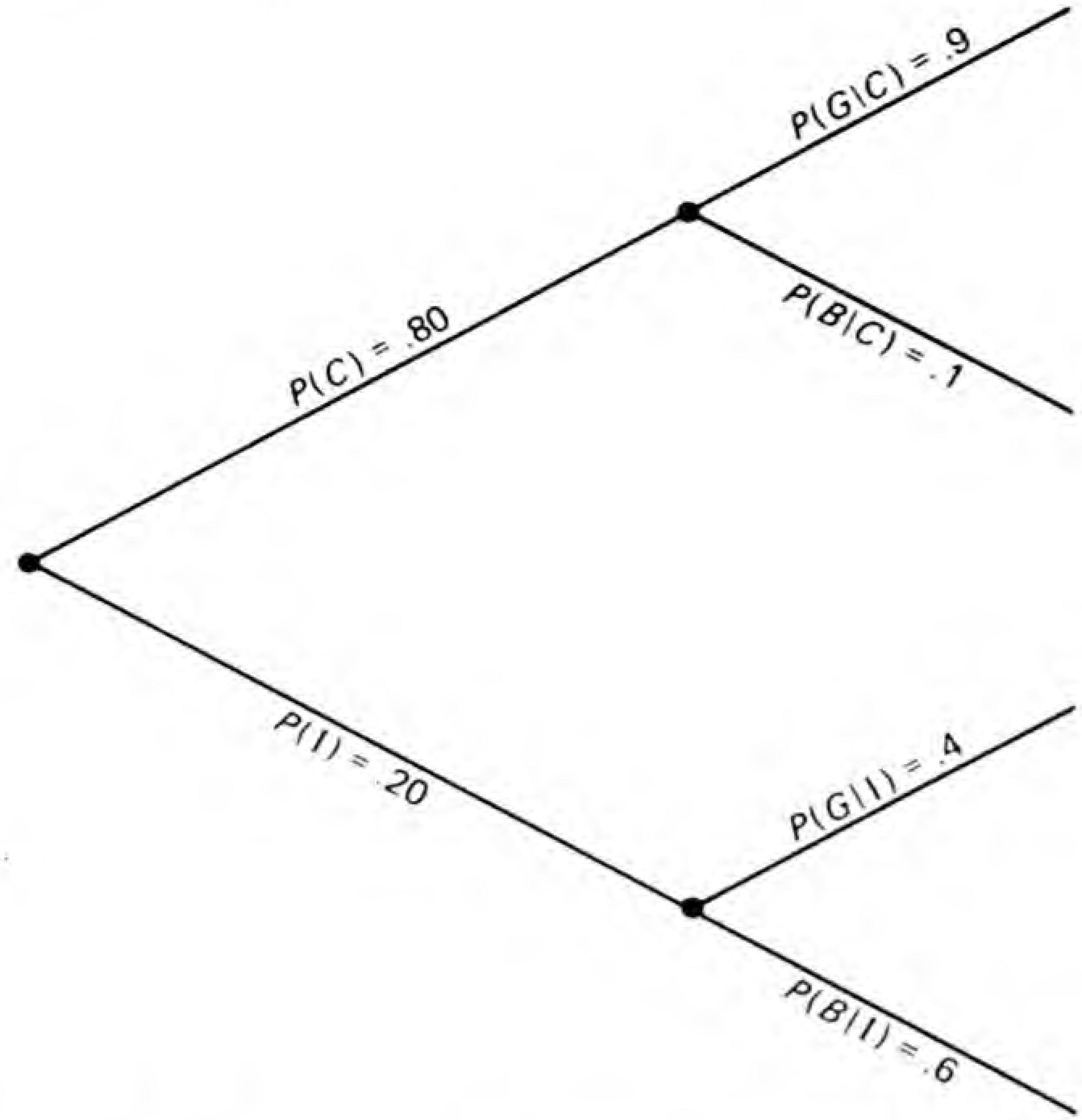


Figure 5-4 Bayesian quality control problem.

If, on the other hand, the machine is incorrectly set up, Frank has informed us that the likelihood that a good part will be produced is .40 and the likelihood that a bad part will be produced is .60.

$$P(G|I) = .40$$
$$P(B|I) = .60$$

Again, these are entered in Figure 5-4.

Now let us suppose that Frank has turned the machine on and observed the first part; it is good. Given this outcome, what can now be said about the likelihood of the machine being properly set up?

To answer this question, the prior probability must be revised. This is accomplished by finding the probability that the machine is properly set up given that a good part has been observed or $P(C|G)$. Following the same steps we took in our first example, we identify all ways in which a good part could have occurred. The relevant paths are crosshatched in Figure 5-5. These include: correctly set up *and* a good part, and incorrectly set up *and* a good part. Their joint probabilities can be computed in the following way.

$$P(C \cap G) = .8 \times .9 = .72$$
$$P(I \cap G) = .2 \times .4 = .08$$

Of these two ways in which a good part could have been generated, we are

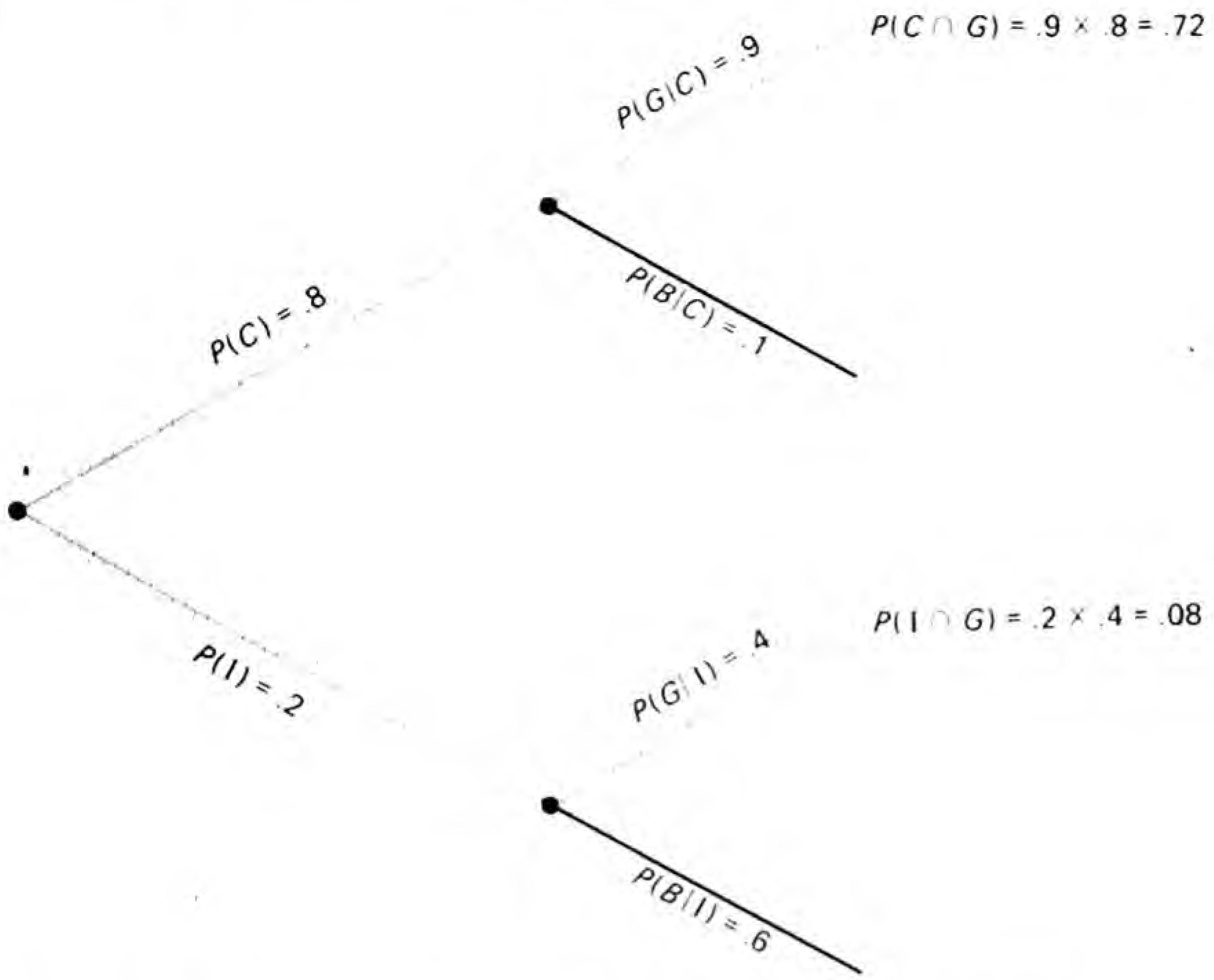


Figure 5-5 All ways in which a good part could have occurred.

especially interested in the likelihood that it was the path in the probability tree represented by "correctly set up and good." The likelihood of this is

$$\frac{P(C \cap G)}{P(C \cap G) \cup P(I \cap G)}$$

This can be simplified to

$$P(C|G) = \frac{P(C \cap G)}{P(G)}$$

which is, once again, Bayes' rule. Finally making the computations, we find

$$P(C|G) = \frac{.72}{.72 + .08} = \frac{.72}{.80} = .9$$

and we conclude that the revised likelihood that the machine is properly set up is .9.

Notice that the original probability estimate that the machine is properly set up has been increased from $P(C) = .8$ to $P(C|G) = .9$. Again this illustrates the impact that additional information may have on prior probabilities. On the basis of this revised probability estimate, Frank should now feel quite confident that the machine is properly set up.

Problem

Suppose that, instead, Frank's first sample yielded a bad part. Now revise the probability that the machine is correctly set up.

$$P(C|B) = \frac{P(C \cap B)}{P(C \cap B) \cup P(I \cap B)}$$
$$= \frac{.8 \times .1}{(.8 \times .1) + (.2 \times .6)}$$
$$= \frac{.08}{.20} = .40$$

Again we can see that there is a substantial change from the prior probability.

THE USE OF BAYES' RULE IN MEDICAL DIAGNOSIS

Jack Turner has just arrived for his appointment with Dr. Woodman. Jack's symptoms are such that Dr. Woodman suspects there is a 1 percent chance that he has a certain rare disease (R). He suspects that there is a 4 percent chance that some other milder disease (M) is causing these symptoms. And finally he suspects that there is a 95 percent chance that nothing is wrong with him (H). These, then, are Dr. Woodman's prior beliefs.

Probability of rare disease:	$P(R) = .01$
Probability of milder disease:	$P(M) = .04$
Probability of healthy condition:	$P(H) = .95$

They are entered on the probability tree of Figure 5-6.

Dr. Woodman is hesitant to base his diagnosis and therapy on these prior beliefs. Instead, he decides to administer a test to his patient and then revise his prior probability as to the likelihood of this rare disease.

The test, however, is not a perfect one. Like many other medical tests, it can register positive (the patient does have the disease) when in fact the patient does *not* have the disease. The reliability of the test depends upon the actual condition of the patient. If the patient actually has the rare disease, the probability that the test will register positive is .90. If the patient has other milder diseases, the likelihood that the test will register positive is .60. And if the patient is actually healthy, the likelihood that the test will register positive is .20. Summarizing these conditional probabilities, we have the following:

$$P(\text{pos}|R) = .90$$
$$P(\text{pos}|M) = .60$$
$$P(\text{pos}|H) = .20$$

They are entered in Figure 5-6.

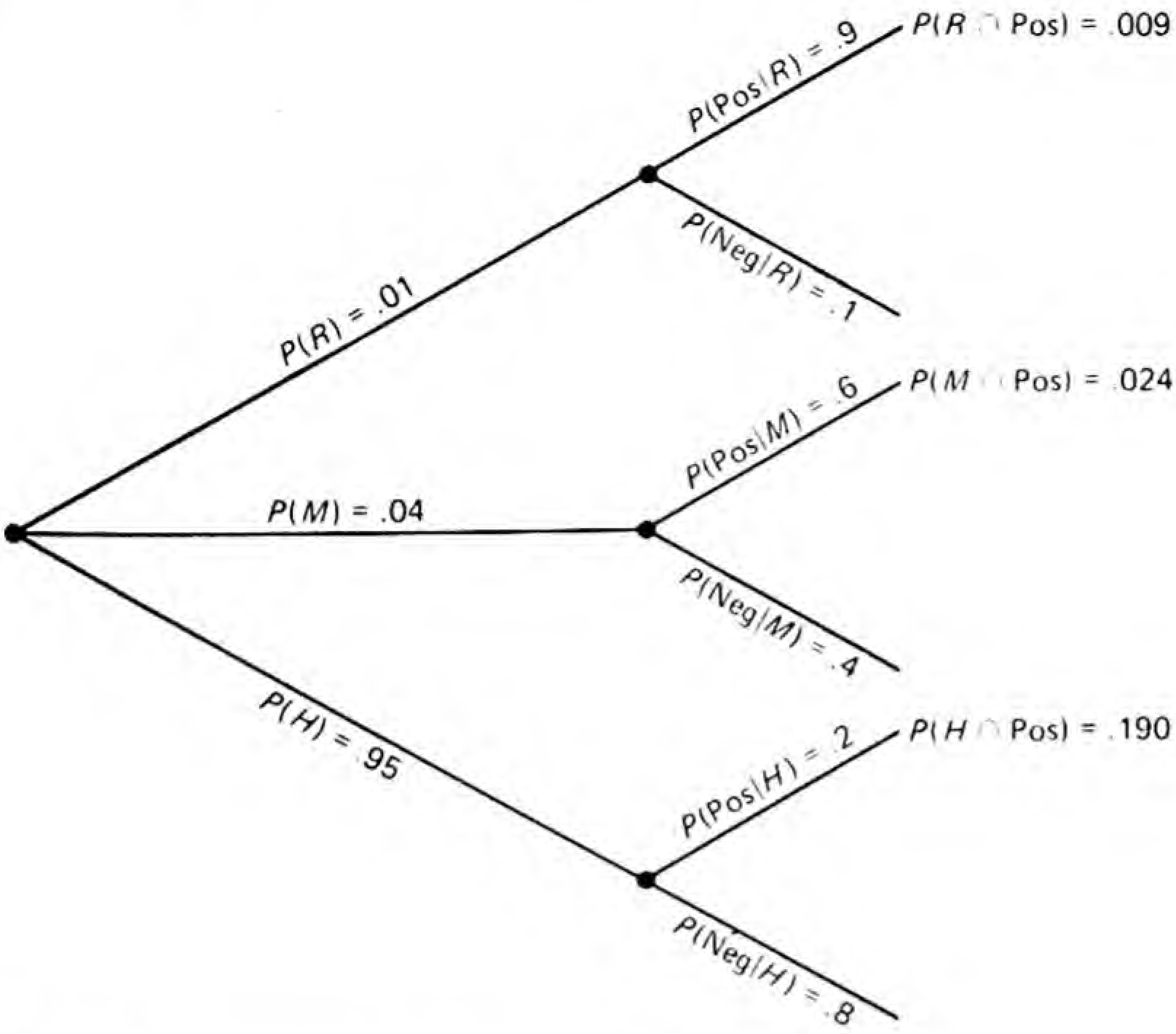


Figure 5-6 Medical diagnosis problem.

After having the test administered, Dr. Woodman is informed that the results were positive. His decision problem is this: Given that the test results were positive, what is the revised probability that the patient has the disease?

$$P(R|\text{pos}) = ?$$

From our tree we see that the following paths will result in a positive result: rare disease and positive result, milder disease and positive result, and healthy and positive result.

Of these three ways in which a positive result can be generated we are especially interested in the likelihood that it was the path in the probability tree represented by "rare disease and positive result." The likelihood of this can be determined in the following way:

$$P(R|\text{pos}) = \frac{P(R \cap \text{pos})}{P(R \cap \text{pos}) \cup P(M \cap \text{pos}) \cup P(H \cap \text{pos})}$$

This can also be simplified to

$$P(R|\text{pos}) = \frac{P(R \cap \text{pos})}{P(\text{pos})}$$

which is, again, the use of Bayes' rule to revise prior probabilities.

Finally, making the computations, we find

$$P(R|\text{pos}) = \frac{.01 \times .9}{(.01 \times .9) + (.04 \times .6) + (.95 \times .2)}$$
$$= \frac{.009}{.009 + .024 + .190} = \frac{.009}{.223} = .04$$

and we can conclude that, based on the outcome of the test, the likelihood that the patient has the rare disease is .04. Indeed this is an increase compared to the prior estimate of .01, but it may not be a high enough probability for Dr. Woodman to make a conclusive decision. Perhaps the same test could be run again with .04 as the new prior probability.

SUMMARY

In some decision-making situations the decision maker may hold prior probabilities that are not strong enough to support a decision. Consequently it might be advantageous to collect some additional information for the purpose of revising these prior probabilities. The purpose of this chapter has been to model these situations and to explore the use of Bayes' rule in analyzing these models.

QUESTIONS

1 Differentiate between a prior and posterior probability.
2 Prior probabilities which are subjective estimates are not as effective as prior probabilities which are objective estimates. Comment on this statement.
3 In the classical approach to statistical sampling absolutely no weight is given to prior probability estimates. Only the information contained in the sample is considered relevant. But in Bayesian analysis prior information is combined with sample information. Both sources of information are preserved in the result. Which method more closely parallels the informal process which many decision makers already use? Which one would you recommend? Why?
4 Do you think that prior probabilities should influence the quality control decision in the Howard Company case, or should the decision be based strictly on the outcome of the sample part. Why?
5 Do you think that the conclusions reached by some physicians are influenced by prior probabilities?

PROBLEMS

5-1 In a game of chance a die is thrown and then, on the basis of the outcome of the die, a marble is drawn from one of two urns.

If a 3 is thrown on the die, then urn 1 is chosen; otherwise urn 2 is chosen. There are 8 red and 2 green marbles in urn 1, and there are 5 red and 5 green in urn 2.

Suppose you have just been told that a player had drawn a red marble. Revise the probability that urn 1 was chosen.

5-2 In problem 5-1, identify how the following concepts are used in the development and solution of the problem.

- **a** Prior probabilities
- **b** Conditional probabilities
- **c** Joint probabilities
- **d** Law of multiplication
- **e** Law of addition
- **f** Mutually exclusive
- **g** Revised prior or posterior probabilities
- **h** Bayes' rule

5-3 From past experience it has been determined that the likelihood that a machine is properly set up is 75 percent. If it is properly set up, the likelihood of a defect is 5 percent. If, on the other hand, it is improperly set up, the likelihood of a defect is 25 percent.

Given that the machine has just produced a defect, revise the prior belief that the machine is correctly set up.

5-4 According to quality control records, a particular machine has been properly set up 80 percent of the time.

When the machine is properly set up, 90 percent of the parts produced are acceptable. When the machine is improperly set up, 60 percent of the parts produced are acceptable.

Given that a bad part has just been observed, what is the likelihood that the machine is indeed properly set up?

According to your result, what would you do?

5-5 The Bridgton Police Department is currently investigating a robbery which took place last week at the Trading Post. They have three suspects and are about to decide whether one or all of them should be called in for questioning.

The police chief, Frank Bell, thinks the likelihood that the crime was committed by suspect 1 is 25 percent; the likelihood that it was committed by suspect 2 is 40 percent; and the likelihood that it was committed by suspect 3 is 35 percent.

Just as he is about to make his decision, Sergeant Henson uncovers some additional evidence. He finds that the weapon used in the robbery was a sawed-off shotgun.

Chief Bell feels that if suspect 1 had committed the crime, the chance that he would have used a sawed-off shotgun is but 5 percent; if suspect 2 had committed the crime, the likelihood that he used such a weapon is 50 percent; and if suspect 3 had committed the crime, the likelihood that he used a sawed-off shotgun is 70 percent.

Given this new piece of information, determine the likelihood that it was suspect 1, 2, or 3.

Do you suppose that police departments use this process (formally or informally) to reach decisions?

5-6 Records indicate that a machine is properly set up 90 percent of the time. If indeed it is properly set up, the likelihood of good parts being produced is 80 percent. If, on the other hand, the machine is improperly set up, the likelihood of a good part is only 40 percent.

A sample of a single part has been taken from the machine and found to be

bad. In the light of this new information, revise the likelihood that the machine is properly set up.

This revised probability is really not strong enough in one direction or the other to lead to a decision as to whether or not the machine has been properly set up. Consequently, another sample is taken and this sample turns out to be good. What can now be said about the likelihood of a properly set up machine? Should another sample be taken?

5-7 Suppose that the revised probability of the medical test described in the chapter was not high enough for Dr. Woodman to reach any conclusive decision. The test was repeated and again it was positive. What is the newly revised posterior probability that the patient has the rare disease?

5-8 The Kirkland Company is considering the introduction of a new product in its consumer-goods line. The marketing department thinks that there is a 60 percent chance that the product will turn out to be a good product in the line and that there is a 40 percent chance that it will be an average product.

To provide more information, the marketing department plans a market test. The product is to be introduced in five major markets.

Given that the product is a good one, marketing feels that there is a 90 percent chance that the sales in this test market will be high. If, on the other hand, the product is an average one, the department feels there is but a 40 percent chance that sales will be high in this test market.

Suppose that low sales are recorded during the test period. What is the probability that the product is a good one?

5-9 The owner of the Lenox Electronics Company feels that there is a 70 percent chance that he will win the bidding on a certain government contract.

Recently he attended a meeting with all those who had submitted bids. At this meeting government representatives discussed the contract and explained the criteria which would be used to guide their final selection. After this meeting the owner of Lenox was reevaluating his chances of winning the contract.

Given that he would eventually win the contract, he feels that there was a 90 percent chance that the government representatives would have responded positively to him at the meeting. Given that he would not eventually win the contract, there was a 10 percent chance that the government representative would have responded positively.

Reflecting back on the meeting, he feels that the representatives did indeed respond positively to him.

What is his chance of winning the contract?

5-10 Mr. Frank Glober was hired by the Lenox Company over 4 weeks ago. At his employment interview it was determined that there was an 80 percent chance that he would turn out to be a good worker.

Recently his performance has been reevaluated. If he really is a good worker, there is an 85 percent chance that this review would show that his actual output is satisfactory. If he is really an unacceptable worker, there is a 30 percent chance that this review would show that his actual output is satisfactory.

Suppose that his review showed his output to be unsatisfactory. What is the likelihood that he is an unacceptable employee?

When the likelihood that a worker is unacceptable is above 55 percent, the worker is fired. What action should be taken in Mr. Glober's case?

CASE STUDY: First National Bank

The First National Bank is a large commercial bank located in Baltimore, Maryland. The services which it offers include demand deposits (checking accounts), savings accounts, trust accounts, loans, and credit cards.

Each of these services is managed by a separate department within the bank, and each of these departments has its own set of operating procedures and controls. The purpose of these procedures and controls is to ensure that bank personnel carry out their responsibilities correctly, completely, and honestly. Commercial loan procedures, for example, require that the following steps be taken when a loan is granted. First, the credit rating of the debtor must be established; second, the interest rate must be determined; third, the collateral required for the loan must be specified; fourth, the authorized officers from the borrowing corporation must sign the loan; and fifth, the authorized loan officers of the bank must approve and sign the loan agreement.

After the loan has been granted, the commercial loan department must continue to monitor the loan. Have payments been made regularly? Is the market value of the collateral sufficient to cover the balance of the loan? Are the bank records up-to-date, accurate, and complete? If the answer to any of these questions is no, the loan department must initiate whatever corrective action is warranted by the situation.

But the operations of a department are not exclusively under its own control. Federal banking laws require that commercial banks maintain an independent internal audit staff for the purpose of examining the controls and procedures used by each operating department. These periodic examinations must conclude that the department either is adhering to these controls and is therefore "in control" or is not adhering to these controls and is therefore "out of control." During these examinations the internal auditor must also be on the lookout for any fraudulent activity within the department.

At the present time Tom Martorano and Diane Richmond have just started to audit the commercial loan department. Tom has performed this audit four times in the last year, but Diane has been on other assignments. From his experience with the department, Tom feels that there is a 90 percent chance that it is adhering to its current procedures and controls. Diane, however, is more skeptical.

The audit was started by selecting a commercial loan and examining its documentation and collateral. Tom feels that if the department was "in control," the probability is just 2 percent that any error will be uncovered during the examination. But if the department was *not* "in control," the probability of uncovering an error increases to 50 percent.

The loan they examined was made 1 year ago to the Acme Corporation. Their examination showed that the credit rating of the debtor was properly established, that the interest rate was sufficient, and that the authorized signatures of the bank's loan officers were obtained. But they found that one

of the required signatures from Acme Corporation was missing. According to a corporate resolution, which First National Bank retained in its files, both the vice president of finance and the treasurer of the Acme Corporation were required to sign every loan agreement. The present loan was signed only by the vice president of finance.

Both Tom and Diane agreed that additional loans must be examined. They selected a second loan and found everything in order. A third loan was examined and again everything was in order.

Tom said, "I am satisfied that the loan department is adhering to its procedures and controls. Before we started the audit I was very confident that it was. Even though we did find one error, the result of the next two samples restored my confidence. I think we should give the department our approval and move to the next audit."

"I don't agree," replied Diane. "We found one error in three accounts. This could mean that one-third of the accounts in its files are in error. I think that something could be wrong with its controls and that we ought to continue the audit by examining additional records."

QUESTIONS

1 Should the prior knowledge which Tom Martorano has influence his decision?
2 If Tom's prior knowledge is used, what is the revised probability that the loan department is "in control" after the first sample has been taken?
3 What is the revised probability that the loan department is "in control" after the second sample has been taken.
4 What is the revised probability that the loan department is "in control" after the third sample has been taken?
5 What conclusion would you reach after the third sample?
6 Is Diane Richmond's position justifiable?
7 In addition to the three loans which have already been audited, how many more loans with no errors would it take before the revised probability that the department is "in control" reaches 90 percent?

APPENDIX A: Revision of Binomial Prior Probabilities

When bayesian statistics are used in statistical sampling, the conditional probabilities are sometimes obtained from the binomial distribution. The following example illustrates this situation.

A sampling plan for a specific manufacturing process requires that a sample of 20 items be taken from a machine. The results of the sample are then used to determine whether the process is properly in control.

In the past the machine has been properly set up 85 percent of the time. When it is properly set up, 90 percent of the parts produced are acceptable and 10 percent are unacceptable. If it is improperly set up, 50 percent of the parts are acceptable and 50 percent are unacceptable.

Today 20 items have been sampled from the machine, and 15 were found to be good. The task at hand is to revise the prior probability that the machine is correctly set up in the light of this new information. In terms of Bayes' rule we would like to determine the probability that the machine is correctly set up (C), given our sample result of 15 good ones out of 20 sampled.

$$P(C|15 \text{ good ones out of } 20) = ?$$

First, the probability tree must be drawn. This is done in Figure 5A-1. The only problem in completing the tree occurs when the conditional probabilities must be

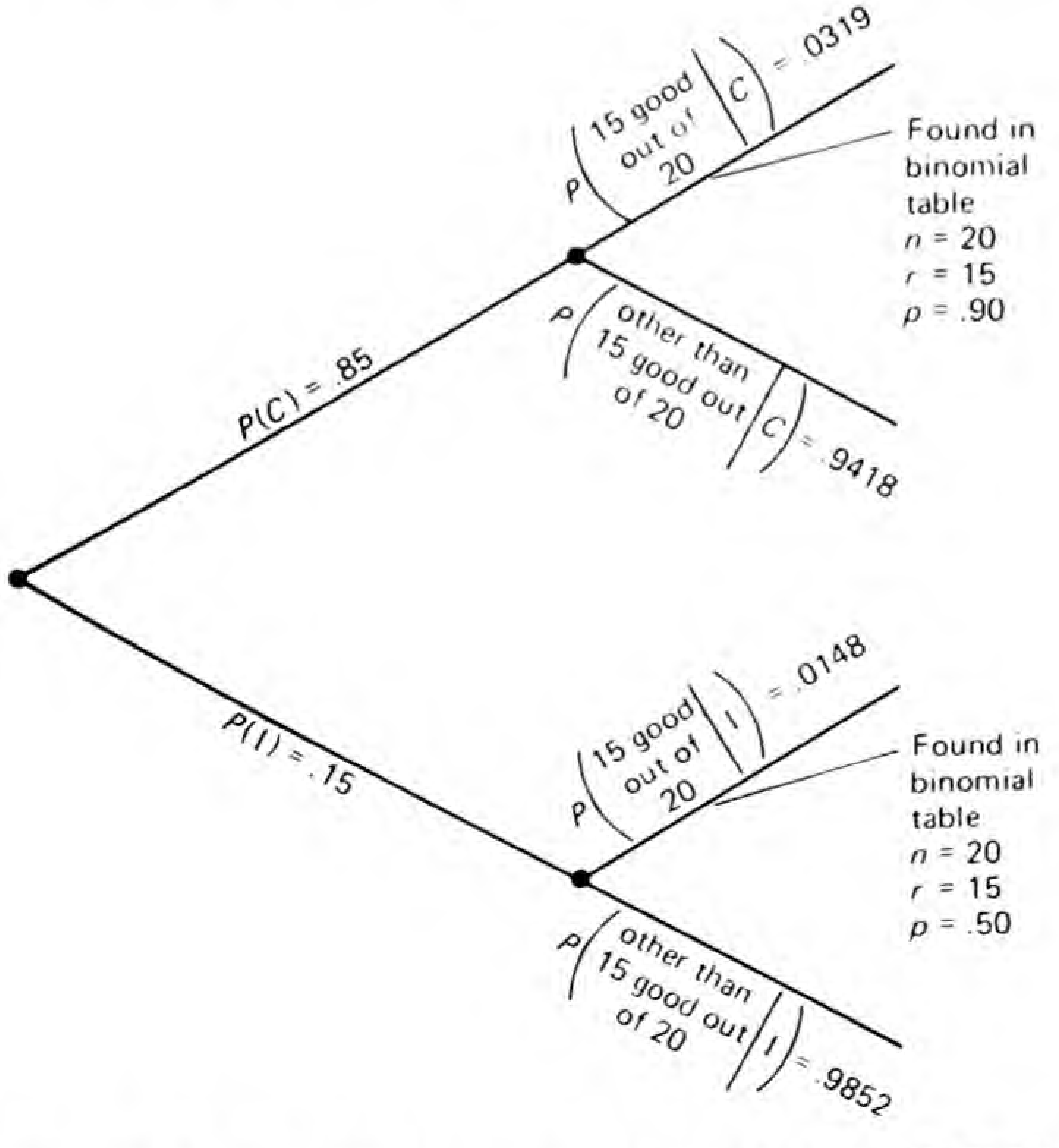


Figure 5A-1 Using the binomial to compute conditional probabilities.

entered. Examples in the early part of the chapter dealt with one good or one bad part. But in this example we have 15 good pieces out of 20 sampled. How do we find the probability of 15 good pieces out of 20 sampled when we know that the probability of one good one (given that the machine is correctly set up) is .90? The answer is to use the binomial probability table. From it we find that when

$$n = 20$$
$$r = 15$$
$$P = .90$$

the conditional probability is

$$P(15 \text{ good out of } 20 | C) = .0319$$

and we can conclude that the likelihood of observing 15 good pieces out of 20 when the machine is properly set up is .0319.

Correspondingly, we can find the likelihood of observing 15 good out of 20 when the machine is improperly set up.

$$n = 20$$
$$r = 15$$
$$P = .50$$
$$P(15 \text{ good out of } 20 | I) = .0148$$

We can now return to the bayesian question: Given that 15 good ones were observed from a sample of 20, what is the revised probability that the machine is correctly set up?

$$\begin{aligned} P(C | 15 \text{ good out of } 20) &= \frac{P(C \cap 15 \text{ good out of } 20)}{P\left(C \cap \begin{matrix}15 \text{ good} \\ \text{out of } 20\end{matrix}\right) \cup P\left(I \cap \begin{matrix}15 \text{ good} \\ \text{out of } 20\end{matrix}\right)} \\ &= \frac{.85 \times .0319}{(.85 \times .0319) + (.15 \times .0148)} \\ &= \frac{.027}{.027 + .002} = .93 \end{aligned}$$

PROBLEMS

5A-1 Find the following binomial probabilities from the table at the back of the book:
- **a** $P = .3, r = 5, n = 10$
- **b** $P = .8, r = 20, n = 20$
- **c** $P = .1, r = 10, n = 100$
- **d** $P = .9, r = 90, n = 100$

5A-2 The likelihood that a machine is correctly set up is 80 percent. If it is correctly set up, 90 percent of the parts produced will be acceptable. If it is incorrectly set up, 40 percent of the parts will be acceptable.

A sample of 20 items has just been taken from the machine, and 15 were

found to be acceptable. Revise the likelihood that the machine is correctly set up.

5A-3 The likelihood that a machine is correctly set up is 60 percent. If it is correctly set up, 95 percent of the parts produced will be acceptable. If it is incorrectly set up, 40 percent of the parts will be acceptable.

A sample of 50 items has just been taken from the machine, and 40 of them were found to be acceptable. Revise the likelihood that this machine is properly set up.

5A-4 The Craftworks Company will soon bring out a new product. It feels that the product has an 85 percent chance of being successful.

To confirm its optimism, the company plans to conduct a market survey. It will ask 100 consumers chosen at random whether they intend to purchase this product. If the product is destined to be a success, the company would expect 40 percent of those interviewed to respond that they intend to purchase the product. If, on the other hand, the product is destined to fail, 25 percent of those interviewed would intend to purchase the product.

Suppose that the result of the survey is that 35 people expressed the desire to purchase the product. What can now be said about the success of the product?

APPENDIX B: Revision of Normal Prior Probabilities

INTRODUCTION

In the following section we will consider the development of a prior normal probability distribution and then its revision based on sample information.

DEVELOPMENT OF A NORMAL PRIOR PROBABILITY DISTRIBUTION

A new industrial product which is currently in the preproduction stage will soon be scheduled for full-scale production and assembly. But before this can be done, it will be necessary to establish assembly standards. These standards will be used to determine the number of workers required, to develop a piece-rate incentive system, and to develop a production schedule.

Management, however, is unsure of the average assembly time necessary to complete the job, but it feels that it can subjectively estimate a prior probability distribution for this average. In addition to this subjective estimate it also can sample the actual assembly times of those workers assembling the preproduction models, and then revise its prior probability distribution on the basis of this sample information.

It will be shown later in the appendix that this revised estimate will be better than either the prior estimate or the sample results used separately.

How can the prior distribution be established? One way is to ask the following question. Between what values (assembly times) would you expect the *average* assembly time to fall 50 percent of the time? Suppose the answer is 13 and 17 minutes.

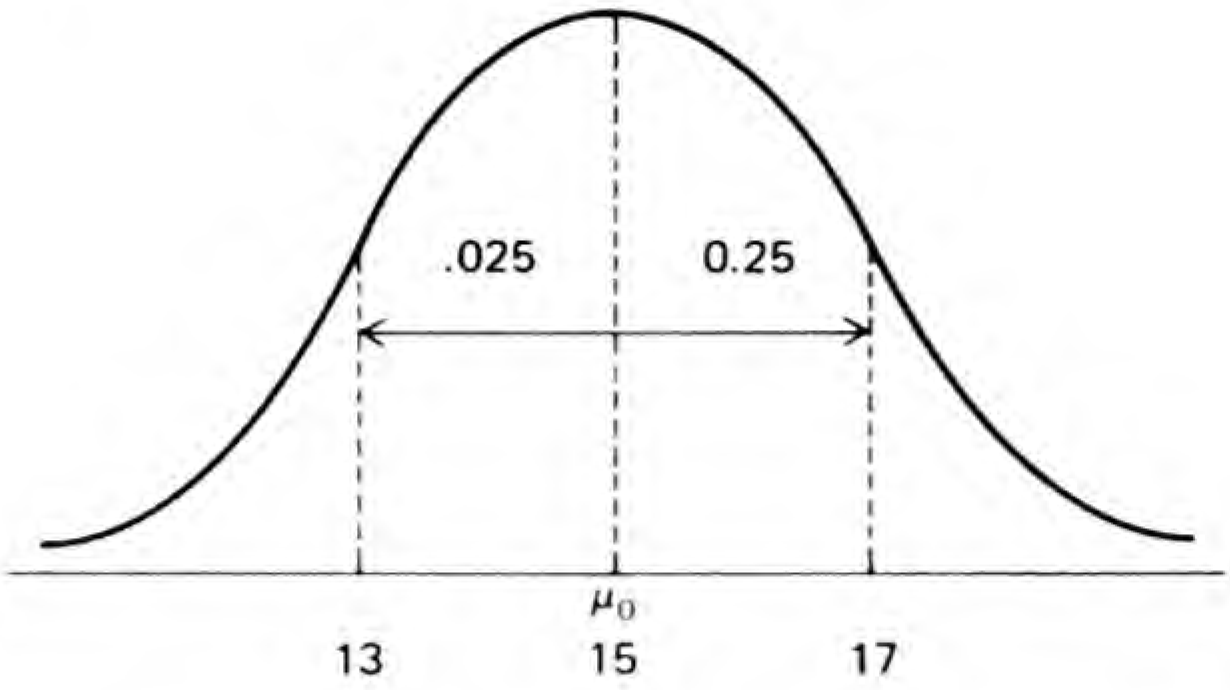


Figure 5B-1 Prior probability distribution for *average* assembly times.

A graphic description of this response is shown in Figure 5B-1. A very reasonable assumption to make is that this distribution of averages is normal.

The prior mean (μ_0) of this distribution of sample means is therefore 15 minutes. Reference to Figure 5B-1 and Table 1 at the end of the book, finds that 25 percent of the area is included between the mean and .67 standard deviations from the mean.

$$.67 \text{ STD} = 2 \text{ minutes}$$

Therefore

$$1 \text{ STD} = \frac{2}{.67} = 3 \text{ minutes}$$

We can conclude that the standard deviation (σ_0) of the distribution is 3 minutes.

To summarize, the prior distribution of assembly times has the following characteristics:

$$\mu_0 = 15$$
$$\sigma_0 = 3$$

where μ_0 represents the mean of the prior and σ_0 represents the standard deviation of the prior.

REVISION OF A NORMAL PRIOR DISTRIBUTION

In the previous section it was estimated that a certain prior probability distribution has the following characteristics:

$$\mu_0 = 15$$
$$\sigma_0 = 3$$

Suppose that a sample of 16 observations is taken and the following sample results are obtained:

$$n = 16$$
$$\bar{x} = 13$$
$$S = 5$$

We must now determine how to revise the prior results in the light of this new sample information.

Reliability

We define the information content or reliability of a distribution as the reciprocal of its variance. Therefore, the reliability of the prior distribution of means R_0 is computed in the following way:

$$R_0 = \frac{1}{(\sigma_0)^2} = \frac{1}{(3)^2} = .111$$

The reliability of the sampling distribution of the means from which we have observed a single sample can be computed in the following way:[1]

$$R_x = \frac{1}{S^2/n} = \frac{1}{(5)^2/16} = .64$$

In the reliability of both the prior and sampling distributions, the reliability increases as the variance decreases. Correspondingly, the reliability decreases as the variance increases.

Revision of the Mean

The mean is revised by taking a weighted average of the prior and sample mean. These means are weighted by their relative reliability.

$$\mu_1 = \left(\frac{R_0}{R_0 + R_x}\right)\mu_0 + \left(\frac{R_x}{R_0 + R_x}\right)\bar{x}$$

where μ_1 is the revised mean. For our example we have the following:

$$\begin{aligned}\mu_1 &= \left(\frac{.111}{.111 + .64}\right)15 + \left(\frac{.64}{.111 + .64}\right)13 \\ &= 2.217 \qquad\qquad + 11.078 \\ &= 13.295\end{aligned}$$

The revised mean is 13.295. Clearly, the informational content or reliability of the sample was so large in comparison to the reliability of the prior distribution that the sample information was weighted very heavily and the revised mean is quite close to the sample mean.

[1] The relationship between the variance of a sample and the variance of the sampling distribution from which it came is the following:

$$\sigma_{\bar{x}}^2 = \frac{S^2}{n}$$

Revision of the Variance

The revised reliability, R_1, can be computed in the following way.

$$R_1 = R_0 + R_x$$

This is equivalent to

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{1}{S^2/n}$$

where σ_1^2 is the revised variance. In our example we have the following:

$$\frac{1}{\sigma_1^2} = \frac{1}{(3)^2} + \frac{1}{(5)^2/16}$$

$$\frac{1}{\sigma_1^2} = .111 + .64$$

$$\frac{1}{\sigma_1^2} = .751$$

$$\sigma_1^2 = 1.33$$

$$\sigma_1 = 1.153$$

We can conclude that the standard deviation of the revised distribution is 1.153. Notice that this revised estimate of the standard deviation is much lower than if we had used the prior or sample information separately.

THE RESULTS AND THEIR USE

The revised mean and standard deviation are summarized below:

$$\mu_1 = 13.295$$

$$\sigma_1 = 1.153$$

Suppose the production standard was to be set at the mean plus 2 standard deviations.

$$\mu_1 + 2\sigma_1$$

$$13.295 + 2(1.153) = 15.601$$

The production standard is therefore 15.601 minutes. There would be a 2.28 percent chance (Table 1 at the back of the book) that the average was actually higher.

PROBLEMS

5B-1 A normal prior probability distribution of means has a mean of 15 and a standard deviation of 2. A sample of size 9 is taken. The sample mean is 20 and the sample standard deviation is 5. What are the mean and standard deviation of the revised distribution of means?

5B-2 A normal prior probability distribution of means has a mean of 50 and a standard deviation of 5. A sample of size 16 is taken. The sample mean is 35 and the sample standard deviation is 10. What are the mean and standard deviation of the revised distribution of means?

5B-3 The Anderson Company manufactures and distributes a line of wrought-iron kitchen furniture. It decided to broaden its line and offer wrought-iron patio furniture as well.

The company felt that an average of 20 sets (table and 4 chairs) of patio furniture could be sold per month to each of its retail outlets. There was a fifty-fifty chance that the average number of sets sold would be between 15 and 25.

Before it began production, however, the company decided to take a sample of 25 retail outlets. The result of the sample was that the retail stores felt they could sell an average of 15 per month. The standard deviation of the sample was 9.

Suppose that the Anderson Company would like to have at least a 90 percent chance of meeting the average demand; how many patio sets should they schedule each month over the next few months?

Chapter 6

Decision Trees

INTRODUCTION

In Chapter 3 we explored a method for analyzing certain decision situations under risk. In those situations the decision maker was confronted with a choice between several alternative strategies. Each of these alternatives led to one of several possible outcomes, and the outcome which finally occurred depended upon the prevailing environmental state.

Those decision situations consisted of but *one* set of alternatives and one set of environmental states. Both sets were depicted on a conditional payoff table.

But not all decisions under risk involve a choice between a single set of alternatives. In this chapter we will examine those situations where the decision maker is confronted with a *series* of interdependent sets of alternatives—often separated by time—between which outcomes beyond the control of the decision maker can occur. Rather than a payoff table, these decision problems must be depicted on a decision tree.

To illustrate, analyze, and solve these more complex decision problems, two case studies will be used. In both of these cases the following sequence of steps will be followed.

1 Uncover the structure of the problem by developing a decision tree
2 Determine the probabilities associated with each outcome

3 Determine the payoffs associated with the outcomes

4 Use these data to determine which of these alternative strategies will result in the highest expected payoff

CASE STUDY: Scott Company

The Scott Company of Des Moines, Iowa, produces and distributes a line of cosmetics which includes lipstick, nail polish, eye shadow, and hair spray. Scott's hair spray product was originally a leader in the field. At its peak it accounted for 20 percent of all hair sprays sold. Recently, however, it has been losing ground. Its market share is now down to 7 percent.

In response to this decline in market share the manager of marketing, David Conway, arranged to have a consumer survey taken at several large department stores throughout the country. The analysis of the survey suggested that Scott's hair spray was inferior to the brands being sold by competitors.

David Conway felt that as a result of this survey he had to choose between three alternatives: first, to engage in a research and development project whose goal it would be to improve the product; second, to discontinue the hair spray altogether; or third, to continue without any changes.

If the outcome of the research and development effort was positive, he would choose to market the new product. But if the result of this R and D effort was negative, the choice would be between discontinuing the old product or continuing to market the old product.

If the result of the R and D effort was positive and the new product was introduced to the market, sales of the new product would most likely be high rather than low. But if the result of the R and D effort was negative and the old product was continued, sales would probably continue at a low level.

David realized that he faced a series of interdependent sets of alternatives which were separated by outcomes beyond his control.

He was unsure of the best decision.

Structuring the Case

David Conway faces a complex decision problem. In the next several pages this problem will be carefully structured piece by piece. First the relationships between each set of alternatives and outcomes will be clarified and these relationships illustrated on a decision tree. Then some additional information on probabilities and payoffs will be given. Finally all of this information will be used to analyze the model and reach a decision.

The First Set of Interdependent Decision Alternatives

David Conway feels that he must choose between one of three action alternatives. The first is to engage in an R and D program to improve the

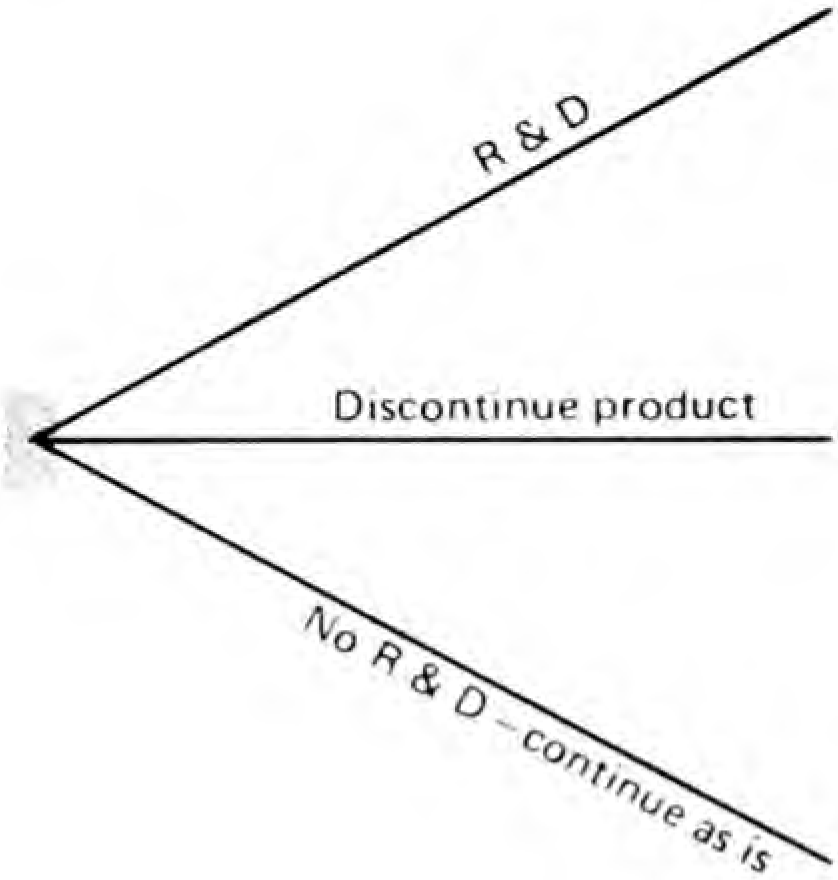


Figure 6-1 First set of interdependent decisions, "action alternatives."

product; the second is to continue as is with the present product; and the third is to discontinue the hair spray altogether.

Action alternatives are defined as the first set of alternatives between which a decision maker must choose. They may be followed in time by other alternatives, but nonetheless they are the first. In fact, the most important result of the completed analysis is to recommend which of these initial alternatives to take. For this reason they are called action alternatives.

These alternatives can be depicted on a decision tree as shown in Figure 6-1. The rectangle preceding the branches in this tree implies a decision point, and each branch represents the alternatives from which the decision maker may choose.

The First Set of Outcomes

Given That the R and D Alternative Is Chosen If David chooses to embark on an R and D program, one of two outcomes[1] may occur; the results of the R and D program may be either positive or negative.

We say that these are outcomes because they are dependent upon the occurrence of an environmental state which is beyond the control of the decision maker. In this case they are dependent upon the result of several chemical experiments.

To depict these outcomes a circle is drawn in Figure 6-2, followed by two branches representing each of the possible outcomes.

Given That Management Continues as Is If David chooses to continue as is, a different set of outcomes is possible. In Figure 6-3 we see that if this alternative is chosen and the company continues without undertaking any product changes, future sales for this product may be either high or low.

Sometimes called "events" in the literature.

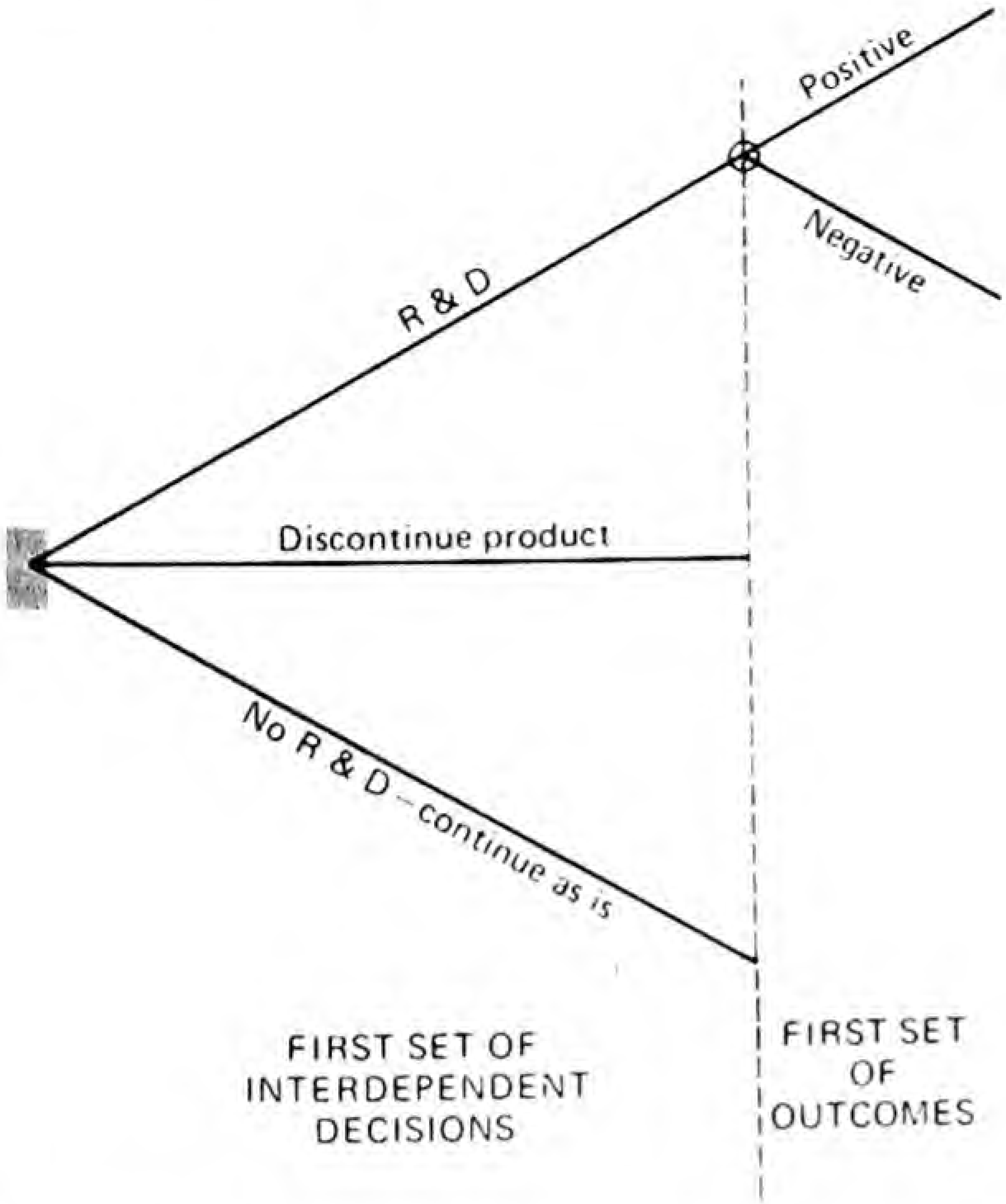


Figure 6-2 First set of outcomes given that the R and D program is undertaken.

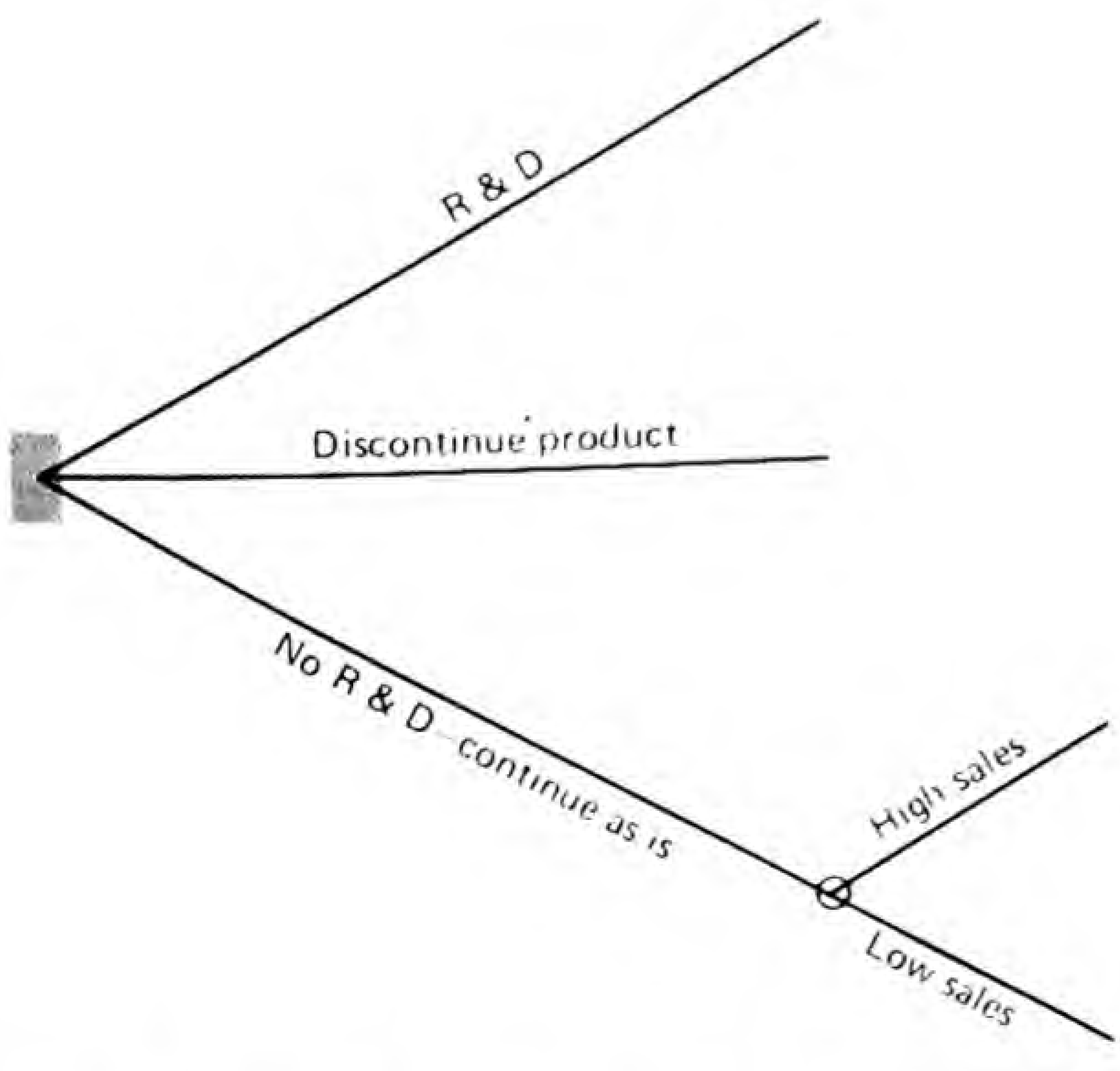


Figure 6-3 First set of outcomes given that management continues as is.

The Second Set of Interdependent Decision Alternatives

Given That the Results of the R and D Program Are Positive If the results of the R and D program are positive, David has two alternative decisions open to him. First, he can elect to market the new product (a choice we might reasonably expect him to take), or second, he can elect not to market the product (a choice we would *not* expect him to take, but we still include it on our decision tree in the interest of being complete). These alternatives are shown in Figure 6-4.

Given That the Results of the R and D Program Are Negative If the results of the R and D program are negative, David will have two alternatives open to him. He can choose to either market or discontinue the old product. Again these alternatives are added to the decision tree in Figure 6-4.

Second Set of Outcomes

Given That the New Product Is Marketed If David markets the new product, the sales that result could be either high or low. Again these

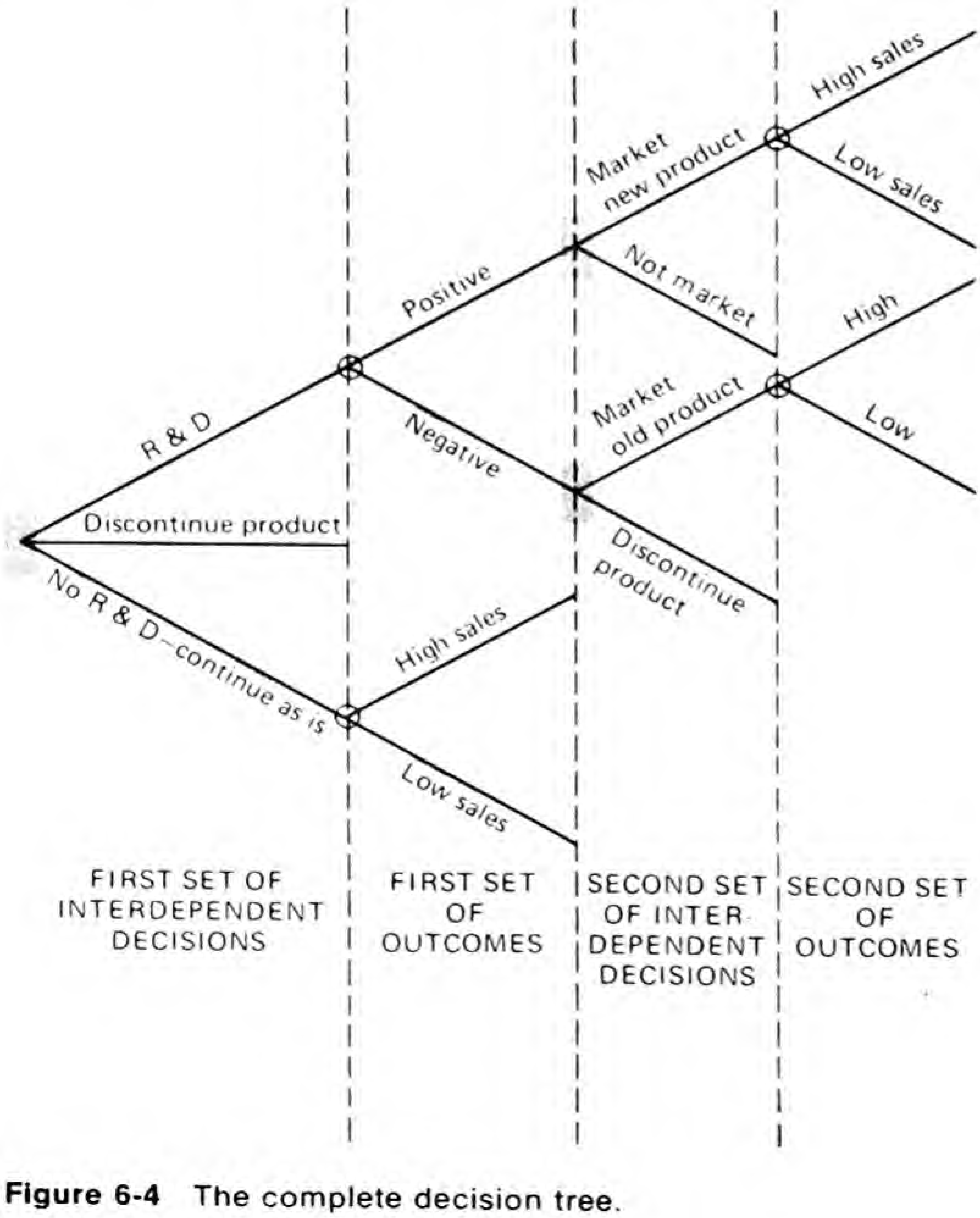


Figure 6-4 The complete decision tree.

outcomes are considered to be beyond the control of the decision maker. They are shown in Figure 6-4.

Given That the Old Product Is Continued If the R and D results are negative and David decides to continue with the old product, the sales level for this product could be either high or low. These outcomes are also shown in Figure 6-4.

The Decision Complete

The first step has been completed. The decision problem has now been expressed as an interdependent series of alternatives and outcomes. This is an important step, for it dissects a complex problem into its basic elements and illustrates the interrelationship between these elements on a decision tree. Without such a tree the relations between these alternatives and outcomes might remain vague in the minds of the decision makers, and discussion of the decision problem could lack any clarity.

The second step in analyzing this decision problem requires that the probability associated with each outcome be determined.

Assessing the Probabilities for Each Outcome

Outcomes are beyond the control of the decision maker. Nevertheless, the decision maker can often determine the likelihood that each outcome will occur. Only on rare occasions will the decision maker be totally ignorant of these likelihoods.

Consider the outcome fork which follows the act "R and D." We see that either a positive or a negative result can occur. To uncover the likelihood of these outcomes, David Conway asked the R and D staff a series of questions. From their answers he was able to estimate subjectively the likelihood of a positive and negative result. He concluded that the likelihood of a positive result was 80 percent and the likelihood of a negative result was 20 percent. These subjective estimates are entered in the branches of Figure 6-5.

This same process must continue for all outcomes. Somehow management must determine subjective probabilities for each of them. Granted, it may be difficult to make these explicit. Nonetheless a thorough job of decision analysis demands that this be done.

Continuing, we turn to the outcomes which follow the alternative "No R and D." David feels that if no R and D is undertaken and if the company continues as is, the likelihood of achieving high sales is 30 percent and the likelihood of achieving low sales is 70 percent. These are entered in Figure 6-5.

To turn to the outcomes which follow the alternative "market new product," David feels that a new improved product would have a 90 percent chance of achieving high sales—given that R and D results have proved positive. Correspondingly, its chances of achieving low sales would be 10 percent. Again, these figures are entered in Figure 6-5.

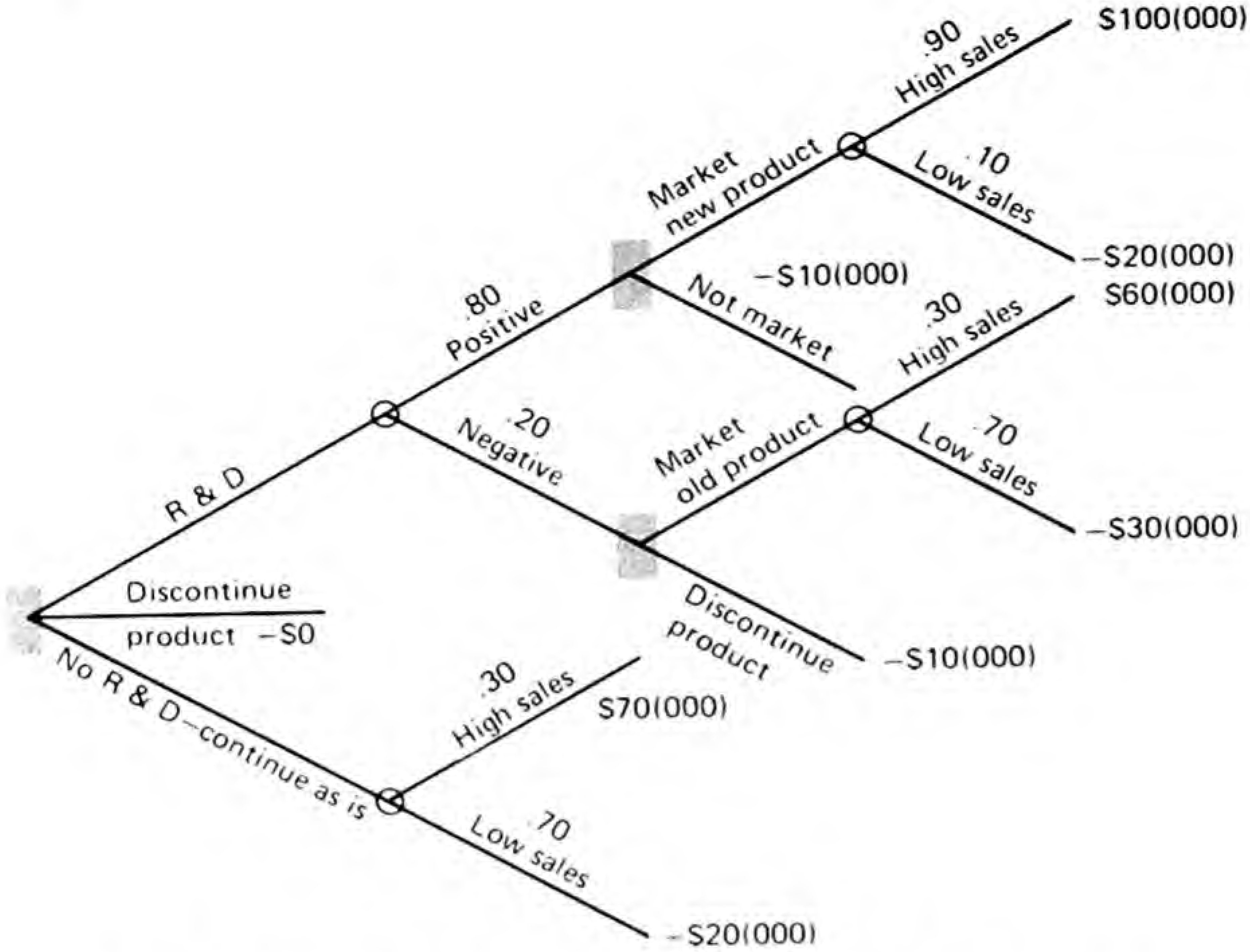


Figure 6-5 Decision tree with subjective probabilities and payoffs.

If R and D is undertaken, a negative result occurs, and the decision is to market the old product, the chance of achieving a high level of sales is 30 percent. The chance of low sales is 70 percent. These are entered in Figure 6-5.

We have now completed the second step: the determination of the likelihood of each outcome in the decision tree. Next, we will estimate the payoffs for each branch in the tree.

Estimating the Payoffs for Each Branch

Each path through the tree—each sequence of alternatives and outcomes—may lead to a different payoff for the firm. These payoffs must be estimated so that the action alternative which leads to the highest expected payoff can be identified.

The Payoffs for Our Example Through careful financial analysis it has been determined that if the new product is marketed, and a high level of sales is achieved, the dollar payoff will be \$100(000). If, on the other hand, the new product achieves only a low level of sales, the payoff will be − \$20(000). This negative payoff can be attributed to the fact that sales revenues less manufacturing costs are not sufficient to cover R and D and marketing expenses for this new product. These figures are entered in Figure 6-5.

If the R and D results are positive and the company decides not to

market the new product, the payoff is – \$10(000), which reflects the expenditure for the R and D effort.

If the old product is marketed after R and D efforts prove negative, the payoff is expected to be \$60(000) if the old product achieves high sales and – \$30(000) if low sales are achieved. If the R and D effort proves negative and the old product is discontinued, the payoff is – \$10(000).

If the action alternative chosen is to discontinue the product with no R and D effort, the payoff will be zero.

Finally, if the company does not engage in product R and D and continues as is, a \$70(000) payoff can be expected if sales of the old product are high and a – \$20(000) payoff if the sales are low.

This completes the estimate of all relevant payoffs, and in fact completes all the information that is needed before the decision tree can be analyzed.

Summarizing the Steps in the Design of a Decision-Tree Model

We have just undertaken the first three basic steps in the design of a decision-tree model. These included the following:

1 The complete description of the decision problem as a series of interrelated decision alternatives and outcomes
2 The determination of the probabilities associated with each outcome
3 The determination of the payoffs associated with each branch of the decision tree

The next step is directed at analyzing the tree in such a way as to identify the action alternative with the highest expected payoff.

The Analysis of the Decision Tree

The analysis of a decision tree begins at the end of the tree and works backward toward the action alternatives. The result of this analysis will be an expected payoff for each of these action alternatives.

Expected Payoffs For each set of outcomes at the end of the decision tree an expected payoff is computed. To turn to the outcomes which may occur if the new product is marketed, the expected value of these outcomes is

$$[.90 \times \$100(000)] + [.10 \times -\$20(000)] = \$88(000)$$

In Figure 6-6 this expected value is entered above the circle from which this set of outcomes emerges.

Choosing between Acts at This Stage By referring to Figure 6-6, it can be seen that the expected value of the alternative "market the new product" is \$88(000) and the value of the alternative "not market" is – \$10(000). At this stage in the decision sequence these two alternatives are the only ones open

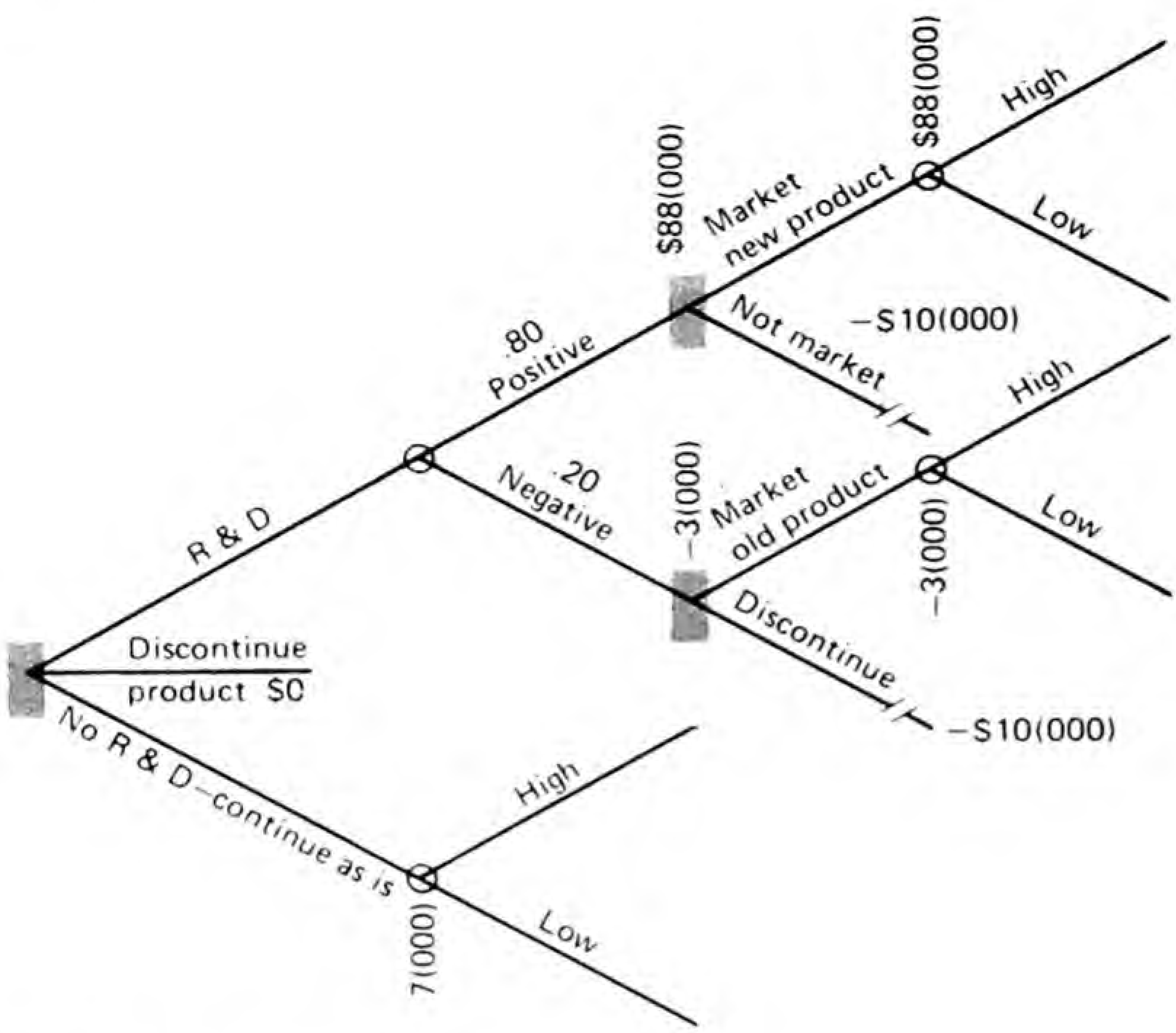


Figure 6-6 The "folding back" process.

to the decision maker, and since alternatives are under the decision makers' complete control, the one with the highest payoff would naturally be chosen. The alternative "market new product" has the highest payoff and is therefore chosen. The alternative "not market" is eliminated. It is now possible to write the $88 expected payoff above the rectangle from which the alternatives "market new product" and "not market" emerge. This expected payoff represents the average payoff that a decision maker could expect from that stage in the decision problem onward.

The Process Continues Returning again to Figure 6-5, we will now fold back the branches associated with a negative result from the R and D effort. If the old product is marketed after the negative R and D result is known, there is a 30 percent likelihood that the payoff will be $60 and a 70 percent likelihood that it will be −$30. The expected value of these outcomes is

$$(.30 \times \$60) + (.70 \times -\$30) = -\$3$$

This expected value is then entered in Figure 6-6 above the circle from which these two outcomes emerge.

Moving back from this outcome to the preceding alternative, we see that the decision maker is faced with marketing the old product with an expected payoff of −$3 or discontinuing the old product at a payoff of −$10. As the lesser of the two evils the old product would be chosen and the loss limited to $3. We

then enter the $-\$3$ above the rectangle from which these alternatives emerge. This is done in Figure 6-6.

Next, we will evaluate the outcomes which emerge from the act "no R and D—continue as is." Returning to Figure 6-5, we see that there is a 30 percent chance that the payoff will be \$70 and a 70 percent chance that the payoff will be $-\$20$. The expected payoff for these outcomes will therefore be

$$(.30 \times \$70) + (.70 \times -\$20) = \$7$$

This expected value is entered in Figure 6-6.

Now we must return to the "R and D" branch and continue folding back. From Figure 6-6 we see that there is an 80 percent chance of an \$88 payoff and a 20 percent chance of a $-\$3$ payoff. The expected value of these outcomes is

$$(.80 \times \$88) + (.20 \times -\$3) = \$69.8$$

This is entered in Figure 6-7.

We have now completed the folding-back process, and from Figure 6-7 the decision maker can compare the expected payoffs for each of the action alternatives. If the "R and D" alternative is chosen, the expected payoff will be \$69.80; if he chooses "discontinue product," the expected payoff will be \$0; and if he chooses "no R and D—continue as is," the payoff will be \$7. To maximize his payoff, he would then choose to engage in R and D.

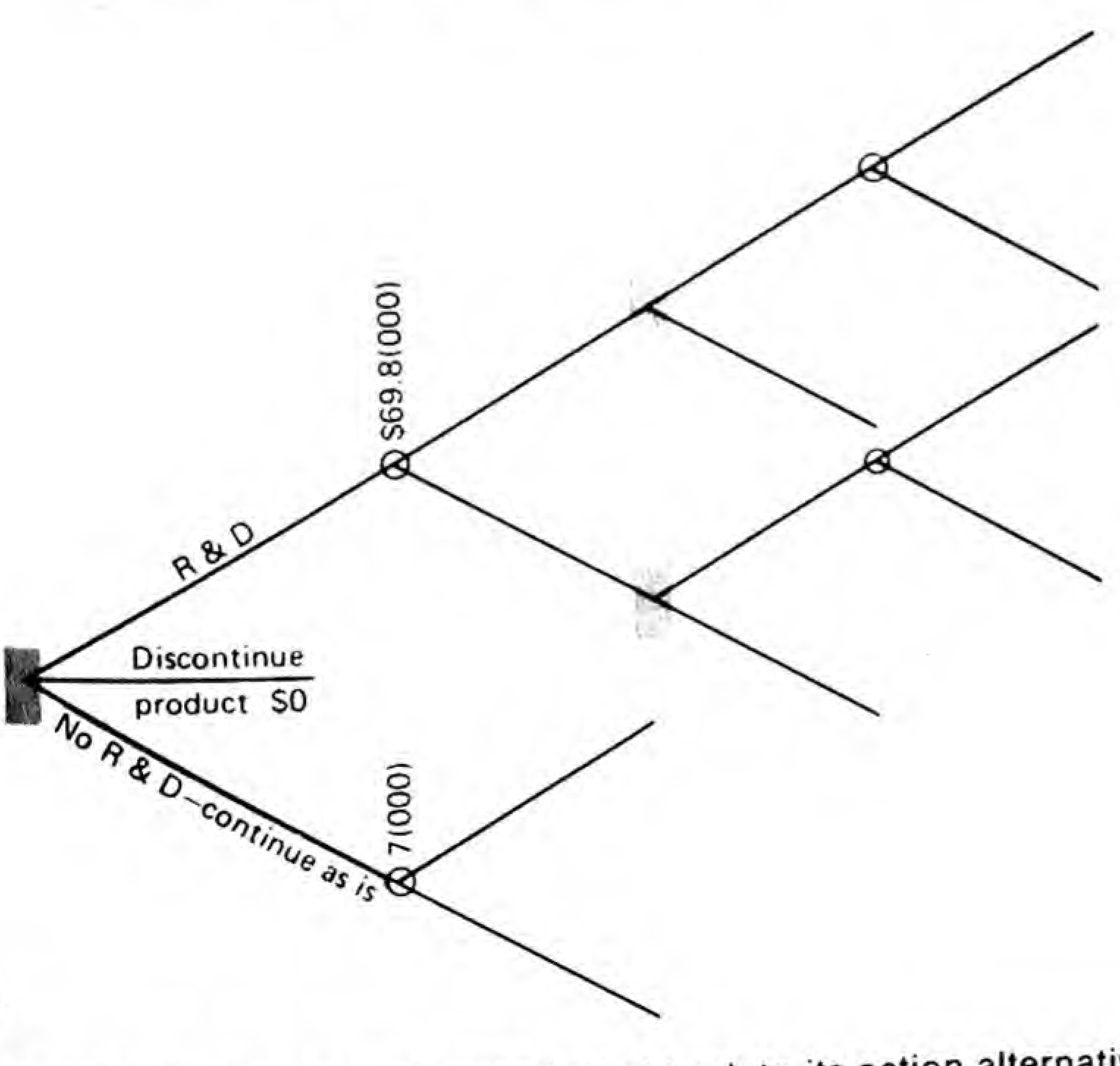


Figure 6-7 The decision tree folded back to its action alternatives.

Summarizing the Process

In analyzing the decision tree we folded back outcomes and alternatives from the end of the decision tree to the initial action alternatives. This was accomplished in the following way:

1 At each outcome fork the expected value of those outcomes was computed.
2 At each alternative fork the alternative with the highest expected payoff was selected.

In this way expected payoffs were eventually computed for each of the initial action alternatives. Finally, the action alternative with the highest expected payoff was selected as the best decision.

Payoffs and Utilities

In the Scott Company case an action alternative was measured in terms of its expected dollar payoff. Expected dollars, however, may not be the sole criterion by which these alternatives should be compared.

Consider the following two alternatives. Alternative A is an investment which has two possible outcomes. The payoff from the first outcome is \$500 and its likelihood of occurrence is 60 percent. The second possible outcome has a likelihood of 40 percent with a payoff of $-$\$100. The expected dollar payoff of alternative A is therefore \$260.

$$(.6)500 + (.4)(-100) = \$260$$

Alternative B, the second alternative, has two possible outcomes. The payoff associated with the first outcome is \$1000 and its likelihood is 50 percent. The second outcome is $-$\$480 and its likelihood is also 50 percent. The expected dollar payoff of alternative B is therefore \$260.

$$(.50)1000 + (.50)(-480) = \$260$$

If you were the decision maker, which would you choose, alternative A or B? Both of them have the same expected dollar payoff, but you would, nevertheless, have some difficulty in choosing between them. The reason for this is that the two alternatives offer different exposures to risk and gain.

A decision maker who is risk-averse may choose alternative A. Choosing this alternative holds the possible loss to a minimum. That is, at most the decision maker can lose \$100. However, alternative A also limits the gain—to \$500. There are countless reasons why a decision maker may behave in this way and exhibit risk-averse behavior. Certainly a firm on the verge of receivership would be risk-averse.

Alternative B might be chosen by a risk-prone decision maker. This decision maker sees the chance of a \$1000 gain and is willing to take the risk. For him, the consequences of losing \$480 would not be too serious.

It should therefore be clear that in the minds of many decision makers alternatives A and B are not best represented by expected dollar payoffs. They are not, indeed, *equal*.

As we have just seen, when the differences between the gains and losses are wide, it may not be appropriate to make a decision based on expected dollar payoff. The utility or desirability that the decision maker holds for these outcomes must be taken into consideration. In the appendix to this chapter a scheme for maximizing the decision maker's utility rather than his expected dollar payoff is presented. For many applications, however, it is enough to maximize the decision maker's expected dollar payoff.

CASE STUDY: Briggs Foundry

The Briggs Foundry has been supplying engine castings to the automotive industry for over 25 years. Lately several defective castings have been returned by its customers. In response to these returns the foundry has decided to reevaluate its testing program.

The production process for this product includes the following steps. First, molten metal is poured into a mold; second, the casting is released once the metal has solidified; third, the casting is tested; and, fourth, it is shipped to the customer or scrapped depending upon the outcome of the test.

The purpose of the test is to determine whether the casting has any flaws. Visible flaws are easy to detect by visual examination, but it is the hidden flaws that are not only the most difficult to detect but also quite costly if they are not detected before the casting is shipped.

If there are hidden flaws inside the casting, and if it is shipped to the customer, either of two events can occur. The first is that the automotive manufacturer will eventually uncover the flaw during the machining process. For example, if the manufacturer were machining a cylinder hole and a flaw were uncovered which left a large pocket in one of the cylinder walls, the casting would have to be scrapped. The second event is that the flaw is in a harmless location and will not be uncovered by the manufacturer; in this case the casting is actually considered to be good. We can therefore conclude that it would be beneficial to uncover those flaws which would leave the casting functionally defective. Only these will be considered bad castings.

The Decision Problem

The manager of the foundry, Thomas Johnson, has spent a considerable amount of time on this testing issue. As he sees it, he has three action alternatives from which to choose. The first is to continue to use his imperfect tester, the second is to purchase a newly developed perfect tester, and the third is to perform no test at all on the castings. We will now look at these three alternatives in greater detail.

Imperfect Test For the last few years Mr. Johnson has been using a tester which does a good, but not a perfect, job. The tester uses electronic means for identifying flaws and occasionally passes a casting that has hidden flaws which show up later in the machining of the engine block.

No Test Although Mr. Johnson has never shipped a casting without subjecting it to a test, to forgo testing is a possible alternative. Since the use of the imperfect tester costs money and since it still lets some bad ones get through, he has not ruled out the possibility of not testing his output at all.

Perfect Test Recently he has been approached by a sales representative from a highly reputable electronics firm who has proposed a perfect tester using sophisticated electronic techniques. The test will identify all hidden flaws which would interfere with the subsequent machining of the engine block. It is, of course, quite expensive.

The Decision-Tree Model

The three action alternatives are imperfect test, no test, and perfect test. These start our decision tree and are shown in Figure 6-8.

If the imperfect test alternative is chosen, two outcomes can occur. The casting will either pass or fail the test. These outcomes are entered in Figure 6-8.

If the casting passes the test, management must decide whether to ship or to scrap. These alternatives are entered in Figure 6-8.

If the alternative chosen is to ship those castings which have passed the test, the manufacturer's plant will find the castings to be either good or bad (remember that some of the castings which pass the test can still be bad because an imperfect test has been used). These outcomes are entered in Figure 6-8.

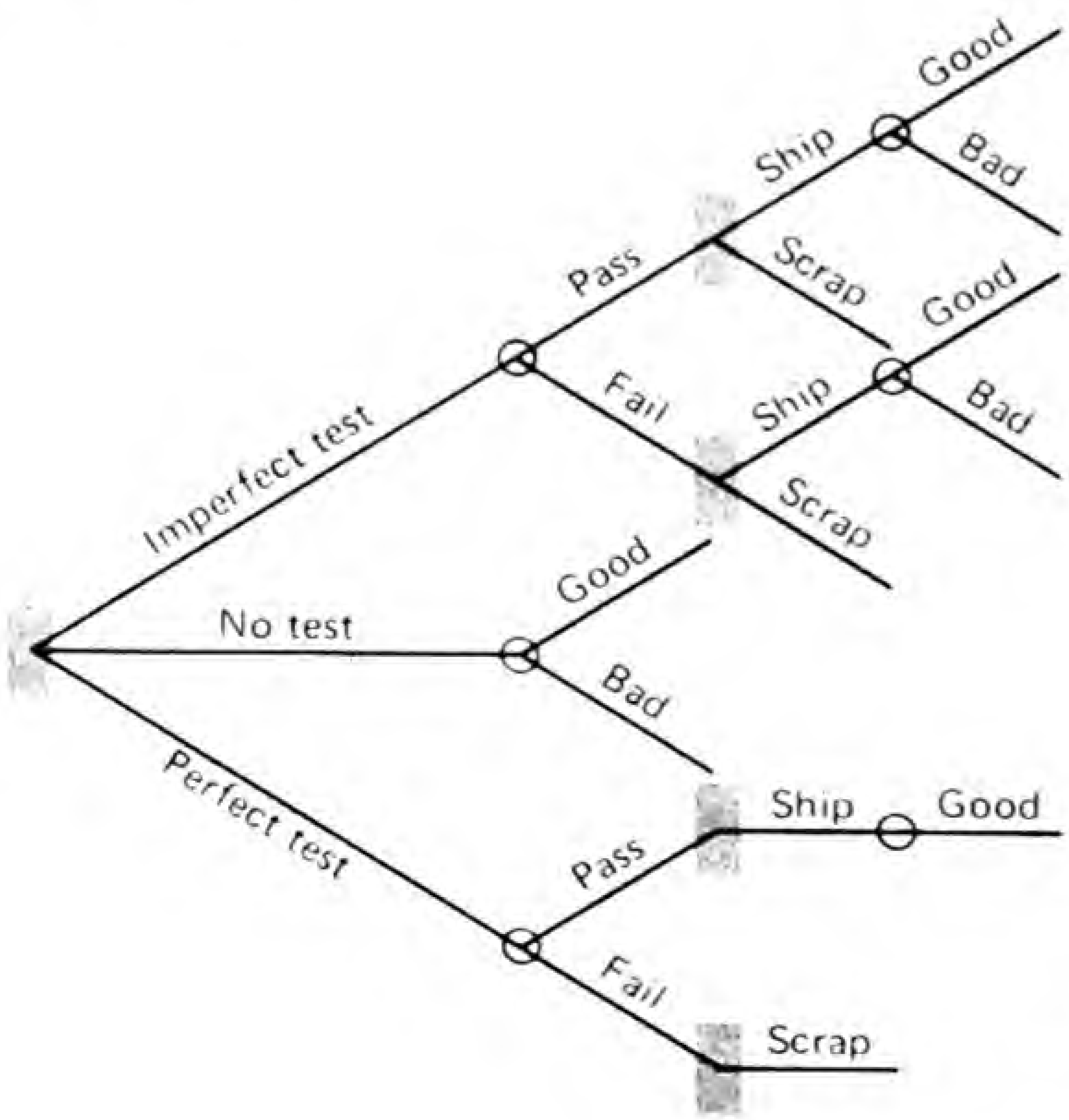


Figure 6-8 Decision tree for Briggs Foundry.

If the casting does not pass the imperfect test, Briggs can still act to ship or scrap. If shipped, the casting will be found to be either good or bad by the manufacturer. This sequence is entered in Figure 6-8.

Next, we will consider the decision alternative "no test." If indeed no test is performed and the casting is shipped directly to the manufacturer, the only relevant outcome will be whether the manufacturer finds this to be a good or bad casting. This sequence of alternatives and outcomes is entered in Figure 6-8.

Finally, we look at the decision alternative "perfect test." If a perfect tester is used, again the casting will either pass or fail the test. But this time if the casting passes the test, it will definitely be shipped because management knows that the automotive manufacturer will find it to be a good one.

On the other hand, if the casting fails the test, it should definitely be scrapped because management knows that it would eventually be found to be bad by the automotive manufacturer. This sequence of alternatives and outcomes is also entered in Figure 6-8.

Assessing the Probabilities for Each Outcome

If the imperfect test is continued, Mr. Johnson sees no reason why current results should not continue. Of all the castings subjected to the imperfect tester, 70 percent pass the test and 30 percent fail. The probabilities are entered in Figure 6-9.

The foundry's records also show that of those which pass the imperfect test and are shipped to the automotive manufacturer 90 percent are good and 10 percent are returned to the foundry as defective castings.

In a recent experiment to determine the effectiveness of the foundry's

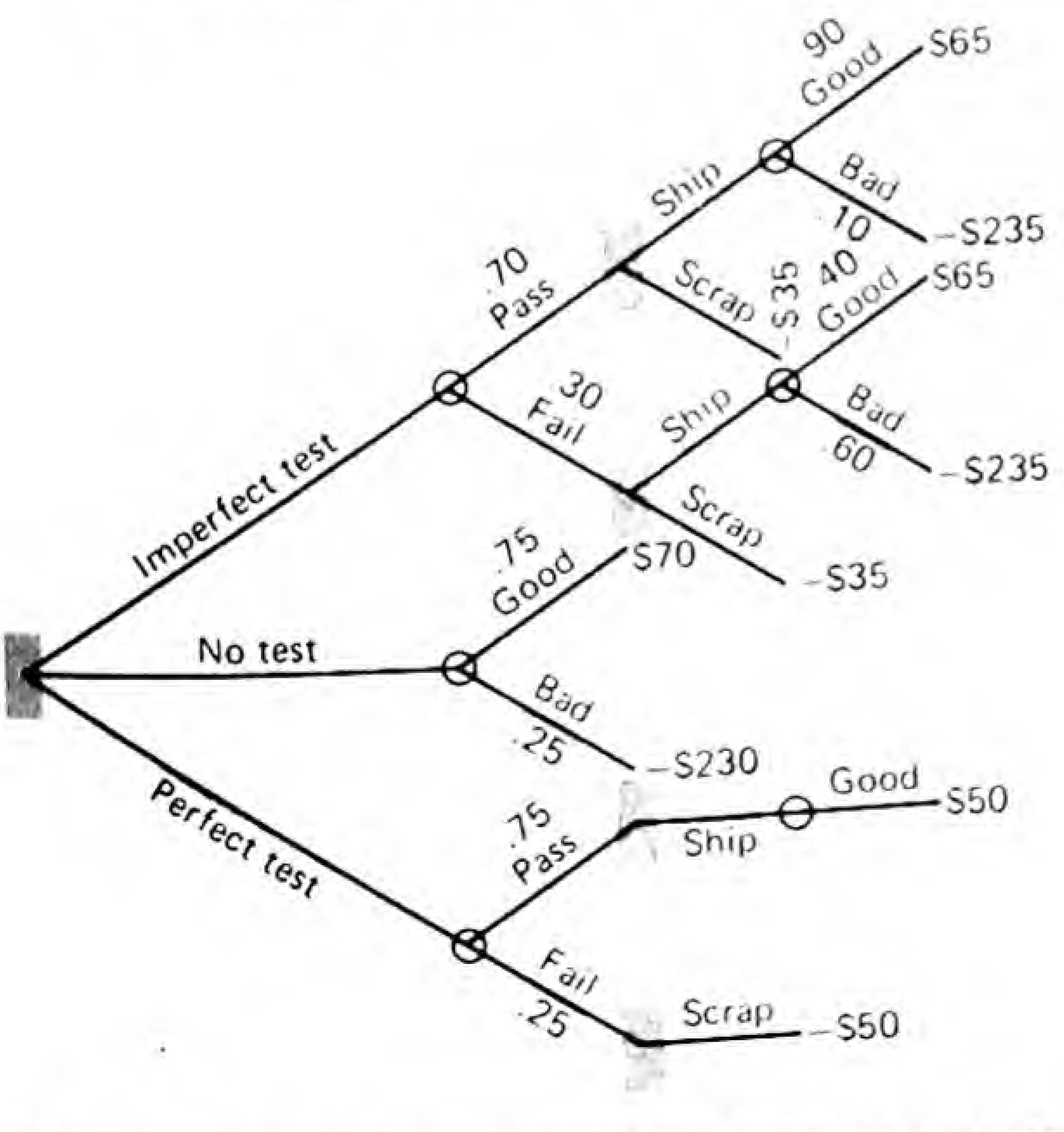


Figure 6-9 Decision tree for Briggs Foundry complete with probabilities and payoffs.

test, Mr. Johnson sent a batch of castings which had failed the imperfect test to the automotive manufacturer. Forty percent of those were found to be good and 60 percent were found to be bad. These figures are entered in Figure 6-9.

Using the results of this experiment, it was determined that if no test were performed, 75 percent of the castings would be found by the manufacturer to be good and 25 percent would be bad.[1]

Turning to the perfect test branch of the decision tree, we can also conclude that a perfect tester would accept 75 percent of the castings and reject 25 percent of them.

Now that the probabilities for each outcome have been determined, the next step is to compute the payoffs for each branch in the tree.

The Payoffs

To compute the net payoff associated with each branch, we must take into consideration the marginal revenue associated with each casting sold, the marginal cost of manufacture, the cost of testing each casting, and finally a penalty cost which is incurred by the foundry whenever a flaw is uncovered by the automotive manufacturer.

Marginal Revenue When castings are found to be acceptable by the automotive manufacturer and no hidden flaws are uncovered, Briggs Foundry receives $100 for each *good* casting. If a flaw is uncovered during the machining process, the customer does not pay this amount.

Marginal Costs The marginal cost of each casting has been computed to be $30. This cost is incurred, of course, whether or not the casting has hidden flaws.

Cost of Testing The cost of using the imperfect test has been computed to be $5 per unit tested; the cost of using a perfect tester has been estimated to be $20 per unit tested. If no test is used, no testing costs are incurred.

Penalty By agreement with the automotive manufacturer the foundry has agreed to incur a $200 penalty whenever a flaw is uncovered during the machining of the engine block. The penalty is to help defray the loss in labor and machine time which the functionally defective casting has cost.

Net Payoffs Computed We will first turn to the branch in which the imperfect test is employed, the unit passes the test, it is shipped, and it is found to be good by the automotive manufacturer. The net payoff will be the following:

[1] Good ones are those which passed the imperfect test and were eventually good (.70 × .90) plus those which failed the imperfect test and were found to be good (.30 × .40). Therefore, the likelihood of a part being good is .63 + .12 = .75.

$$\begin{array}{lccccccc} & \text{Marginal revenue} & - & \text{marginal cost} & - & \text{cost of testing} & & \\ = & 100 & - & 30 & - & 5 & = & \$65 \end{array}$$

This is entered in Figure 6-9.

For the branch below the one just considered, the casting is found to be bad after having previously passed the imperfect test. The net payoff this time will be the following:

$$\begin{array}{lcccccccc} & -\text{Marginal cost} & - & \text{cost of testing} & - & \text{penalty} & & \\ = & -30 & - & 5 & - & 200 & = & -\$235 \end{array}$$

The net payoff for this outcome is therefore a per unit loss of $235.

The net payoffs for the remaining branches are computed in Table 6-1, and the results are entered in Figure 6-9.

Table 6-1 The Computation of Net Payoffs for All 10 Branches in Figure 6-9 Starting from the Top Branch and Working Down to the Tenth Branch

Branch	Marginal revenue	Marginal cost	Test cost	Penalty	Net payoff
1	100	30	5		65
2		30	5	200	-235
3		30	5		-35
4	100	30	5		65
5		30	5	200	-235
6		30	5		-35
7	100	30			70
8		30		200	-230
9	100	30	20		50
10		30	20		-50

Analysis of the Decision Tree

Now that the payoffs and probabilities have been determined, the last step involves the folding-back process. This is accomplished by starting at the end of the tree, computing the expected value for each outcome fork, and then at each alternative fork choosing the alternative with the highest expected value. This folding-back process continues until the expected value for each of the three action alternatives has been determined.

This process is illustrated in Figure 6-10. By comparing the expected values for each of the three action alternatives, it can be seen that the perfect tester should be used. This decision alternative will yield an expected payoff of $25 per unit manufactured.

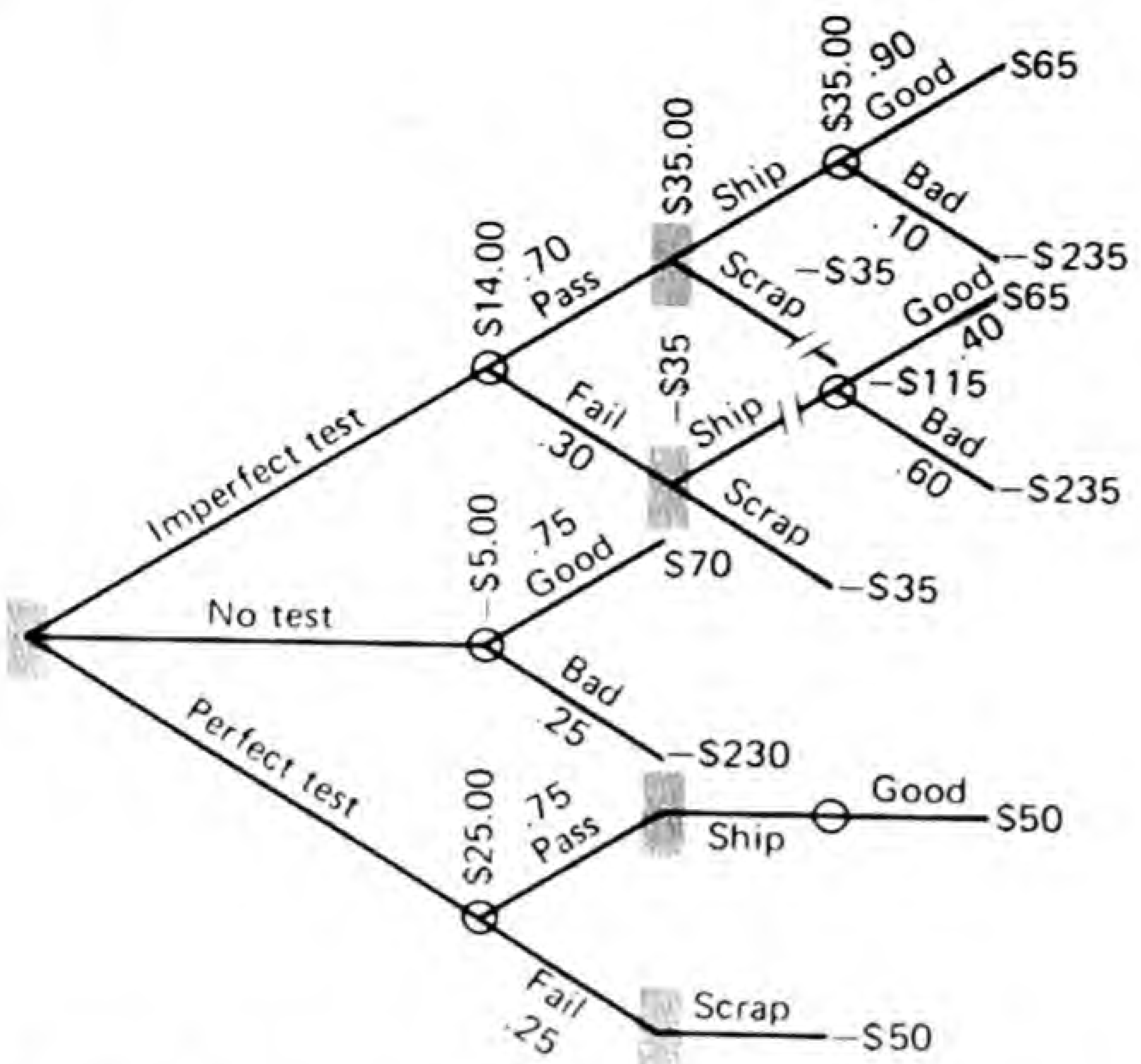


Figure 6-10 The folding-back process for Briggs Foundry and the solution.

SENSITIVITY ANALYSIS

In most decision-tree problems subjective estimates are used to determine both the likelihood of outcomes and the payoffs associated with each branch. Oftentimes these subjective estimates are nothing more than educated guesses. When this is true, it seems only natural to question the sensitivity of the result to these estimates. For example, in the consumer-product case presented at the beginning of the chapter it was determined that the best strategy was to enter an R and D phase directed at improving the product. This conclusion was based on the subjective estimates for outcome probabilities and payoffs. Suppose David Conway was unsure of the payoff estimates which he made. He would certainly find it useful to explore the consequence of other payoffs. Is it possible that these changes might point to a different action alternative? To answer this question, sensitivity analysis could be performed on the decision tree. The relevant changes could be made to the payoffs and the tree, once again, folded back. It could then be determined if these changes would affect the choice of an action alternative.

IS DECISION-TREE ANALYSIS USEFUL?

In the last 10 years there has been a steady increase in the use of decision-tree analysis. In fact, many analysts feel that its potential impact on administration will be great.[1]

[1] Rex V. Brown, "Do Managers Find Decision Theory Useful?" *Harvard Business Review*, May–June 1970, p. 78.

It has been used successfully by many companies, including Du Pont, Ford, Pillsbury, General Mills, and General Electric. The Pillsbury Company has used this technique to determine whether the company should switch from a box to a bag as a package for a certain grocery product. General Electric has used this analysis to increase by a factor of 20 the R and D budget for a new product. In addition, major consulting firms, including Arthur D. Little, Inc., and McKinsey and Company have reported numerous applications for their clients.

Indeed many companies are finding decision-tree analysis useful. They are finding it most useful, however, in the marketing function, including product selection, product development, promotion, pricing, and test marketing. The concept is by no means limited to these applications; as companies gain experience with this technique and as more people become familiar with its use, the list of successful applications will continue to grow.

Some Suggestions

To increase the likelihood that the use of decision-tree analysis will meet with success, the following suggestions should be considered:

1 The chief executive must see decision analysis as useful and necessary. His backing is a must.

2 The key executives who will be involved directly or indirectly must understand what decision analysis can do for them. Often orientation courses are used to accomplish this.

3 A problem should be chosen and a trial run undertaken to illustrate the usefulness of the method. Perhaps an outside consultant can be used during this phase.

4 Inside specialists must be developed. Staff expertise must be available to assist those using decision analysis on a problem.

5 Decision analysis should be kept relevant and meaningful; do not get lost in the technical aspects of the analysis; the techniques should never dominate the analysis—only the decision makers should.

Although these steps will not guarantee successful use of the technique, they will help. Successful results come only after the technique has been accepted, tried, and made familiar to all those involved. This takes time.

SUMMARY

The formulation of a decision-tree model requires that the problem be broken down into a series of alternatives and outcomes, and that the relevant probabilities and payoffs be estimated. By starting at the end of the decision tree and working forward, the expected values for the action alternatives are computed. It is in this way that complex decision problems can be logically and consistently resolved.

Those who have used decision analysis point to three advantages. First,

the use of the analysis focuses thinking on the critical aspects of the decision problem. Second, it forces any hidden assumptions into the open and clarifies their implications. Third, and perhaps most important, the analysis provides an effective vehicle for examining the structure and interrelationships among the relevant elements found in a problem.

QUESTIONS

1 How do the decision problems under risk presented in Chapter 3 differ from the problems presented in this chapter?
2 What is an action alternative, and why does it differ from the other alternatives in a decision-tree model?
3 Under what conditions can a decision be made on the basis of expected dollar payoff?
4 When is sensitivity analysis of a decision tree useful?
5 Draw a decision tree which depicts the relevant alternatives and outcomes which exist after you successfully complete the requirements for your degree. Your decision tree should not look beyond a five-year period. What are the payoffs and probabilities? What decision should be made?

PROBLEMS

6-1 Suppose that the probability of achieving a high level of sales for the old hair spray product in the Scott Company case was revised to 50 percent. What is the expected dollar payoff associated with the act "continue as is"?

6-2 Return to Problem 6-1 and answer the following question. What is the expected dollar payoff associated with the act "R and D"?

6-3 In the Scott Company case the possible loss that could be incurred if the new product only achieved low sales has been revised to \$200,000. What is the expected dollar payoff of the act "R and D"?

6-4 Suppose that the probability of achieving high sales with the new hair spray was revised to 70 percent. What decision should be made by the Scott Company?

6-5 In the Briggs Foundry case Mr. Johnson has just learned of an error made by the manufacturer. It has informed him that when it machined the batch of castings that had failed the imperfect test 60 percent were found to be good and 40 percent were found to be bad.

Recompute the probability that a good casting will be produced by Briggs Foundry.

6-6 Using the information provided in Problem 6-5, determine the expected dollar payoff of the alternative "imperfect test."

6-7 Using the information provided in Problem 6-5, compare the expected dollar payoffs of the three action alternatives.

6-8 In the Briggs Foundry case at most how much should Mr. Johnson be willing to spend for a perfect test?

6-9 In the Briggs Foundry case suppose that the penalty for a bad casting was decreased to \$100. What effect would this have on the decision?

6-10 In the Briggs Foundry case suppose that only 1 percent of the castings which pass the imperfect test are found to be bad by the manufacturer. Does this affect the decision?

CASE STUDY: Durham Company

The Durham Company is a job shop located in the southern New Hampshire town of Nashua. It employs 100 people and has 200 machine tools. By most standards, it is a fairly large job shop.

Durham manufactures parts and subassemblies that its customers find impossible—for one reason or another—to produce themselves. It manufactures only upon receipt of an order and has never manufactured anything on the speculation that the item could be sold later.

The vice president of manufacturing, Bill Abbot, recently received a letter from Artisan, Incorporated, asking Durham Company if it would be interested in manufacturing a small machined part which is to be part of a new industrial product.

The letter explained that Durham, along with three other job shops, was being asked to submit a bid on the job. It was also suggested in the letter that a prototype of the unit be submitted along with the bid. Engineering drawings were enclosed.

The letter went on to say that an order for 10,000 of these parts would be awarded to the job shop with the best proposal.

Bill Abbot spent a considerable amount of time thinking about this letter. His first concern was that he wasn't quite sure of the costs that would be incurred in the development of a prototype. From the drawings he estimated that there was a 20 percent likelihood that the prototype cost would be $1000, a 50 percent likelihood that it would be $1500, and a 30 percent likelihood that it would be $2000.

Mr. Abbot felt that after the prototype was completed, he could decide on the magnitude of the bid. He could bid either high or low. Preparation of the bid would cost $300. If he bid high, he realized that the chances of winning the manufacturing contract would be small, but if indeed it was awarded the contract, Durham Company would be assured of a good profit. If, on the other hand, he bid low, the likelihood of receiving the manufacturing contract would be substantially greater, although expected profits would be lower.

At this stage, Bill felt that if the prototype cost $1500, his high bid would have to be $50,000 and the low bid would be $44,000. If the prototype cost $1000, his high bid would be $45,000 and the low bid $39,000. Finally, a prototype cost of $2000 would lead to a high bid of $60,000 and a low bid of $48,000.

Whether or not the company won the contract would depend only upon the level of the bid.[1] If the bid was high, it would have but a 30 percent chance of winning, whereas if the bid was low, it would have a 70 percent chance of winning.

Manufacturing costs would depend only upon the cost of the prototype.

[1] It is assumed here that if prototype costs are high for Durham Company, they will also turn out to be high for their competitors.

If the prototype cost was $1000, manufacturing costs for all 10,000 pieces would be $12,000. A prototype cost of $1500 would result in a manufacturing cost of $18,000, and a prototype cost of $2000 would result in a manufacturing cost of $35,000.

There was one other alternative which Mr. Abbot had to consider. It was possible for Durham Company not to build a prototype and still submit a bid. Again, a high or low bid could be submitted. Without the experience gained by building a prototype he felt the lowest the bid could be was $48,000, while the high bid would be $60,000. Under this circumstance, the chance of winning, given that the high bid was submitted, would be 10 percent while the possibility of winning, given a low bid, would be 30 percent. In setting these probabilities, he realized that it was reasonable for them to be this low since Artisan, Incorporated, would not have a prototype by which to judge Durham's bid. If the contract was won, there would be a 20, 50, and 30 percent chance that manufacturing costs would be $12,000, $18,000, and $35,000, respectively.

QUESTIONS

1 Structure this problem as a decision tree.
2 On the basis of the expected dollar payoff criterion, what decision should Mr. Abbott make?
3 What shortcoming does the expected dollar payoff criterion have in reaching a decision for this case?

CASE STUDY: Lone Star Oil

The Lone Star Oil Company was founded in 1975, shortly after the Arab boycott led to an oil crisis in the United States.

The company's mode of operation was to lease land, test for oil, drill, and then sell the rights to the oil if oil was discovered. There were no operational departments in the company since any testing or drilling that was required was subcontracted; Lone Star Oil Company, then, was strictly an administrative organization.

Revenue during the first year of operation was low. The net liquid assets of the company have currently been reduced to $130,000, and the president, Ralph Salmi, has been under pressure. "Ralph, I think we had better be damn careful this time," said the vice president, Jim Planter. "We need to analyze our options, compare the outcomes, and carefully make our decision. We can't afford any more mistakes."

Ralph added, "If we blow this one, we're through."

They were referring to an option which Lone Star purchased 2 weeks ago for $10,000. The option gives the company the right to take tests and drill for oil. At any time until the option expires, it can sign a lease agreement on the

property for a fixed fee of \$100,000. The option expires in 3 weeks, and until now no tests or drilling have taken place. At present this happens to be the only option which Lone Star has open.

Jim said, "Ralph, it looks to me like we have three choices. Either we let the option expire and do nothing about it, take seismic tests, or drill without any initial testing. Based on some information which he collected, our geologist feels that there is a 60 percent chance that any tests run in the geographic region covered by the option will lead to positive test results."

Ralph interrupted, "If our seismic tests are positive and we drill, what are our chances of finding oil?"

"I would say 80 percent," said Jim.

"What if the tests are negative? Do you think we should even consider drilling?" asked Ralph.

"I really don't know," replied Jim, "because our chances of finding oil given a negative test result drop to 10 percent."

"Just a minute, Jim," added Ralph. "Let's take a look at the dollar payoffs and expenses. After all, these figures will make or break us."

"According to my latest information," said Jim, "seismic tests will cost us \$30,000, and they can be completed within a few days. Drilling a well will cost us \$100,000. And if we find oil, the rights to this land can be sold for \$500,000."

Ralph thought for a few moments and then said, "Jim, what do you think we should do?"

Jim replied, "I favor taking a test. In my opinion, this is the safest approach. In fact, after speaking to our geologist I feel that the likelihood that seismic tests will be positive is not 60 percent but more like 70 percent. Our geologist was probably too conservative in his estimate."

QUESTIONS

1 Structure this problem as a decision tree.
2 On the basis of the expected dollar payoff criterion, what decision would you recommend?
3 Should the current net asset position of the company be ignored in the analysis?
4 What is the shortcoming, if any, of the expected dollar payoff criterion when used to reach a decision in this case?

CASE STUDY: Wilson Company

The Wilson Baby Food Company has been a nationally recognized producer of baby foods for 45 years. Over these years it has been fortunate to be participating in a growing market. But this will probably not be true in the future.

Recently, sales have been falling along with the drop in the birthrate. In

fact, most long-term predictions foresee this lower birthrate as a permanent phenomenon.

Two months ago at a corporate long-range planning session it was agreed that the company should consider the expansion of its product line beyond baby foods. It was unanimously decided that the first move should be to change the name to the Wilson Company.

Shortly after the meeting a proposal was made to produce and market a line of geriatric foods. These prepared foods would be aimed at the 65-and-over age bracket. Preliminary analysis indicated that this market had above-average growth potential over the next 30 years, and at present few firms competed in this market.

In addition, this analysis showed that if sales were high, an average profit of \$6 million per year could be realized over the first 5 years of the project's life. If, on the other hand, sales were low, the company could incur an average loss of \$3 million per year over the same time period.

On the basis of historical experience, the management of the Wilson Company thought that the likelihood of high sales from *any* project was .3 and the likelihood of low sales was .7.

The manager of marketing, Janet Collinge, was unclear as to what she should do next. She felt she had two alternatives. The first would be to undertake a market survey which would yield additional information from which a better decision could be made. The second alternative would be to omit a survey and base the decision on current information.

If a survey was undertaken, the result would predict either success for the product (high sales), inconclusive results, or product failure (low sales).

Survey results are by no means perfect. Ms. Collinge has conducted countless surveys in the past and has some information which may be helpful in reaching a decision alternative. First, if one considers all those products which have achieved a high level of sales in the marketplace, the results indicate that in 60 percent of the cases the survey forecast success, in 30 percent of the cases the result was inconclusive, and in 10 percent of the cases the survey forecast failure. Next, if one considers all those products which have achieved a low level of sales, these surveys showed that in 20 percent of the cases the survey forecast success, in 30 percent the result was inconclusive, and in 50 percent the survey forecast failure.

QUESTIONS

1 Supposing that Ms. Collinge decides not to take a survey, what is the expected value of the project? Should it be undertaken?
2 What is the probability that the survey will predict success *given* that sales are high?
3 What is the probability of high sales *given* that the survey predicts success?
4 Draw the decision tree.
5 Should Ms. Collinge undertake the survey (ignore survey costs)?
6 Suppose the survey cost was \$30,000. Does this change the decision?

APPENDIX A: Preference Theory

In this appendix the concept of utility or preference will be explored and a mechanism developed to analyze the outcome of an action alternative in terms of its preference. We will return to the first case study presented in the chapter, the Scott Company, and develop a solution which will maximize the decision maker's preference rather than maximizing the dollar payoff.

MATHEMATICAL EXPECTATIONS, CERTAINTY EQUIVALENTS, AND RISK PREMIUMS

In the previous sections of this chapter the objective in solving decision-tree problems was to find that action alternative which led to the highest expected *dollar* payoff.

If the decision maker is comfortable with this approach and, in fact, behaves in this way, then we say he is an "averages player."

Consider the following example of an averages player. A decision maker is faced with a 20 percent chance of making a \$100 profit and an 80 percent chance of a \$10 profit. The expected value of this "gamble" is \$28.

$$.20(\$100) + .80(\$10) = \$28$$

Now, suppose our decision maker is asked if he would be willing to sell this gamble for a given sum of money. If he replies that he would accept \$28 for this gamble, he is behaving as an averages player. If, however, he would be willing to accept something less than \$28, he is not. Suppose he would be willing to accept \$20 certain in exchange for this gamble. That is, he would take the guarantee of less money rather than face the gamble. The \$20 is called the certainty equivalent, and the difference between this and the expected value is called the risk premium.

Expected value		\$28
Less:	certainty equivalent	20
Equals:	risk premium	\$ 8

In this example the risk premium is \$8.

When expected values are unlikely to be the same as certainty equivalents, a decision-tree analysis based on expected values will not maximize the decision maker's preference.

In the next section a simple method will be presented for determining these preferences, and then they will be used in the analysis of a decision-tree model.

PREFERENCE VALUES

The first step in determining preferences is to identify the most attractive and the least attractive payoffs. They will act as the reference points in the analysis; the most attractive or preferable payoff will be assigned the value 100 percent, and the least preferable payoff will be assigned 0 percent.

After these extremes have been identified, all the payoffs are listed in order of preferability from the most preferable to the least.

Next, their relative preferability on the scale from 0 to 100 percent is estimated. For example, the most preferable payoff has a preference value of 100 percent and the next most preferable payoff may have a value of 75 percent.

The assignment of relative preferabilities continues until all payoffs have been assigned percentage values. The analyst should reexamine the completed list and confirm that the relative measures do indeed reflect the preferability of each payoff as compared with the others.

The analyst should bear in mind that the relative preferability of payoffs might bear little relation to their dollar payoffs. For example, outcome A might have a payoff of $15,000 but be less preferable than outcome B, which has a payoff of $14,000. The reason might be that outcome B places the firm in a more competitive position in the long run.

These relative ratings represent the preference that the decision maker holds for each of the outcomes. In the next step these preferences are entered on the decision tree *in place of* the dollar payoffs.

From this point on the solution of the decision process is just as it was before: the tree is folded back; this time, however, it will be the preferences, not the dollar payoffs, which are folded back. At each outcome fork expected preferences are computed, and at each alternative fork the alternative with the highest expected preference is selected. In the final step of the analysis the action alternative with the highest expected preference is identified as the alternative to be recommended.

SCOTT COMPANY REVISITED

In performing a preference analysis for the Scott Company case, the first step is to rank all payoffs from most preferable to least preferable. This is a subjective process, and the ranking presented in Table 6A-1 is just one point of view; other rankings are possible and depend upon the preferences of the decision maker. Next, relative preference values between 0 and 100 percent are assigned to these payoffs. Then these preferences are entered on the decision tree found in Figure 6A-1.

From here the process of folding back should be a familiar one. Turning to the

Table 6A-1 Ranking and Preferences for the Outcomes of the Scott Company Decision Problem

Branch	Dollar payoff	Ranking	Preference, percent
R and D—Positive—Market new product—High sales	100,000	9(best)	100
R and D—Positive—Market new product—Low sales	−20,000	4	20
R and D—Positive—Discontinue	−10,000	3	10
R and D—Negative—Market old product—High sales	60,000	7	70
R and D—Negative—Market old product—Low sales	−30,000	2	5
R and D—Negative—Discontinue	−10,000	5	30
Discontinue product	0	6	40
No R and D—High sales	70,000	8	80
No R and D—Low sales	−20,000	1(worst)	0

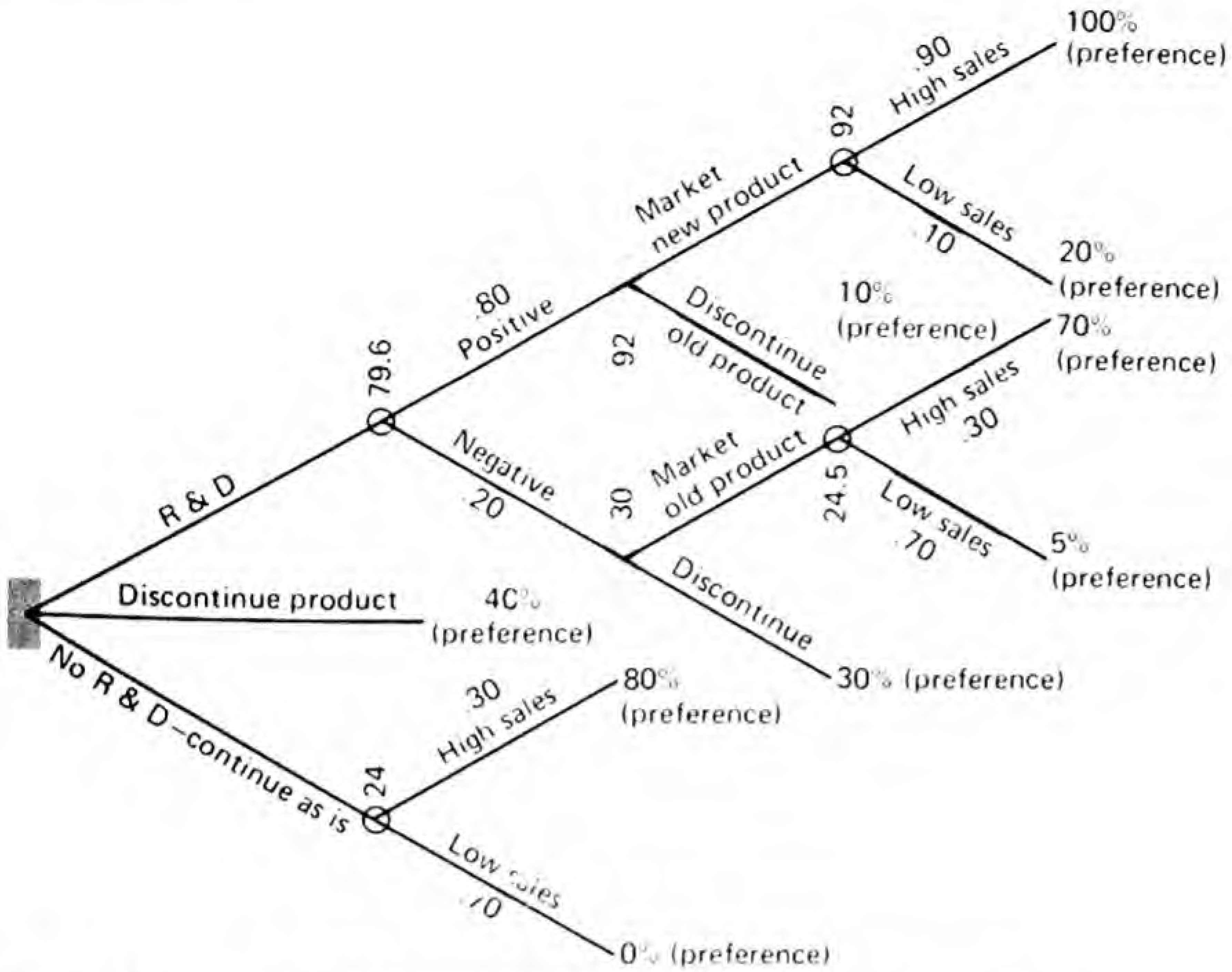


Figure 6A-1 Preference analysis for the Scott Company case.

top branches of the tree, we can compute the expected preference in the following way:

$$.90(100) + .10(20) = 92$$

This expected preference value is entered above the circle depicting the outcome. Moving backward, we are then confronted with an alternative. The decision maker can either market the new product with an expected preference of 92 or discontinue the product with an expected preference of 10. Since he is a preference maximizer, he would choose "market new product."

The process of folding back continues until the expected preferences for the three action alternatives have been determined. The R and D alternative has the highest expected preference and is therefore the one recommended.

In this example the action alternative with the highest expected *payoff* also turned out to be the one with the highest expected *preference*. This will not always happen, but in the Scott Company example the R and D alternative is so dominant that we might expect both analyses to lead to the same conclusion.

Where there is clearly no dominant alternative using the expected dollar payoff method, it would not be surprising to find that a preference analysis would recommend a different action alternative. After all, the preference analysis includes all those imponderables which would have given the decision maker cause for concern if the analysis was strictly based on dollar payoffs.

Chapter 7

Linear-Programming Models: Formulation

INTRODUCTION

Many decision problems—both personal and administrative—can be classified as constrained optimization problems. In these problems there is an objective to be optimized and a set of constraints which cannot be violated.

Ron Lemieux is a bachelor who seems to face every weekend with the same objective: to have the best possible time. But several constraints limit the achievement of this objective. First is a budget constraint: he never seems to have much money. Second is a time constraint: he has only from noon Saturday until Sunday evening. Third is a transportation constraint: he currently owns an old car which he is reluctant to drive beyond the city limits.

A management scientist would say that Ron faces a constrained optimization problem. In fact the form of this problem is quite similar to that of many administrative decision problems. First there is an objective to be achieved, and second there are are constraints which cannot be violated. The topic for this and the next two chapters will be the formulation of models for these constrained optimization situations and their analysis using linear-programming techniques.

Linear programming has been applied to a large number of problems. Some of these include production scheduling, media selection in advertising, job shop

scheduling, investment analysis, project scheduling, inventory management, warehouse location, and commercial bank portfolio management. This list is not exhaustive, nor has the record been unblemished. Some attempts at using linear programming have failed, but countless others, however, have met with success. The extent of these successes has been such that it is imperative that every up-to-date manager know something about this technique.

Objective and Constraints

All problems which can be formulated as linear-programming models share a number of characteristics in common. First, there is an objective. It may be to maximize profit, minimize costs, maximize return on investment, minimize the length of time required to complete a set of jobs, and so on.

Second, there are constraints. They act to limit the achievement of this objective. Without constraints, for example, an automobile manufacturer could manufacture an unlimited quantity of cars. However, budget constraints, demand, a limited pool of trained workers, and limited physical plant and equipment all serve to constrain, or place limits upon, output levels in the short run.

In summary, then, these problems have an objective and a set of constraints. The concern of the decision maker is to achieve the objective while staying within the bounds of the constraints.

By the way of a case study, we will now explore a problem which exhibits these characteristics in greater detail.

CASE STUDY: Jerry Company

The Jerry Company, founded in 1956, is one of the largest manufacturers of tents in the United States. Its product line includes several kinds of lightweight mountain and heavyweight camping tents. The mountain tents are purchased primarily by backpackers who intend to carry the tents on pack frames during extended hiking trips. The camping tents are heavier models, made of canvas, and used primarily by people who camp at roadside areas.

The manager of production, James Grover, is currently in the process of formulating a production schedule for next month. It has already been decided to manufacture but two models during this planning period: one camping model and one mountain model. What remains is to determine how many of each to schedule.

According to the sales department, there is an unlimited demand for both models. That is, the Jerry Company can sell as many as they produce. Constraining production, however, is the fact that the two departments through which both tents must be processed have limited capacities. The first department, fabric cutting, has an expected capacity of 8000 worker-hours of labor during the next month. Every camping model which is processed by this

department requires 2 worker-hours of cutting time, whereas every mountain tent requires 1 hour. The second department is sewing. Here the fabric, mosquito netting, and zippers are sewn. The expected capacity of this department is 16,000 worker-hours. Every camping model which is processed by this department requires 2 worker-hours of labor; every mountain model requires 4 hours.

The objective of the production department is to maximize the profit contribution generated by its products. From a recent analysis Mr. Grover knows that the profit contribution (revenue less variable expenses) for each camping model is \$20 and for each mountain model is \$30.

From these data, how would you advise Mr. Grover to plan production? Certainly there are many alternatives from which to choose. One is to produce nothing; another is to produce one of each kind; and another is to produce 1 mountain model and 2 camping models. The list could go on and on provided that the alternatives do not violate either of the constraints. Then, for each of these alternatives a profit contribution could be computed, and finally the alternative with the highest profit contribution would be selected. But this is a tedious and inefficient process. A more efficient method will be developed in Chapters 8 and 9. First, however, we will consider two requirements which must be met before this method can be used.

THE BASIC ASSUMPTIONS OF LINEAR PROGRAMMING: CERTAINTY AND LINEARITY

Linear-programming models can be used to represent constrained optimization systems only when the value of the parameters of that system is known with certainty. In the Jerry Company case, profit contribution, resource capacities, and resource utilization are all assumed to be known with certainty. For example, if 1 camping and 1 mountain tent are manufactured, the profit generated from this output level will be *exactly*

$$1(20) + 1(30) = \$50$$

The profit contribution for any output level is therefore known with certainty. In addition the capacity of the scarce resources is also known with certainty. For example, the case implies that *exactly* 8000 worker-hours of capacity in the cutting department are available. The utilization of scarce resources is also known with certainty. For example, it is implied that it will take *exactly* 2 worker-hours of the available capacity in fabric cutting to cut a camping tent and 1 worker hour to cut a mountain tent.

We can conclude that in linear programming there are no probability distributions; the assumption is made that all parameters are known with certainty.

The second assumption that must be made before a linear-programming model can be used is that the system must be linear throughout. Implied in the profit contribution for each tent is that the profit generated by the first camping

tent is \$20; by the second one, \$20; by the hundredth one, \$20; by the thousandth one, \$20; and so on. This also holds for the mountain tent. The first one generates \$30 profit, the tenth one \$30 profit, the hundredth one and the millionth one and so on.

Linearity is also assumed in the constraints. For example, in the fabric-cutting department, the first camping tent processed requires 2 hours of labor, the second requires 2 hours of labor, the hundredth requires 2 hours, and so on. Likewise, the first mountain tent requires 1 hour of cutting time, the second one requires 1 hour of cutting time, the thousandth one requires 1 hour of cutting time, and so on. The sewing constraint is also linear and can be interpreted in a similar fashion. Indeed this problem is linear throughout, and hence the word "linear" in linear programming.

THE PROBLEM IN TABULAR FORM

To analyze any linear-programming problem, it is quite useful to summarize the data in tabular form. The data given in the Jerry Company case are summarized in Table 7-1. In the top row of this table is found the profit contribution associated with each activity. Then the constraints are presented. First we have the fabric-cutting constraint. Here it can be seen that each camping tent requires 2 worker-hours per unit produced and each mountain tent requires 1 worker-hour per unit produced. The capacity in worker-hours is recorded in the last column. The data for the sewing constraint are found in the last row.

Table 7-1 Data for the Jerry Company Case in Tabular Form

	Camping tent, per unit	Mountain tent, per unit	Capacity
Profit contribution	\$20	\$30	
Fabric cutting	2 worker-hours	1 worker-hour	8,000 worker-hours
Sewing	2 worker-hours	4 worker-hours	16,000 worker-hours

THE PROBLEM IN MATHEMATICAL FORM

Objective Function

The problem can be stated mathematically by referring to the table. First the objective—to maximize total profit contribution—can be expressed in the following way:

$$\text{Max } P = 20C + 30M$$

where P is profit, C is the number of camping tents produced, and M is the number of mountain tents produced. This mathematical expression implies

that profits are to be maximized and that this profit is equal to $20 times the number of camping tents produced, plus $30 times the number of mountain tents produced. It is called the *objective function*. For example, if 10 camping tents and 15 mountain tents were produced, profit would be

$$P = 20(10)+30(15)$$
$$P = 650$$

Since the objective is to maximize profit, we would like to make the value of this objective function as large as possible. But the constraints limit the achievement of this objective. Consequently they must also be expressed mathematically.

Fabric-cutting Constraint

In the fabric-cutting department, the 8000 worker-hours of shop time that are available cannot be exceeded. We are free to use less than this amount, but we cannot use more. Each camping model uses 2 worker-hours of this total, and each mountain model requires 1 worker-hour of this total. We can, therefore, state the *combined* utilization of this available capacity in the following way:

$$2C + 1M$$

If we produce 100 camping tents and 300 mountain tents, we have used

$$2(100)+1(300) = 500$$

hours of the 8000 hours available. What remains is to specify explicitly that the number of worker-hours used in this department must be *less than* or *equal to* 8000. This can be done in the following way:

$$2C + 1M \leq 8000$$

This is called an inequality and specifies that the left-hand side of the expression must be less than or equal to the right-hand side. It cannot, however, exceed the right-hand side. Let's check two strategies. First, we will schedule 500 camping tents and 3000 mountain tents. We therefore have:

$$2(500) + 1(3000) \leq 8000$$
$$4000 \leq 8000$$

and we see that 4000 worker-hours are required to accomplish this plan and that this is indeed less than the available 8000 worker-hours. Therefore, there is no violation of this constraint. Consider another example, where 4000 camping tents and 2000 mountain tents are scheduled. This time we have:

$$2(4000) + 1(2000) \le 8000$$
$$10{,}000 \nleq 8000$$

and we see that 10,000 worker-hours are required, thereby violating the constraint. This strategy would therefore not be permitted.

In summary, then, our mathematical expression of this constraint specifies that the use of this resource be less than or equal to its capacity.

Sewing Constraint

Next the constraint for the sewing department will be expressed mathematically. We have

$$2C + 4M \le 16{,}000$$

where the left-hand side of the expression represents the total worker-hours that are used in the sewing department for a particular production strategy and the right-hand side represents the capacity of that department.

Nonnegativity Constraints

To be complete, we must finally add nonnegativity conditions. These can be written in the following way:

$$C \ge 0$$
$$M \ge 0$$

They ensure that the values of C and M are greater than or equal to zero. This prevents the occurrence of negative numbers. After all, what would it mean to produce minus three tents? Therefore, our nonnegativity conditions ensure that only nonnegative numbers are possible.

A Summary of the Problem

The mathematical statement of this linear-programming problem can be summarized in the following way:

Max $P = 20C + 30M$

subject to the constraints:

$$\begin{aligned} 2C + \ 1M &\le 8000 \\ 2C + \ 4M &\le 16{,}000 \\ C \qquad &\ge 0 \\ M &\ge 0 \end{aligned}$$

Although such a formulation might be new and unfamiliar to you now, it will

be an old friend in just a few weeks. For you will find that the mathematical statement of the objective function, followed by the constraints and nonnegativity conditions, is common to all linear-programming problems.

SUMMARY

The formulation of linear-programming models requires that the criterion or objective associated with the problem be established and then expressed as an objective function. Next the constraints that are to be included in the model are determined and expressed in inequality form.

The methods behind the analysis and solution of these models will be presented in the next two chapters. Meanwhile the problems and the case at the end of this chapter will give you the opportunity to become familiar with the process of formulating these models. In fact, the formulation of the model is often the most difficult step.

QUESTIONS

1 What is a constrained optimization problem?
2 Explain the basic assumptions of linear-programming models.
3 Suppose that each camping tent produced beyond 50 tents requires less and less time in the sewing department of the Jerry Company. Is a linear-programming model still an appropriate way in which to analyze this case study?
4 Why are nonnegativity constraints necessary?
5 Would it be possible for the Jerry Company to produce 2000 camping tents and 4000 mountain tents? Why?

PROBLEMS

7-1 Suppose that in the Jerry Company case the profit per mountain tent was revised to \$40 and the length of time required to sew each of these tents was increased from 2 hours per unit to 3 hours per unit. Formulate the revised problem in mathematical form.

7-2 Suppose that in the Jerry Company case the demand for camping tents is limited to 1000 units. Revise the mathematical model to include this constraint.

7-3 The Alliance Appliance Company produces toasters and small electric broilers. Demand for its product is so great that it can sell all it produces. For each toaster sold the contribution to profit is \$5, and for each broiler the contribution is \$7. These products must pass through two manufacturing departments: stamping and finishing. It is these departments which seem to constrain output. Each toaster requires 2 hours in the stamping department and 1 hour in the finishing department; each broiler requires 1 hour in the stamping department and 2 hours in the finishing department. The capacity of the stamping department is 10,000 worker-hours, and that of the finishing department is 12,000 worker-hours.

Set this problem up as a linear-programming model.

7-4 The Ace Trucking Company produces trucks in two models. Model A contributes \$150 each to profits, and model B contributes \$200 each. Three depart-

ments constrain output: stamping, engine manufacture, and assembly. Each model A requires 1 hour in the stamping department, 2 hours in the engine department. The capacity of the stamping department is 5000 worker-hours; of hours in stamping, 2 hours in the engine department, and 3 hours in the assembly department. The capacity of the stamping department is 5000 worker-hours; of the engine department, 8000 worker-hours; and of the assembly department, 7000 worker-hours.

Set this problem up in the mathematical form.

7-5 The Able Rug Company manufactures two types of rugs: nylon and wool. Each of these is made in both a *good* quality and a *better* quality. The profit contribution from a nylon rug of good quality is \$20, and from the nylon rug of better quality it is \$25. For the wool rugs in Able's line, the good-quality model contributes \$23 and the better model \$40 to profit.

The manufacturing process for these rugs is such that each rug must pass through each of two departments: weaving and shipping. The nylon rug of good quality requires 2 hours of weaving and ½ hour of shipping time; the nylon rug of better quality requires 3 hours of weaving and 1 hour of shipping. The wool rug of good quality requires 2½ hours of weaving and 2 hours of shipping; the wool rug of better quality requires 3½ hours of weaving and 2 hours of shipping. The capacity of the weaving department for the next month is 5000 worker-hours, and that of the shipping department is 3000 worker-hours.

The marketing department has undertaken a demand study and found that no more than 4000 nylon and 3000 wool rugs can be sold during the next month.

Set this up as a linear-programming model.

Hint: Let N_g represent the number of good-quality nylon rugs scheduled, N_b represent the number of better-quality nylon rugs, W_g the number of good-quality wool rugs, and W_b the number of better-quality wool rugs.

7-6 A manufacturer produces three products, X, Y, and Z. Each product must be processed through four departments, A, B, C, and D. The manufacturing time for each of these products is given below:

	Product		
Department	**X**	**Y**	**Z**
A	3	4	2
B	2	6	3
C	3	5	7
D	8	3	1

The capacities for these departments are:

Department	**Capacity**
A	2000
B	2500
C	1800
D	3000

If the per unit profit contributions for X, Y, and Z are \$1, \$2, and \$1.50, respectively, set up this problem in linear-programming format.

7-7 The United Airplane Company has six model 707 jets, nine model 727 jets, and twelve model DC-8 jets, which all operate from a central airline terminal in New York. Their passenger load capacities are 150 for the model 707, 100 for the model 727, and 125 for the DC-8. All are available for today's flights.

United dispatches planes to three cities: Boston, Chicago, and Miami. According to reservation records, Boston needs 350 seats, Chicago 400, and Miami 385. An empty seat has no value, and a plane can fly only once a day out of New York.

The cost of sending a plane from the New York terminal to any of the three cities is given below:

	Equipment		
Destination	**707**	**727**	**DC-8**
Boston	25	20	30
Chicago	30	30	35
Miami	35	45	34

Hint: Let X_1, X_2, X_3 denote the number of jets of each type dispatched to Boston; Y_1, Y_2, Y_3 the number of jets of each type dispatched to Chicago; Z_1, Z_2, Z_3 the number of jets of each type dispatched to Miami.

Set this problem up as a linear-programming model.

7-8 The Evergreen Company must select from five possible investment alternatives for the coming year. The net present value of these investments is given below:

Investment	**1**	**2**	**3**	**4**	**5**
NPV	25	10	14	32	16

The objective of the firm is to maximize the net present value of its investment mix; however, several factors constrain its choice of investments. First it must ensure that adequate capital is available for the included investments over the next 2 years, since each of these investments requires cash over both of these years. The cash outflow requirements for each of these investments is:

	Investment				
	1	**2**	**3**	**4**	**5**
Capital outflow, year 1	2	17	14	12	9
Capital outflow, year 2	7	11	14	7	11

The budgeted available capital is 35 in the first year and 30 in the second year.

In addition, these projects require a workforce, and labor is also in scarce supply. Below are given the workforce requirements for each of the investments over the first 3-year period, after which it is felt that an adequate labor supply will be available for whatever investments are selected.

	Investment				
	1	2	3	4	5
Workforce, year 1	7	4	3	8	5
Workforce, year 2	11	2	4	8	4
Workforce, year 3	3	2	2	3	7

Workforce capacity in the first year is 20; in the second year it is 25; and in the third year it is 30.

Set this problem up as a linear-programming model.

7-9 The Wichita Company operates two factories and three warehouses. The output from the factories is shipped to the regional warehouses, and from there the product is distributed to retailers.

The two factories have limited capacities. Factory A can produce no more than 5000 units per month, and Factory B can produce no more than 3000 units per month. The warehouses have just submitted their demand for the next month. Warehouse 1 needs 2000 units, warehouse 2 needs 1500 units, and warehouse 3 needs 4000 units.

Per unit production and shipping costs are given below:

	To warehouse		
From factory	1	2	3
A	\$5	\$6	\$4
B	2	9	5

For example, a unit manufactured in factory A and shipped to warehouse 1 will incur a cost of \$5.

Management is about to develop a production and shipping schedule for next month.

Formulate a linear-programming model that will help management make this decision.

7-10 The Economy Auto Rental Company offers budget car rentals in six Northeastern cities. At the present time the company has more cars than it needs in three cities and a shortage of cars in the other three. Cities 1, 3, and 6 have an excess of 92, 40, and 35 cars, respectively. Cities 2, 4, and 5 require at least 50, 80, and

37 cars, respectively. The cost of transporting each car between these cities is given below.

From \ To	2	4	5
1	25	14	6
3	17	16	12
6	31	21	15

Management must determine the quantity of cars that must be shipped from the oversupplied locations to the understocked locations.

Formulate a linear-programming model which will help management make this decision.

Set this problem up as a linear-programming model.

CASE STUDY: Exeter Office Equipment Company

The Exeter Office Equipment Company manufactures a line of office equipment including file cabinets, desks, chairs, and tables. Production facilities are located in Pittsburgh, Cleveland, and St. Louis, and at the present time all three facilities have some excess capacity. The manufacturing managers from these plants and the corporate vice president of manufacturing have scheduled a meeting in Cleveland next week. Their problem is how to utilize this excess capacity effectively.

Three weeks ago it was learned from the vice president of manufacturing that demand for two types of file cabinets had been exceeding the scheduled quantity in production. The larger unit, model A, has eight drawers and sells for \$25; the smaller unit, model B, has four drawers and sells for \$13.

The manufacturing control department at that time was scheduling 2000 units per day of model A and 3000 units per day of model B. But the vice president of manufacturing suggested that this could be raised to 2600 units per day of model A and 3500 units per day of model B.

The file cabinets have been manufactured only at the Pittsburgh plant. Telephone calls to the manufacturing managers at the other two plants, however, have revealed that they have the capability to produce these models. Both agreed that it would take a week to set up the production line. No new equipment would be needed, and there would be enough foremen to set up and supervise the additional production lines.

Manufacturing costs for each model would be the same in all plants. Labor and raw material costs would be \$13 per unit for model A and \$4 per unit for model B.

According to the manufacturing managers of these plants, the workforce which could be assigned to these products is limited. The Pittsburgh plant could produce at most 500 additional units per day, regardless of the model or combination of the models involved; the Cleveland plant could produce 600 units in total per day; and the St Louis plant could produce 300 units in total per day.

The manufacturing managers were also concerned about their available storage space. The Pittsburgh plant could allocate at most 9000 additional square feet, the Cleveland plant could allocate at most 8000 square feet, and the St. Louis plant could allocate 3500 square feet. Each model A requires 20 square feet and each model B requires 12 square feet. Each plant can store on the average 1 day's worth of production.

None of the manufacturing managers wanted to accept the responsibility for producing all of this additional output. They felt that a sudden drop in demand would have too great an impact on production and employment levels. A preliminary agreement was that no one plant should be assigned more than 40 percent of the additional units, regardless of models.

QUESTIONS

1 Can this problem be modeled? If so, formulate the model.
2 What will the output of the model tell you?
3 To what extent will the output be useful when the managers meet next week in Cleveland?
4 The St. Louis plant's production facility can rent an additional 5000 feet of storage space at a cost of $20 per day. Is there some way that the model could be used to determine whether this space should be rented?
5 Upon hearing of these plans the vice president of marketing has sent a letter to the vice president of manufacturing. See Exhibit A. Reformulate the model to take into account the marketing strategy. How can the outcome of these models be compared to select between these two alternatives?
6 If a computer code is available, solve questions 1, 4, and 5 on the computer. Is the marketing strategy more profitable?

Exhibit A

TO: Robert Scott, VP Manufacturing

FROM: Frank Anderson, VP Marketing

SUBJECT: Demand for file cabinets

It has recently come to my attention that you are considering the expansion of production capacity for models A and B. Before you make any moves in that direction, perhaps you should hear our plans.

Since delivery of these units to our customers has been less than acceptable, we have decided to change our pricing strategy. We expect this to reduce the level of demand.

Effective next month model A will sell for $26 while model B will sell for $14. This should reduce demand for A to at most 2200 units per day and to 3300 units per day for model B.

Before you formalize your expansion plans, you should give me a call.

Chapter 8

Linear Programming: Graphical Method

INTRODUCTION

In Chapter 7 the focus was on the formulation of linear-programming models. Now we will turn to the ways in which these models can be analyzed.

When linear-programming models have only two variables or activities, as in the Jerry Company case, graphical methods can be useu to analyze the model. These methods are simple and, above all, provide valuable insights into the more advanced methods that are required for models having more than two variables. The advanced methods will be covered in Chapter 9.

THE FOUR STEPS OF THE GRAPHICAL METHOD

There are four basic steps in the graphical method. They are:

1. Outlining the graph
2. Drawing the constraints
3. Drawing a series of objective functions
4. Identifying the solution

These steps will now be illustrated by the Jerry Company case.

OUTLINING THE GRAPH

The first step is to identify the two activities or variables on the axes of a graph. In Figure 8-1 this has been done for camping (C) and mountain (M) tents.

DRAWING THE CONSTRAINTS

Fabric-cutting Constraint

First the fabric-cutting constraint will be drawn. To accomplish this, two end points are identified in the following way. Suppose all the available time (8000 worker-hours) in the fabric-cutting shop is devoted to making camping tents. Since it takes 2 worker-hours of fabric-cutting to make one camping tent, at most 4000 of them can be processed.

$$^{8000}/_2 = 4000$$

One end point therefore is $M = 0$; $C = 4000$. This point is marked on the graph. Now we turn to the other end point. Suppose the entire capacity of the fabric-cutting shop is devoted to producing mountain tents. Through this shop we could then process a maximum of $^{8000}/_1 = 8000$ tents. The second end point is therefore $M = 8000$; $C = 0$. It is also plotted on the graph.

The next step is to connect these two end points with a straight line, and this line becomes the fabric-cutting constraint. We can interpret all points

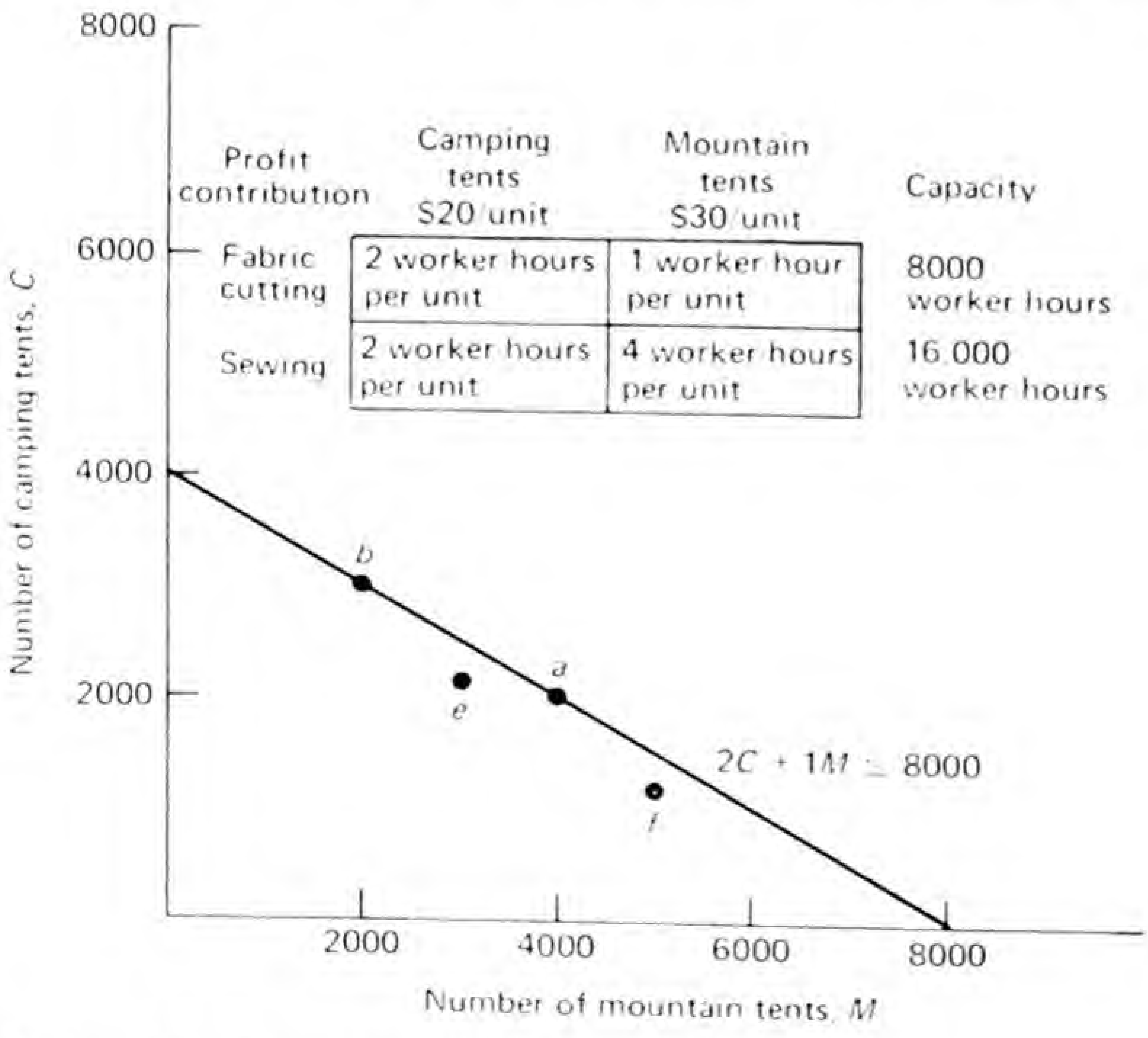


Figure 8-1 Drawing the first constraint.

which lie *on* this line as a combination of camping and mountain tents which will *exactly* use up all the capacity in the fabric-cutting shop. For example, consider point *a*. It lies on the line and represents the processing of 2000 camping and 4000 mountain tents. This combination requires

$$2(2000) + 1(4000) = 8000$$

worker-hours of capacity and therefore exactly uses up the capacity of the shop. As another example, consider point *b*, where

$$C = 3000$$
$$M = 2000$$

This combination will require

$$2(3000) + 1(2000) = 8000$$

worker-hours and again uses up the capacity of the shop. We can therefore conclude that points which lie on this line represent some combination of camping and mountain tents which will exactly utilize the available capacity.

Our constraint, however, was specified as an inequality and as such allowed less than the capacity of the shop to be used. Therefore a point such as *e does* indeed represent a possible or *feasible combination* of products in that shop. At *e* we have 2000 camping and 3000 mountain tents. Together they utilize

$$2(2000) + 1(3000) = 7000$$

or 7000 of the 8000 available hours. Point *f* also represents another feasible combination of products which uses up less than the available capacity. In fact we can say that the entire space lying to the left and below the line (and including the line itself) represents feasible combinations of camping and mountain tents. That is, these combinations will either use up less than or exactly the available capacity but never more. In Figure 8-2 this region is shown shaded.

Next the sewing constraint will be graphed. While this is done, the fabric-cutting constraint which we have just graphed will be temporarily ignored. Our strategy will be to graph each constraint separately. Eventually they will be combined.

Sewing Constraint

To draw the sewing constraint, we simply repeat the steps followed for the fabric-cutting constraint. First the end points are located, then they are connected with a straight line, and finally the region representing all feasible combinations is shaded.

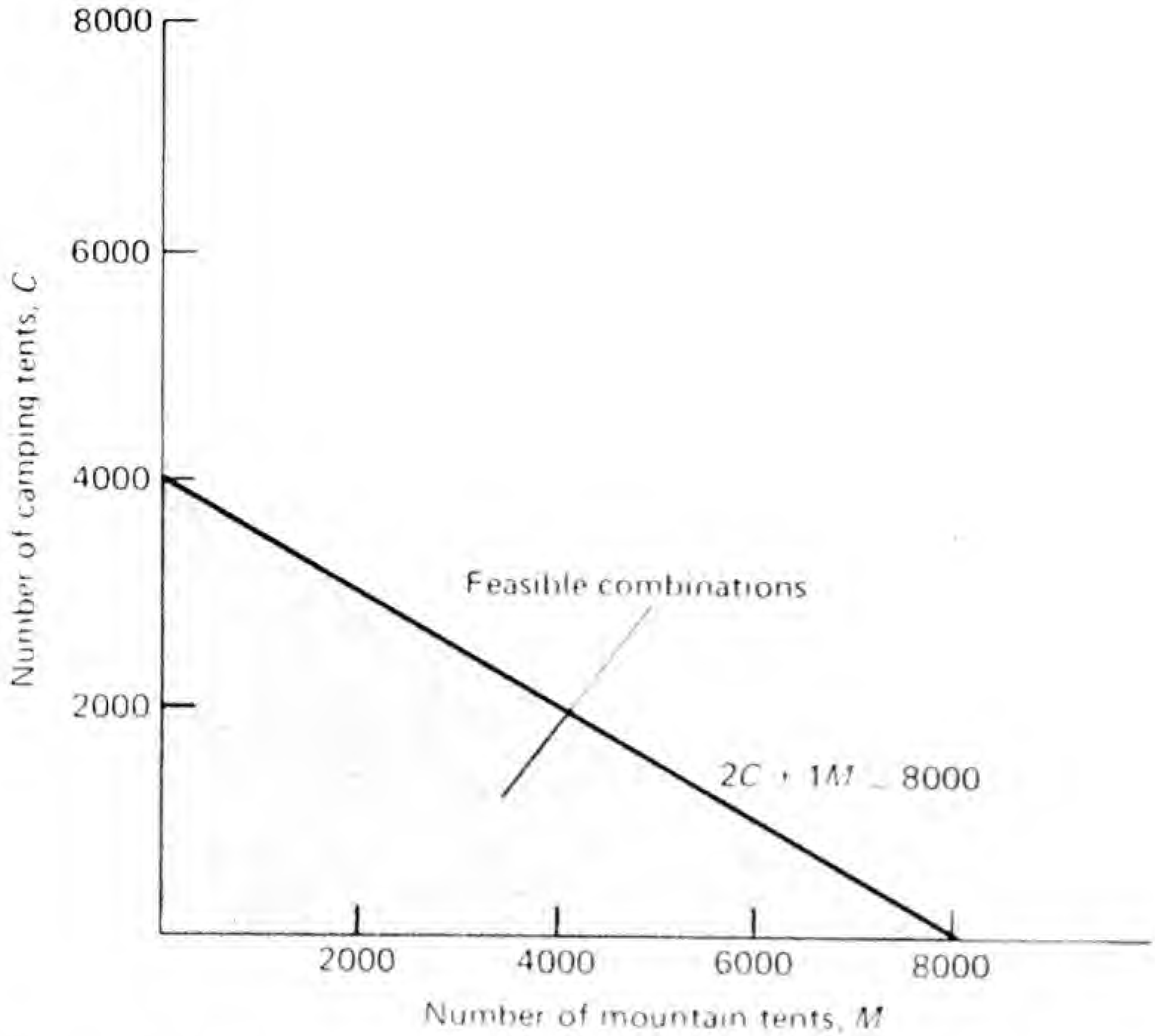


Figure 8-2 Feasible combinations for the first constraint.

To find the first end point, we assume that only camping tents will be produced. This being the case, the most that could be produced would be $^{16,000}/_{2} = 8000$ tents. The first end point is therefore $C = 8000$; $M = 0$. This point is drawn in Figure 8-3. To find the second end point, we ask how many mountain tents could be produced if no camping tents were scheduled. We could produce as many as $^{16,000}/_{4} = 4000$ tents. This point is also recorded in Figure 8-3. Now a line joining these points can be drawn and any point *on* this line represents some combination of camping and mountain tents which will *exactly* use up the available capacity in the sewing shop.

Points to the left and below the line represent combinations of camping and mountain tents which will require less than the 16,000 worker-hours of productive capacity. The feasible combinations for this shop therefore include the line and area below it. Any combination of products which falls within this region can be feasibly processed through the sewing shop.

Feasible Region

Now we must step back and consider the *combined* capabilities of the fabric-cutting and sewing departments. To accomplish this, both constraints are drawn on the same graph. This is done in Figure 8-4. First we investigate point *a*. It represents the processing of 2000 camping and 2000 mountain tents. From our analysis of the fabric-cutting constraint, we know that such a combination is feasible in that department. That is, the point lies below and to the left of the constraint. Furthermore we also know that this combination is

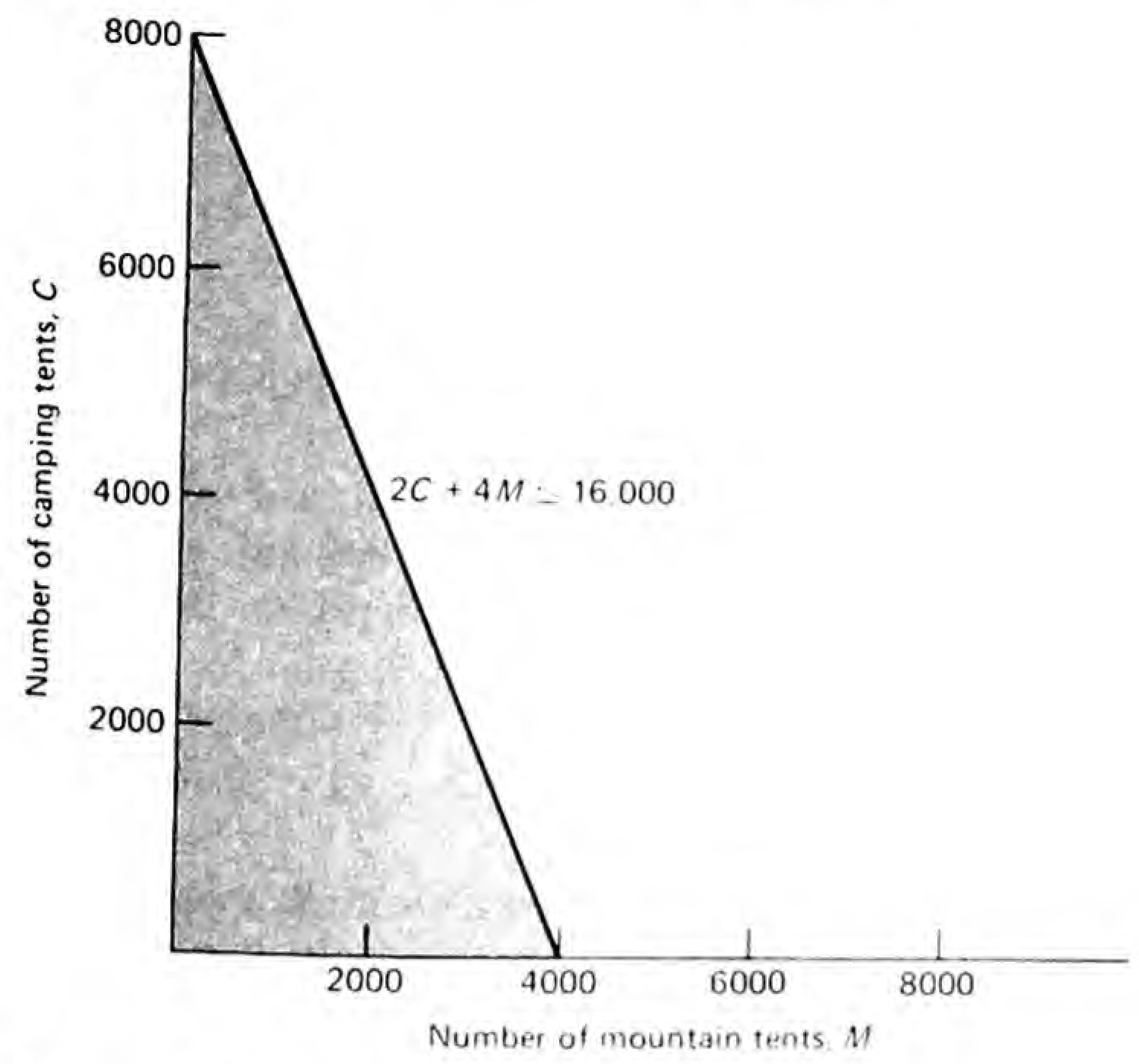


Figure 8-3 Sewing constraint.

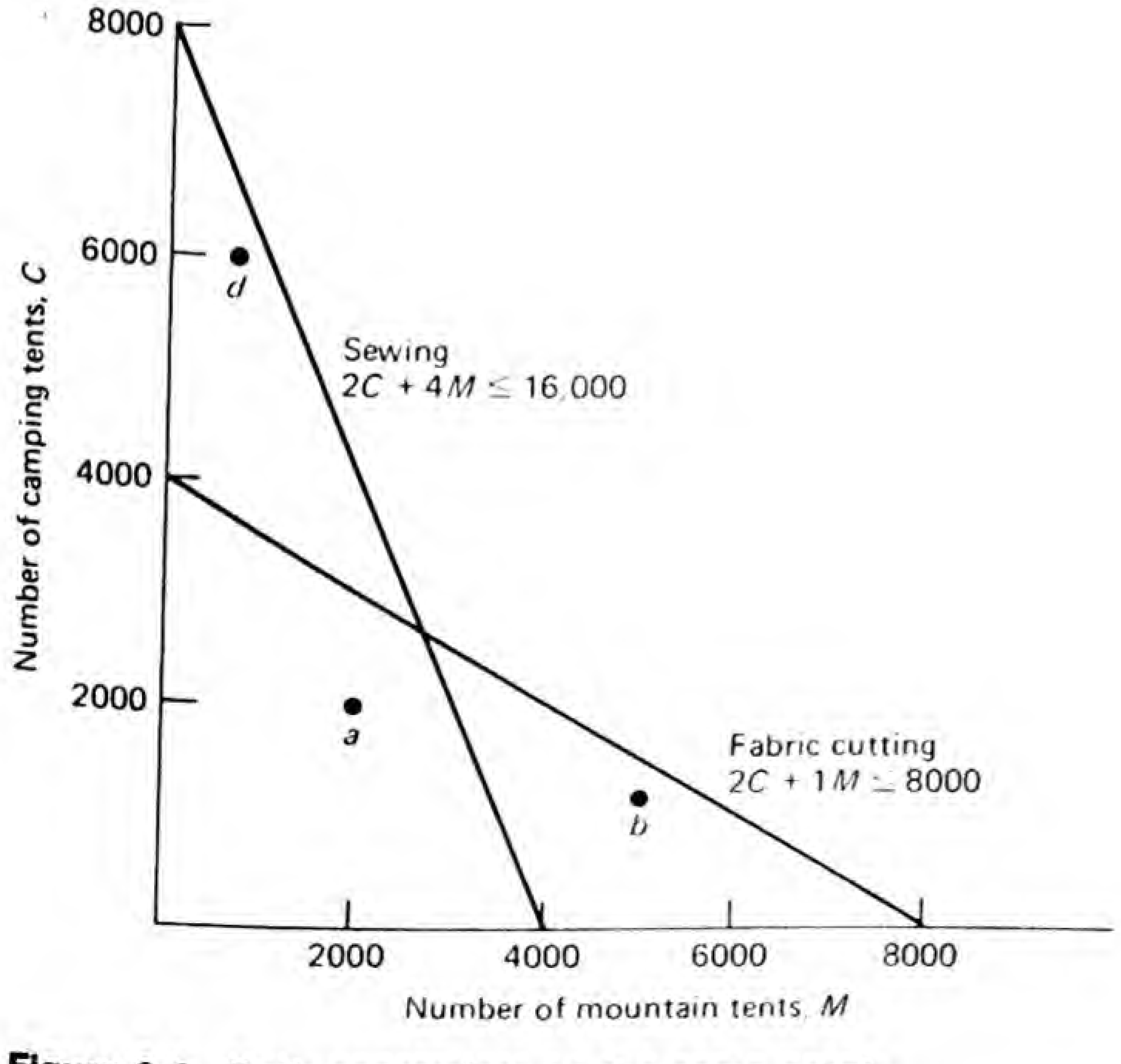


Figure 8-4 Both constraints on the same graph.

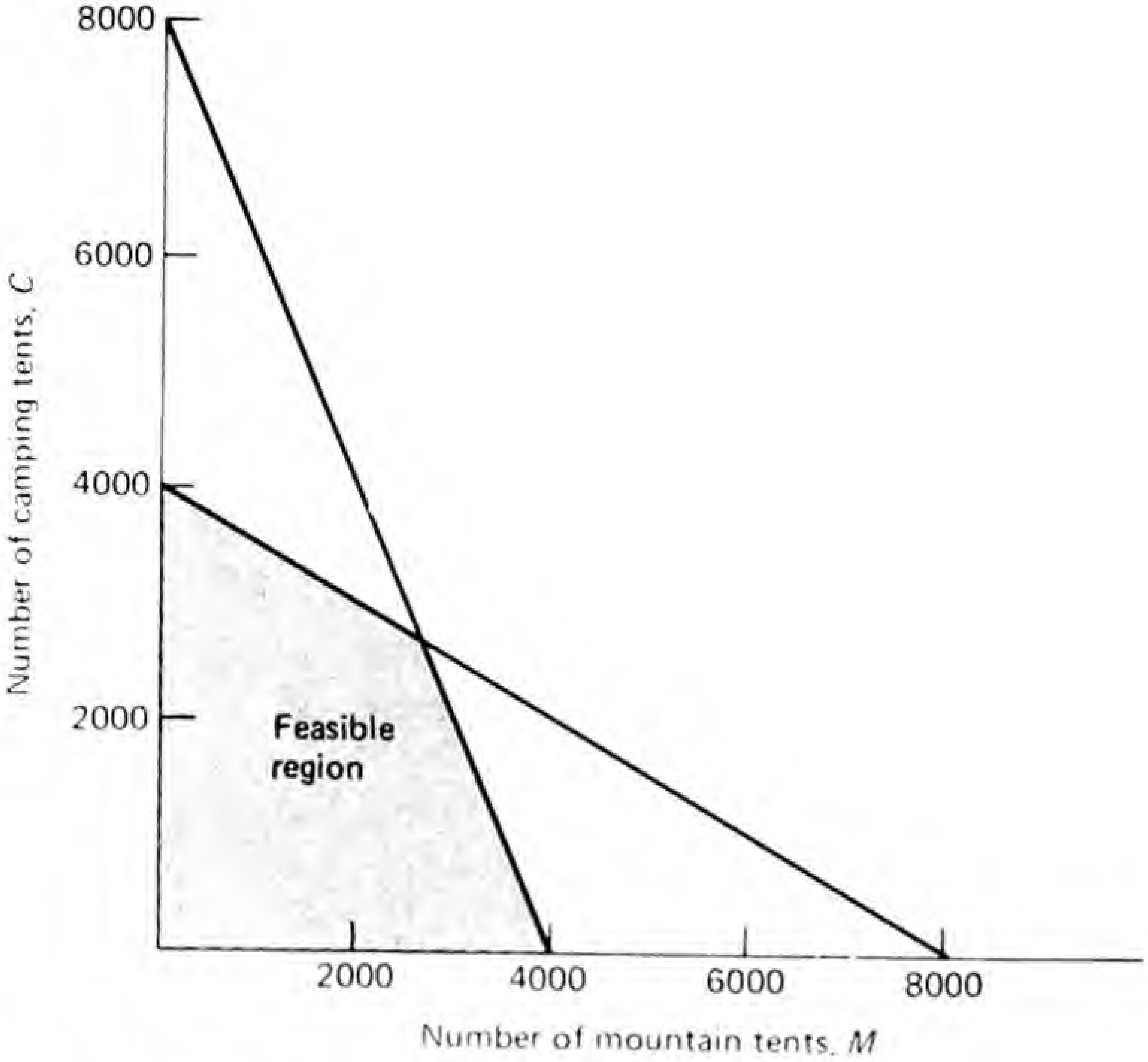


Figure 8-5 Feasible region.

feasible for the sewing department since point *a* also lies below and to the left of the sewing constraint. We can therefore conclude that the output combination represented by point *a* is feasible in both departments. Next we will examine point *b*. This output combination is feasible in the fabric-cutting department since it lies below and to the left of the constraint. This is not true, however, for the sewing department. The capacity in this department is exceeded because point *b* lies beyond its shaded area of feasible combinations. Therefore the combination represented by point *b* violates one of the constraints and as such is not an admissible combination. Point *d* also violates a constraint. Although the combination represented by this point is feasible in the sewing department, it is not in the fabric-cutting department.

Perhaps you have already discovered which area is feasible to both departments. It is that area which both constraints share in common. This is shown in Figure 8-5. The common area is often referred to as the *feasible region*. Any point within this space and including its borders represents a combination of camping and mountain tents which can be processed within the constraints of *both* departments.

What remains is to find the best or optimal point in this feasible region.

GRAPHICAL PORTRAYAL OF THE OBJECTIVE FUNCTION: ISOPROFIT LINES

Up to this point we have considered only those factors which constrain output. To this we must now add profit data, and then we will be able to

determine where in this feasible region the highest profit can be obtained. To accomplish this, several profit lines are constructed. They are drawn at arbitrary locations on the graph in the following way. *Suppose* we produced 3000 camping tents (the 3000 is an arbitrary choice) and no mountain tents. The profit generated would be

$$P = 20(3000) + 30(0) = \$60{,}000$$

Next we ask how many mountain tents would have to be produced to generate this \$60,000. The answer is $^{60{,}000}/_{30} = 2000$ tents. Now two points are identified on Figure 8-6.

$$\left.\begin{matrix} C = 3000 \\ M = 0 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} C = 0 \\ M = 2000 \end{matrix}\right.$$

Connecting them with a straight line, we have what is called an *isoprofit* line. An isoprofit line is a line of equal profits such that any point *on* the line represents some combination of camping and mountain tents which will *exactly* generate \$60,000.

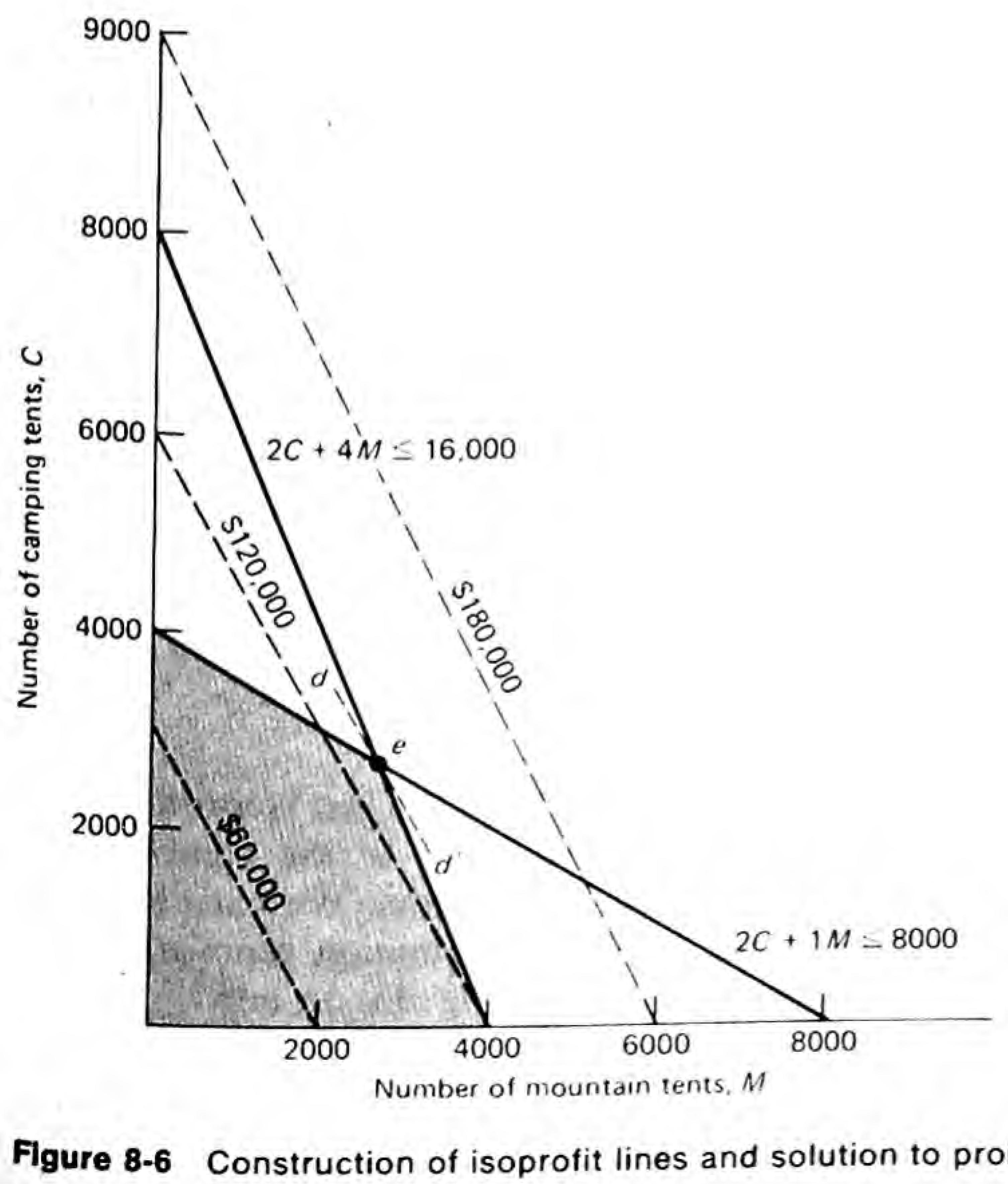


Figure 8-6 Construction of isoprofit lines and solution to problem.

Again, another isoprofit line will be drawn. This second arbitrary choice will be 6000 camping tents. Its associated profit is

$$P = 20(6{,}000) + 30(0) = \$120{,}000$$

Once again we ask how many mountain tents would have to be produced to generate this profit. The answer is $^{120{,}000}/_{30} = 4000$ tents. The two new points are therefore

$$\left.\begin{matrix} C = 6000 \\ M = 0 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} C = 0 \\ M = 4000 \end{matrix}\right.$$

They are also plotted in Figure 8-6 and then connected by a straight line. Points on this isoprofit line represent combinations of camping and mountain tents that will generate exactly $120,000 of profit.

Characteristics of Isoprofit Lines

Countless other isoprofit lines could be drawn in the same manner, and we would find that they share common characteristics. First, isoprofit lines are all parallel to each other. Second, isoprofit lines of higher profit lie to the right and above isoprofit lines of lower profit. Since our objective is to maximize profit, we would favor isoprofit lines of higher profit. That is, scheduling some combination of output which falls on the $180,000 isoprofit line is more attractive than the $60,000 isoprofit line. The problem with choosing the $180,000 isoprofit line, however, is that nowhere does it cross the feasible region. Consequently there is no combination of camping and mountain tents which can be feasibly processed by both departments and also generate $180,000 profit. All along the $60,000 isoprofit line, however, lie numerous feasible combinations. Therefore we *could* choose to schedule a combination which falls on this line, but why not operate on a higher isoprofit line? The $120,000 isoprofit line passes through the feasible region. In fact even higher profits can be achieved by isoprofit lines which lie to the right and above this $120,000 line. On which of these shall we operate? On the highest one which is just about to leave the feasible region.

SOLUTION

From Figure 8-6 it can be seen that isoprofit line dd' is the last isoprofit line to intersect the feasible region. Isoprofit lines which lie to the right do not contain any points which are feasible to both constraints. We can therefore conclude that profit is maximized if the output combination represented by point e is scheduled. From the graph we can estimate this to be

$$C = 2666$$
$$M = 2666$$

and the profit associated with this solution is

$$P = 30(2666) + 20(2666)$$
$$P = \$133{,}333$$

Of course, the better the graph, the more accurate the answer. In Appendix A to this chapter a method is shown whereby the exact point can be computed algebraically.

One final hint: It is generally unnecessary to draw more than one isoprofit line in the search for the optimal point. Since all isoprofit lines are parallel to one another, it is usually possible to identify the optimal point by moving a straightedge away from the isoprofit line until it just leaves the feasible region.

Problem

Now that you have seen one problem solved, take the problem which follows and solve it. So that you will have a way to check your work, the complete solution is given after the problem.

$$\text{Max } P = 10A + 10B$$

subject to the constraints:

$$\begin{aligned} 1A + 2B &\le 12 \\ 2A + 1B &\le 12 \\ A \quad\; &\ge 0 \\ B &\ge 0 \end{aligned}$$

Solution

First the constraints are drawn. These are shown in Figure 8-7.

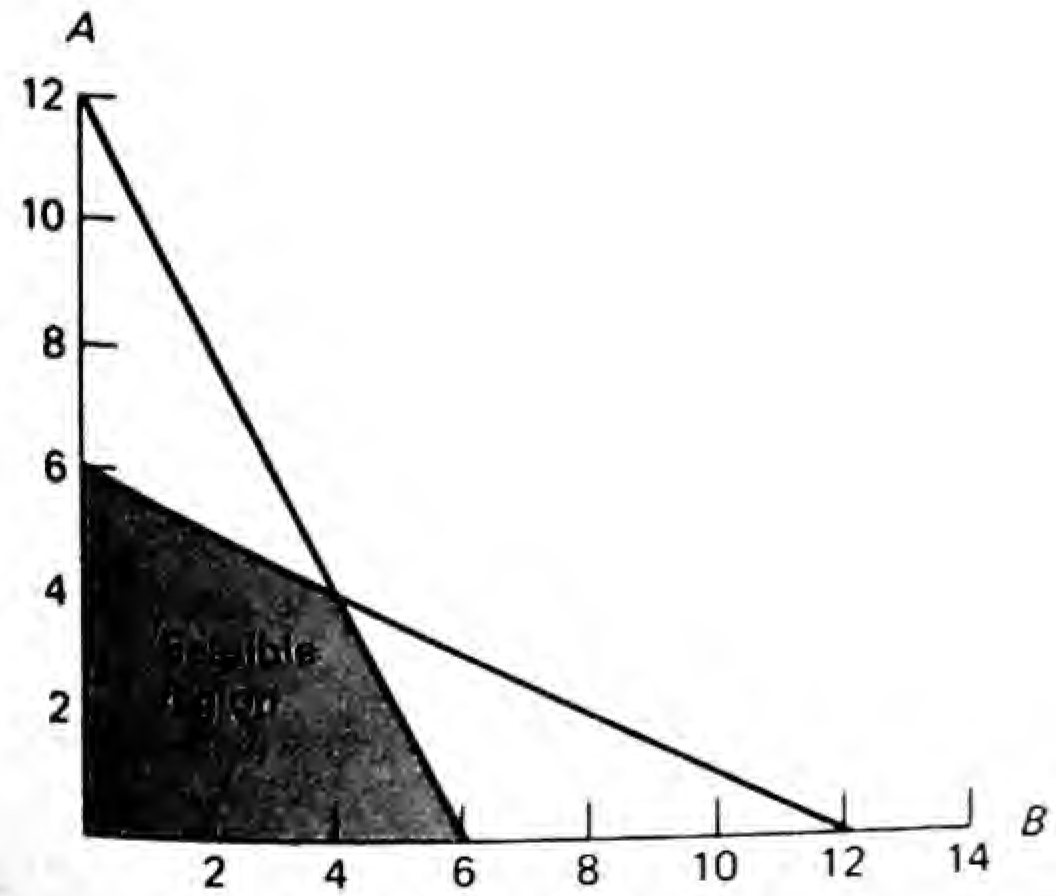


Figure 8-7 Constraints and feasible region.

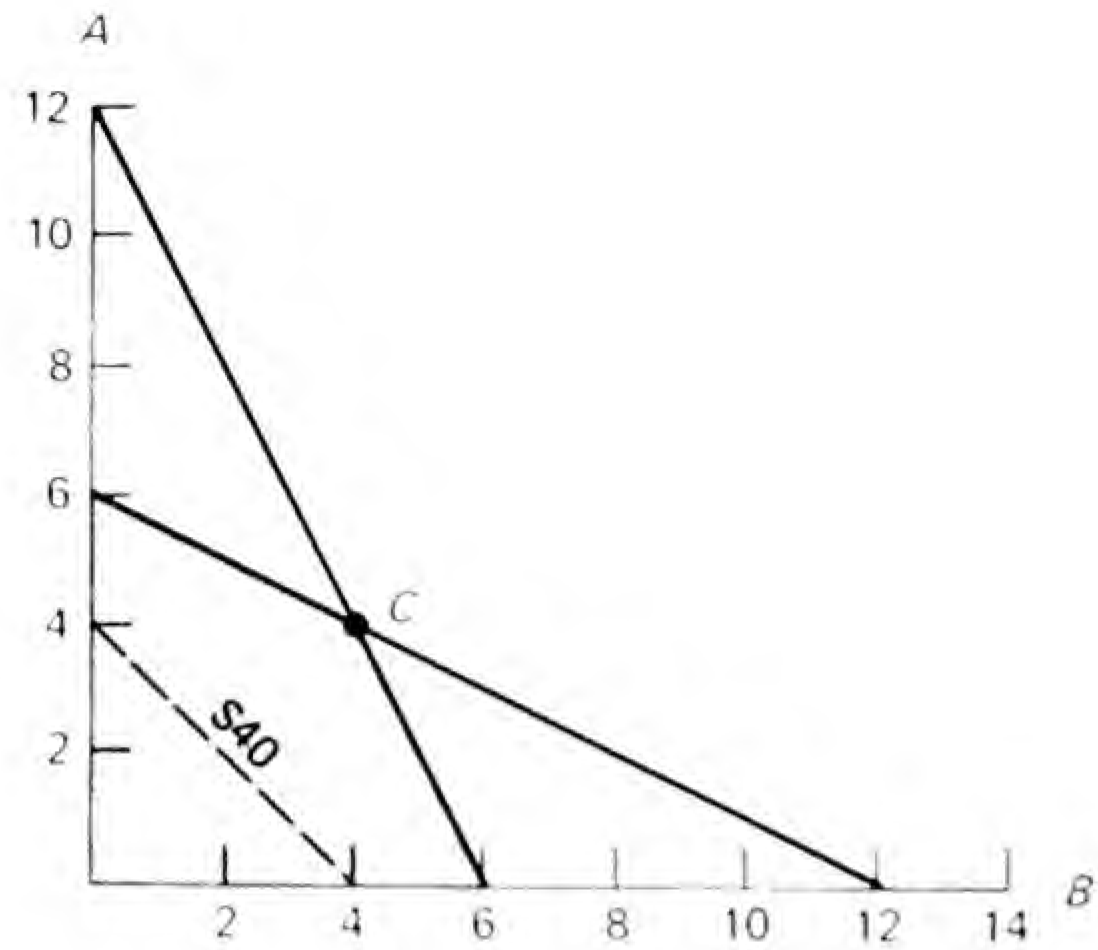


Figure 8-8 Solution to example.

Next an isoprofit line is drawn. This is shown in Figure 8-8. From the isoprofit line we can see that profit will be maximized at point C or

$$A = 4$$
$$B = 4$$

SUMMARY

Graphical methods can be used to solve linear-programming problems which have two variables. First these variables are identified on each axis, then the constraints are drawn, next an isoprofit line is drawn, and finally the extreme point is identified where the highest isoprofit line will just leave the feasible region.

In Chapter 9, the simplex method is developed for solving problems which have more than two variables.

QUESTIONS

1 Define the following terms:
  a Feasible combination
  b Feasible region
  c Isoprofit line

2 Why is graphic analysis inadequate for a model with more than three variables or activities?

3 Will the solution of a graphical linear-programming problem always occur at a corner of the feasible region? Why?

4 Is there a limit to the number of constraints that can be handled in a graphical solution?

5 Is it possible to solve the problem shown in Figure 8-5 without drawing an isoprofit line?

PROBLEMS

8-1 Identify the feasible combinations for *each* of the following constraints:

a $3A + 2B \leq 12$

b $A + B \leq 10$

c $2A + 1B = 14$

d $2A + 0B = 12$

8-2 Identify the feasible region for the following *set* of constraints:

$$\begin{aligned} 2X + 1Y &\leq 10 \\ 2X + 4Y &\leq 20 \\ Y &\leq 8 \\ X &\leq 7 \\ X &\geq 0 \\ Y &\geq 0 \end{aligned}$$

8-3 Identify the feasible region for the following *set* of constraints:

$$\begin{aligned} 4X + 1Y &\leq 16 \\ 2X + 8Y &\leq 32 \\ X + Y &\leq 20 \\ X &\geq 0 \\ Y &\geq 0 \end{aligned}$$

8-4 Solve the following problem:

$$\begin{aligned} \text{Max } P = 3A + 4B& \\ 2A + 1B &\leq 16 \\ 4A + 10B &\leq 20 \\ A &\geq 0 \\ B &\geq 0 \end{aligned}$$

8-5 Solve the following problem:

$$\begin{aligned} \text{Max } P = 2A + 2B& \\ 1A + 2B &\leq 10 \\ 2A + 1B &\leq 10 \\ A &\geq 0 \\ B &\geq 0 \end{aligned}$$

8-6 Solve the following problem:

$$\begin{aligned} \text{Max } P &= 2A + 3B \\ 2A + 3B &\le 12 \\ 3A + 1B &\le 12 \\ A \quad &\ge 0 \\ B &\ge 0 \end{aligned}$$

8-7 Solve the following problem:

$$\begin{aligned} \text{Max } P &= 4A + 3B \\ 2A + 3B &\le 12 \\ 3A + 2B &\le 12 \\ A + B &\le 5 \\ A \quad &\ge 0 \\ B &\ge 0 \end{aligned}$$

8-8 Return to the Jerry Company case and determine the solution if the demand for camping tents was limited to 2500 and the demand for mountain tents was limited to 2000. What excess capacity will exist, if any, in both the sewing and fabric-cutting departments?

8-9 The G-M Bicycle Company produces two types of bicycles. Each one requires four operations. Owing to the recent emphasis on physical fitness, management feels that it can sell as many as are produced. If the profit from bicycle A is \$20 per unit and that from B is \$35, how many of each should the company produce? The maximum number of bikes which can be processed through each department for either bike A or bike B is:

	Bike A	Bike B
Stamping	1200	1400
Assembly	1000	900
Paint	800	1200
Inspection	1000	700

8-10 The Biltmore Company produces golf carts G and snowmobiles S. Per unit profit for each G produced is \$100, and per unit profit for each S produced is \$200. Sales have been so good lately that management thinks all units of G and S produced can be sold successfully. Constraining output, however, is capacity in three departments. In the machine shop each G processed requires 3 worker-hours of labor and each S processed requires 2 worker-hours of labor. Capacity in this shop is 30,000 worker-hours over the planning horizon of interest (3 months). In the assembly department each G requires 3 worker-hours and each S requires 4 worker-hours. Capacity in this shop is 40,000 worker-hours. In the painting shop each G requires 1 worker-hour, and each S requires 1 worker-hour. The capacity of this shop is 10,000 hours. How many of each should be produced to maximize profit?

CASE STUDY: Zamyad Company[1]

The Zamyad Company of Teheran, Iran, produces cars under an agreement with Volvo of Sweden and trucks under an agreement with Nissan Motors of Japan. The company was established in 1966 and has recently moved into spacious quarters on the Karaj road about 20 kilometers outside Teheran. It employs approximately 1000 people and generally produces an average of 30 cars and trucks per day.

Capital investment constraints have limited the nature of Zamyad's manufacturing facilities. Consequently, it is not able to manufacture many of the items required for the assembly of automobiles and trucks. These items are imported from Volvo or Nissan. However, both Nissan and Volvo must limit the quantities of parts shipped to Zamyad because of constraints on their own capacities. Volvo has guaranteed to supply parts sufficient for the assembly of up to 500 cars per month; Nissan has guaranteed to provide parts sufficient for up to 200 trucks per month.

Nissan has just announced several price increases, which have raised the direct manufacturing cost of a Zamyad truck from $800 to $1000.[2] (Direct manufacturing costs include all labor and materials.)

Volvo has not raised prices on purchased parts, so the direct manufacturing cost of a car has remained stable at $800. Zamyad assesses overhead at 100 percent of direct manufacturing cost.

The Ministry of Economics controls the selling price of Zamyad's output: cars sell for $4300 and trucks sell for $6000.

Zamyad's vehicles have a reputation as well-made and dependable products. Demand is so great that the company can sell all the cars and trucks it can produce, and the company expects no change in this situation. Zamyad presently has unfilled orders (already paid) for 150 cars and 100 trucks.

The manufacturing process for both cars and trucks consists essentially of two departments, which limit the number of vehicles which can be produced during any month. These departments are fabrication and assembly. An agreement with the Ministry of Labor has set the minimum labor usage at 14,000 worker-hours for the combined departments.

The fabrication department is organized as a job shop which produces hundreds of different parts on 45 machine tools. A recent analysis has shown that this shop can plan on no more than 12,000 worker-hours of capacity in the coming month. Each car manufactured requires 20 worker-hours of fabrication; each truck requires 40 worker-hours.

The assembly department is set up as a conventional assembly line: vehicles are assembled on trolleys which are pushed by hand from one station to the next. About 10,000 worker-hours of capacity will be available in the

[1] By Barry Shore and Linda G. Sprague.

[2] The Iranian unit of currency is the rial; the 1974 exchange rate was approximately 67.25 rials to the dollar. All figures in this case have been converted to dollars.

assembly department in the coming month. Each car requires 25 worker-hours of assembly; each truck requires only 10 worker-hours.

At this morning's management meeting, Farah Hormozi, the production manager, expressed considerable concern over Nissan's price increases. The next month's production schedule was to be announced tomorrow, and she asked Ali Reza Rastegar, managing director, whether the cost change should affect the planned schedule. Mr. Rastegar replied, "I think we'll just have to absorb the price increases until the Ministry of Economics allows us to increase our selling price. So, let's go ahead with your previous plan—200 cars and 200 trucks."

QUESTIONS

1 Can this problem be modeled in a linear-programming format? If so, formulate the model.
2 Solve the model graphically.
3 Compare the results of the model with the current production plan.
4 Discuss the effects on the optimal solution to Zamyad's production-mix problem if these situations occur:
   a Increasing overhead expenses have caused the accounting department to raise the overhead assessment to 120 percent of direct manufacturing costs.
   b An error in the record keeping indicates that the number of back-ordered trucks is only 85.
   c The engineering department has found a new procedure for assembling the automobiles that will reduce the time for assembly to 20 hours per automobile. This change does not apply to truck production.
   d Owing to a dock strike only 50 percent of the expected parts from the Nissan plant will arrive this month.
   e You have received word indirectly that the Minister of Labor will relax your labor restriction by 5000 hours per month.

APPENDIX A: Solution by Simultaneous Equations

Rather than reading the approximate answer from a graph, it is possible to solve, by the simultaneous equation method, for the exact answer. After constructing an isoprofit line, it should be clear which *two equations* will intersect to form the optimal point. In Figure 8-6 these included the following:

$$2C + 4M = 16{,}000$$
$$2C + 1M = 8000$$

They can be solved by eliminating one variable and solving for the remaining one. This can be accomplished by multiplying the second equation by -1 and then adding the two equations together.

$$\begin{array}{rcr} 2C + 4M &=& 16{,}000 \\ -2C - 1M &=& -8{,}000 \\ \hline 3M &=& 8{,}000 \\ M &=& 2{,}666 \end{array}$$

We can now substitute M back into either equation and solve for C.

$$2C + 4(2666) = 16{,}000$$
$$C = 2666$$

Our solution, then, is:

$$M = 2666$$
$$C = 2666$$

PROBLEMS

8A-1 Find the exact solution to problem 8-4 using the simultaneous equation method.
8A-2 Find the exact solution to problem 8-5 using the simultaneous equation method.
8A-3 Find the exact solution to problem 8-6 using the simultaneous equation method.
8A-4 Find the exact solution to problem 8-7 using the simultaneous equation method.

APPENDIX B: Minimization

Often the problem is stated in terms of the minimization of an objective function. For example, in the blending of livestock feed the objective is to minimize costs while meeting minimum daily nutritional requirements. In production scheduling the problem may be stated as the minimization of production costs; in a media selection problem the objective may be to minimize the expenditure of advertising funds.

A MINIMIZATION PROBLEM

Consider the following problem:

$$\text{Min } C = 4A + 4B$$

subject to the constraints:

$$2A + 1B \geq 10$$
$$1A + 2B \geq 8$$
$$A, B \geq 0$$

Now we turn to solve the problem graphically.

Constraints

First the constraints must be drawn. Since these are "greater than or equal to constraints," feasible combinations lie to the right and above the constraint. The first constraint is shown in Figure 8B-1.

The second constraint is shown in Figure 8B-2. Both constraints are combined to generate the feasible region in Figure 8B-3.

Isocost Lines

Next we must construct an arbitrary isocost line, which is much like an isoprofit line except that it is a line of equal cost. One such isocost line is shown in Figure 8B-4. Since the objective is to minimize costs, isocost lines to the left and below our $40 line are even more desirable. The isocost line which passes through point D is the lowest one which still intersects the feasible region.

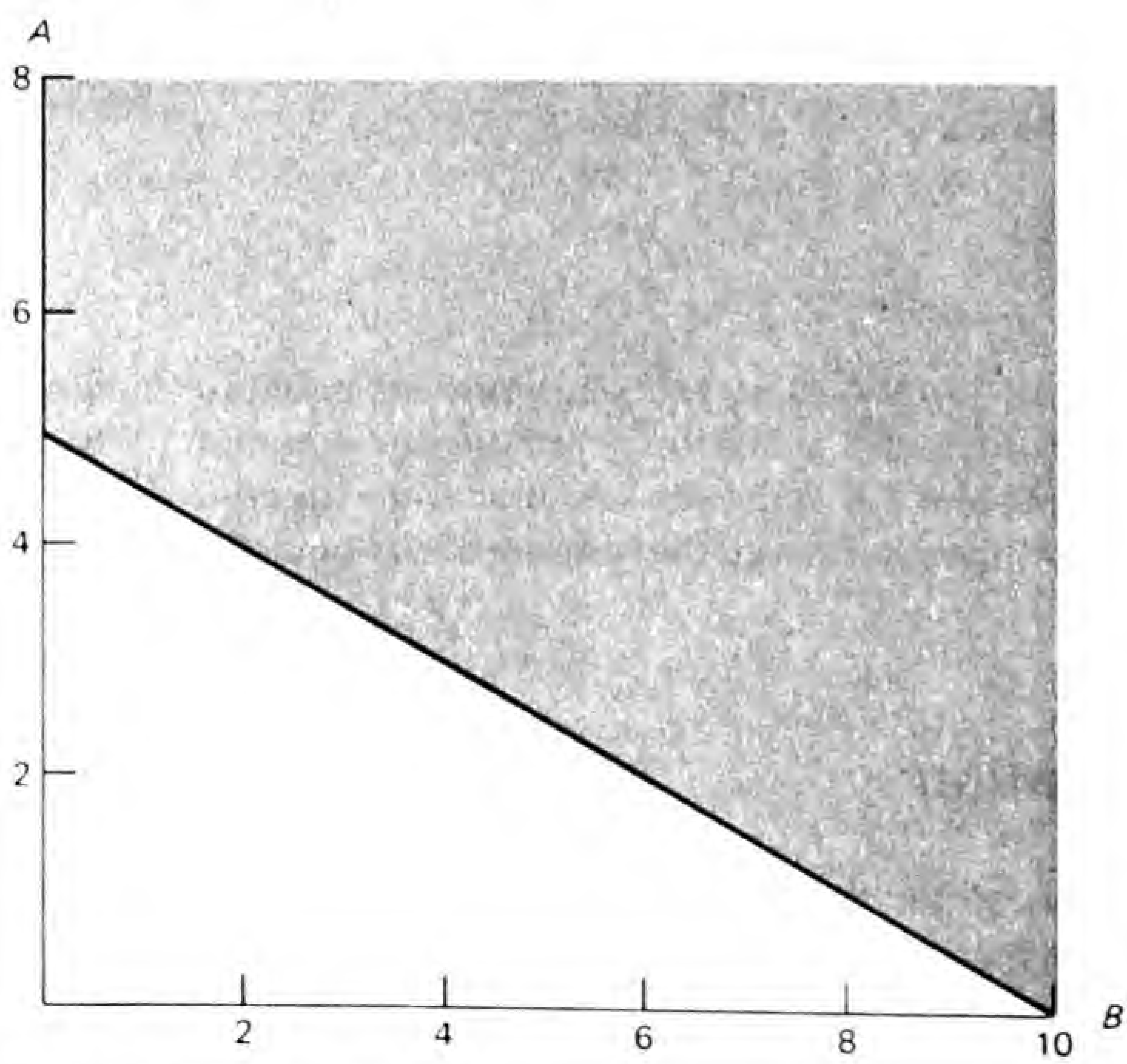


Figure 8B-1 The first constraint.

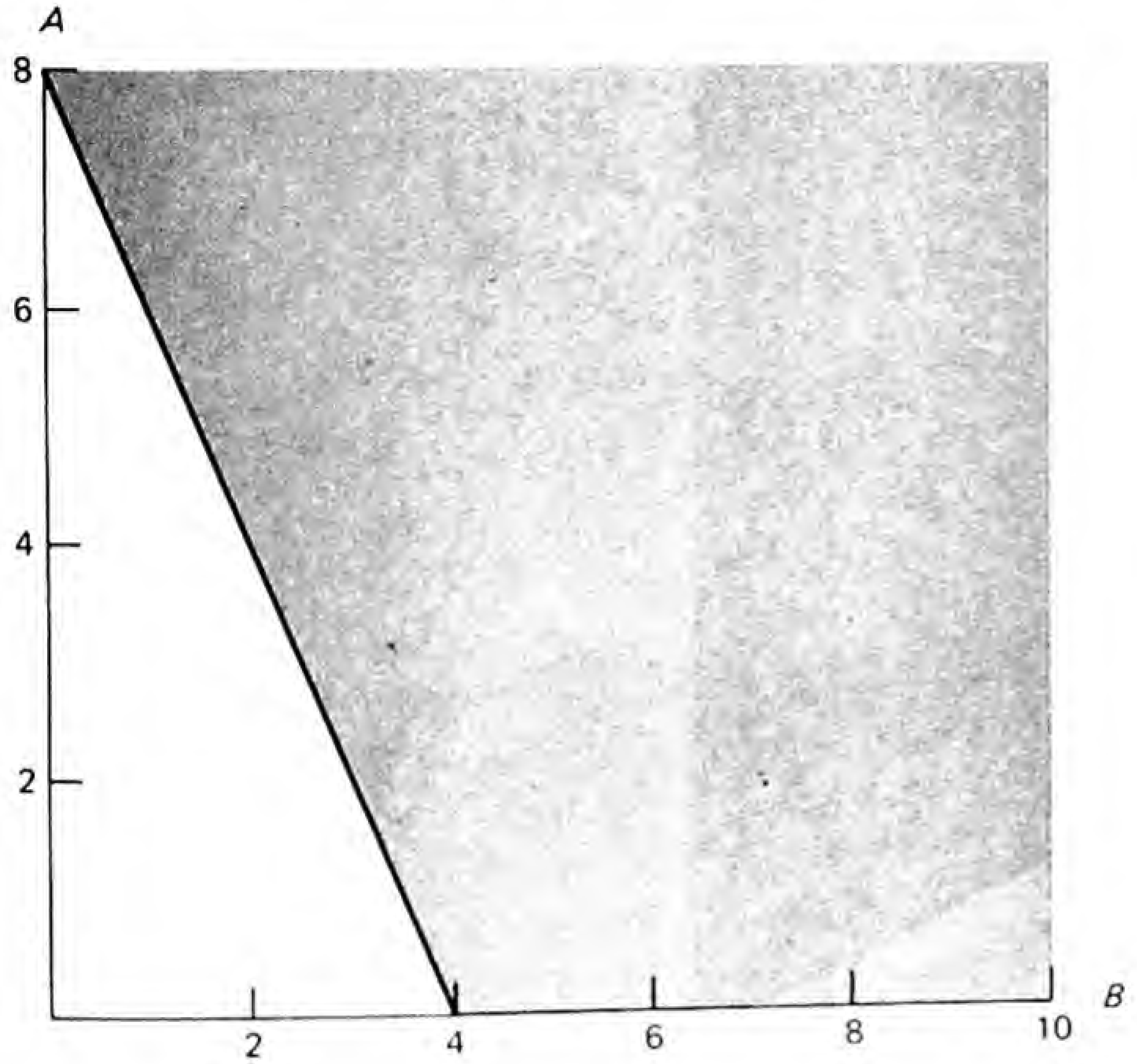


Figure 8B-2 Second constraint.

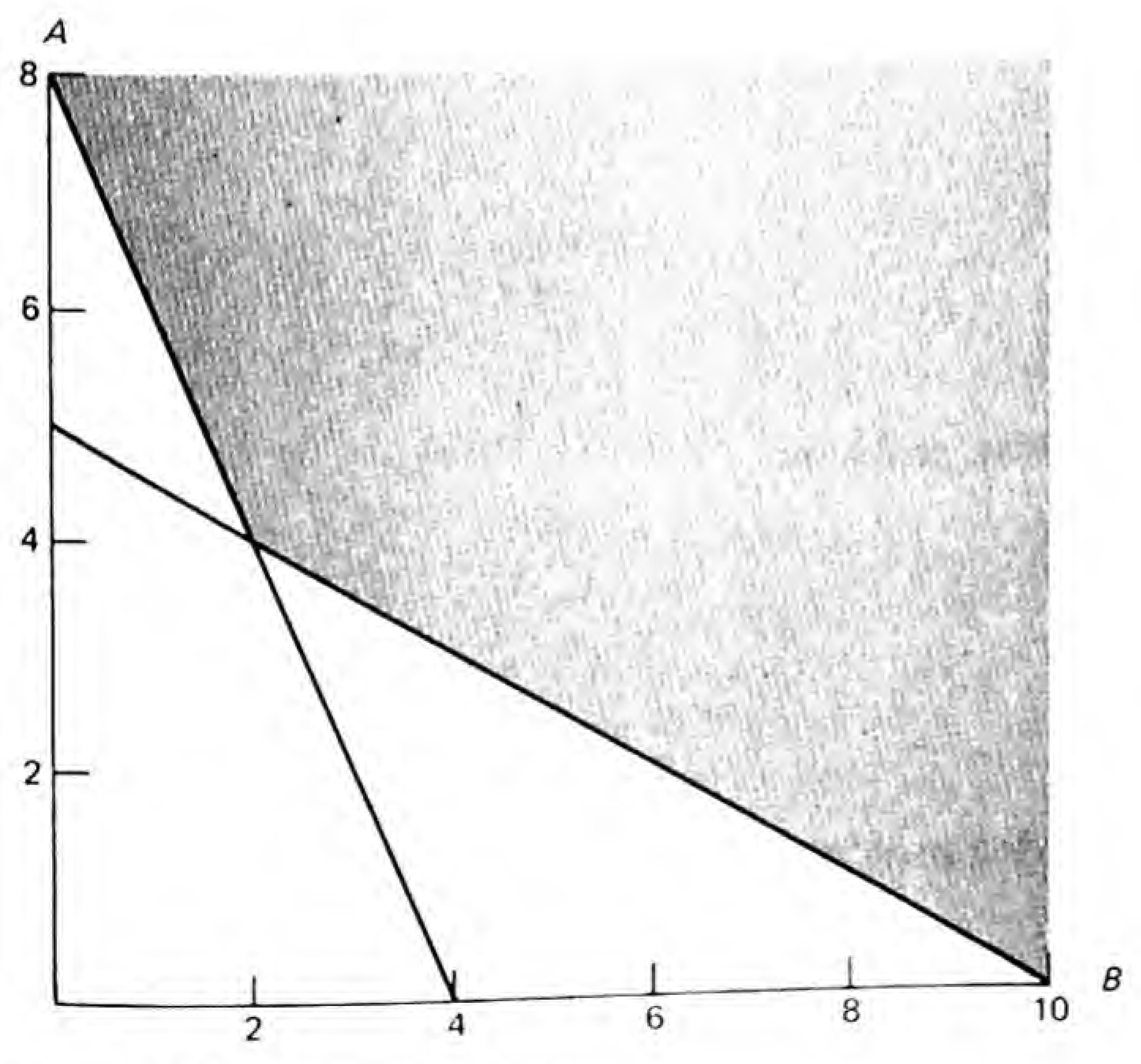


Figure 8B-3 Feasible region.

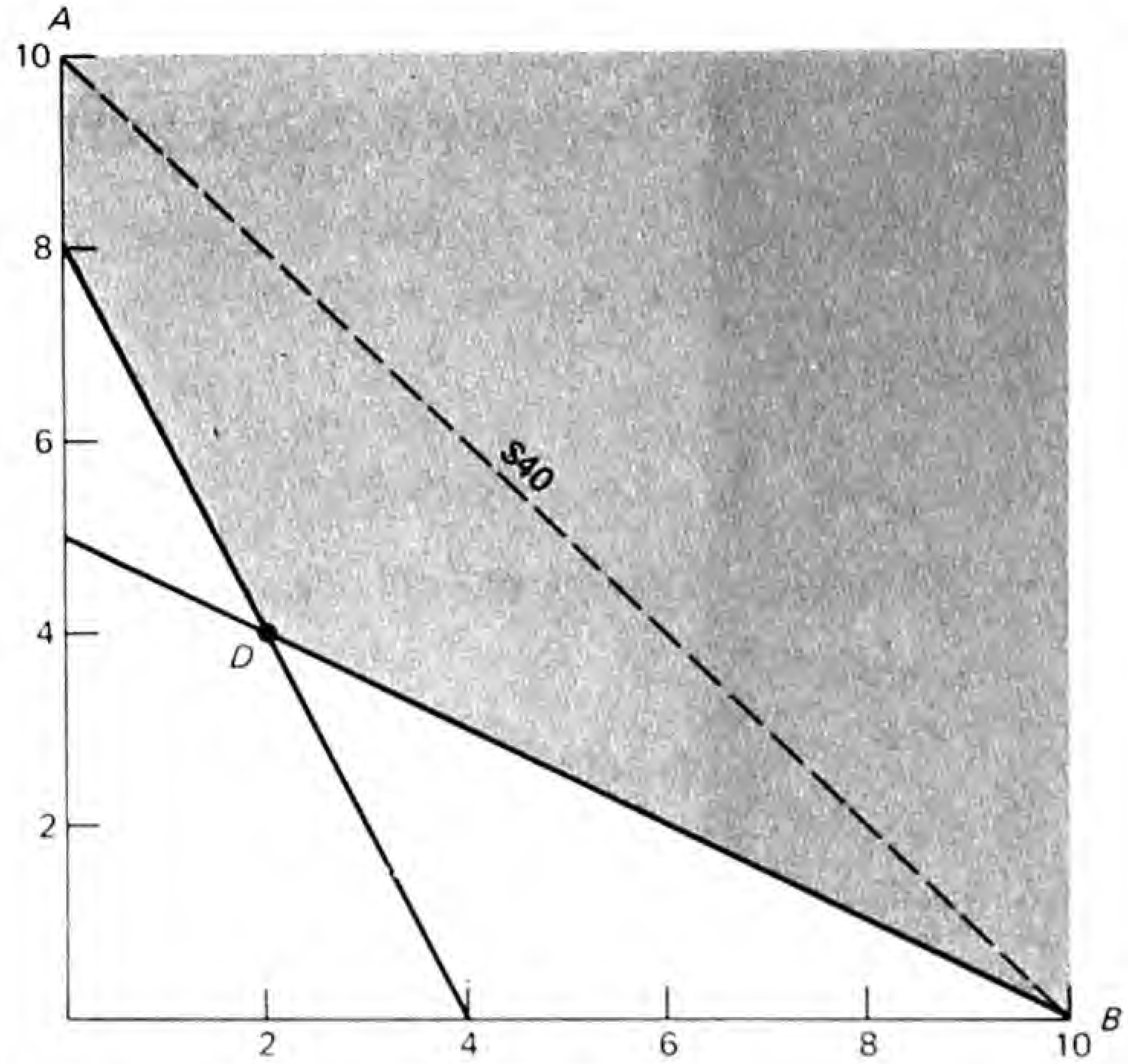


Figure 8B-4 Isocost line and solution.

Solution

We can conclude that the solution is found at point D where

$$A = 4$$
$$B = 2$$

and the cost of this solution is

$$4(4) + 4(2) = \$24$$

PROBLEMS

8B-1 Graphically solve the following problem:

$$\begin{aligned} \text{Min } C &= 1A + 2B \\ 2A + 3B &\geq 5 \\ 3A + 1B &= 6 \\ A, B &\geq 0 \end{aligned}$$

8B-2 Graphically solve the following problem:

$$\begin{aligned} \text{Min } C &= 2X + 3Y \\ 1X + 2Y &\geq 4 \\ 2X + 1Y &\geq 4 \\ X, Y &\geq 0 \end{aligned}$$

8B-3 Graphically solve the following problem:

$$\begin{aligned} \text{Min } C = 2A + 4B& \\ 3A + 2B &\geq 10 \\ 1A + 1B &= 5 \\ A \qquad &\leq 2 \\ A, B &\geq 0 \end{aligned}$$

Chapter 9

Linear Programming: Simplex Method

INTRODUCTION

The Limitation of the Graphical Method

The graphical method developed in the last chapter is limited to linear-programming models which include only two products or activities. The Jerry Company case, for example, included only camping and mountain tents.

The graphical method would have been cumbersome had the Jerry Company decided to schedule three products. To analyze this larger problem, a three-dimensional graph would have been necessary: one dimension for each product.

What if the company had decided to schedule more than three products? Then it would have been impossible to use graphical analysis.

Fortunately, the simplex method, covered in this chapter, can be used to analyze any linear-programming problem regardless of size.

The Role of Corner Points in the Simplex Method

The backbone of the simplex method is that only the extreme points or corners of the feasible region need to be searched for the optimal solution of a problem. This can be seen graphically by returning to the Jerry Company case. Mathematically the problem was expressed in the following way.

$$\text{Max } P = 20C + 30M$$
$$2C + 1M \le 8000$$
$$2C + 4M \le 16{,}000$$
$$C \ge 0$$
$$M \ge 0$$

The graphical solution is repeated in Figure 9-1. From it certain observations can be made. Because the feasible region is formed by linear constraints, a polygon—or many-sided figure—results. Consequently as the isoprofit line is moved out as far as possible, it will generally leave at a corner or extreme point. In the Jerry Company case, it left at corner *d*.

On the rare occasion when the isoprofit line is parallel to one of the constraints, the optimal solution will be found all along a portion of the constraint *including* its two corners. For example, the isoprofit line shown in Figure 9-2 is parallel to the constraint $2A + 1B \le 2$. Consequently as the isoprofit line is moved further and further out to identify the optimal solution, it does not leave the feasible region at just a corner but all along *ba*. It can therefore be concluded that there are multiple solutions to this problem. Any point which falls along this segment is considered just as good—or as profitable—as any other. But notice that included in these multiple solutions are the corner points *b* and *a*. This leads us to establish a very important principle. In the solution of a linear-programming problem, only the corners need to be searched. The corner which is found to have the largest objective

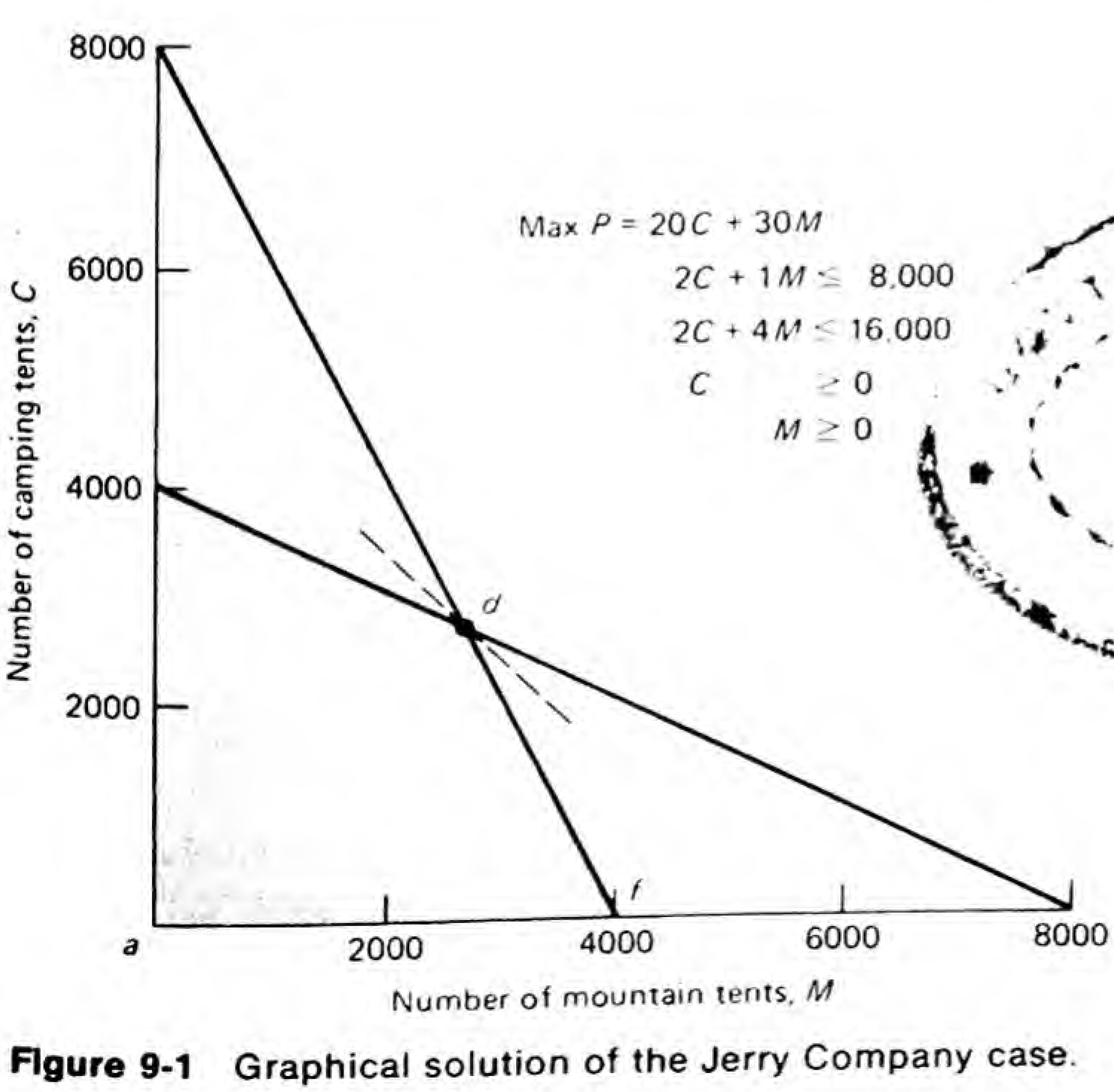


Figure 9-1 Graphical solution of the Jerry Company case.

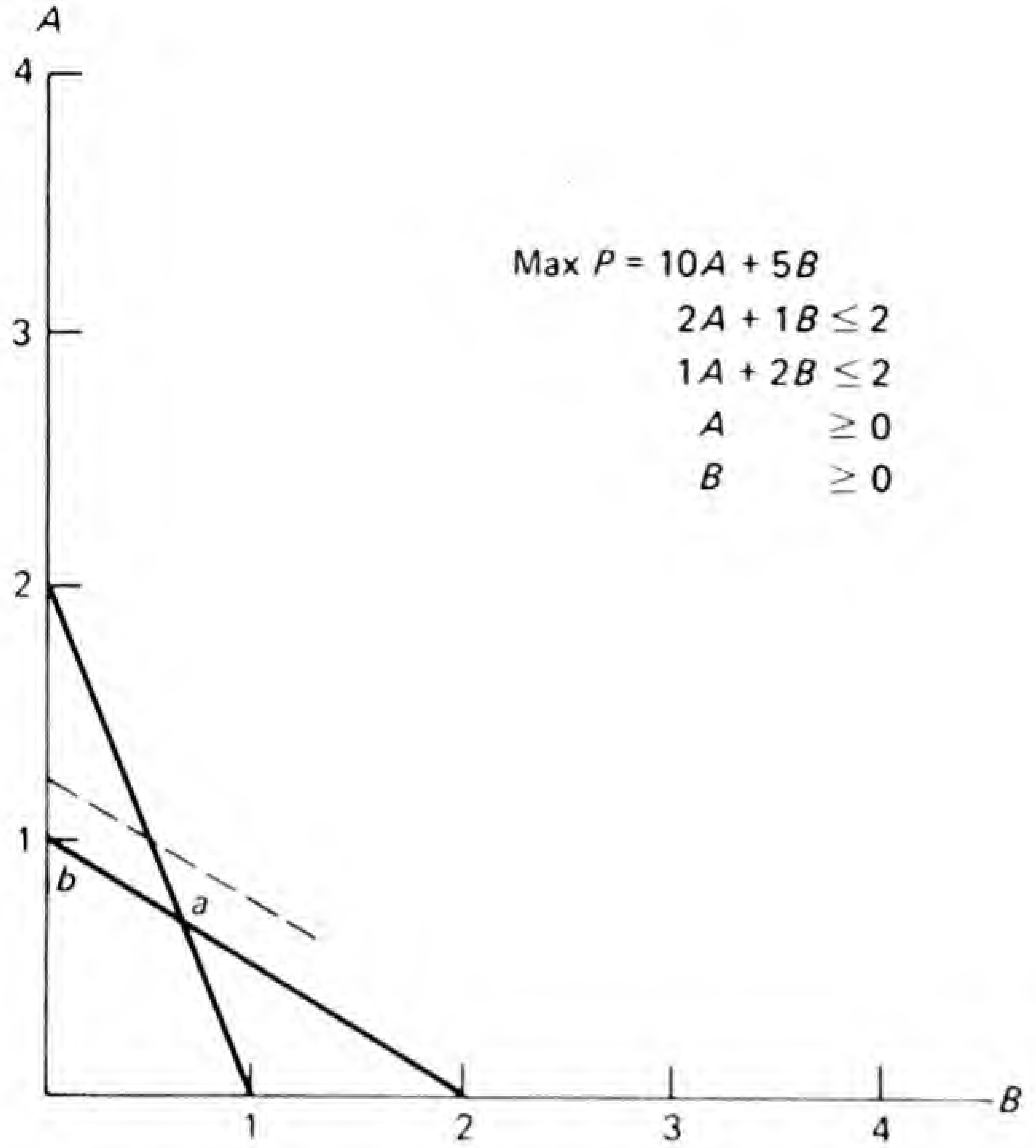


Figure 9-2 Isoprofit line parallel to constraint.

function value represents the optimal value of the problem. This principle is the backbone of the simplex method.

An Intuitive Explanation of the Simplex Method

Before we proceed into the mechanics of the simplex method, perhaps it will be helpful if we explore the intuition behind the steps which make up this technique.

In the last section it was concluded that the simplex method needs to be concerned only with corner points. To begin, the simplex method starts in the safest corner—a corner which is always feasible, the origin. For example, in the problem illustrated in Figure 9-1, the simplex method would start at corner *a* where

$$C = 0$$
$$M = 0$$

From there it proceeds in the most profitable direction toward another corner. Two choices exist: to bring in some units of camping tents C or some units of mountain tents M. It is important to realize that the simplex method will proceed at this stage by either introducing some number of camping tents *or* some number of mountain tents, but not some combination of the two. Since the per unit profit contribution of mountain tents at $30 is larger than the per

unit profit contribution of camping tents at \$20, the simplex method will proceed to introduce as many mountain tents as possible until a constraint is met. This will occur at corner *f*, where

$$M = 4000$$
$$C = 0$$

Then the simplex method determines whether profit can be increased by continuing to move toward another corner. If so, camping tents will be introduced until corner *d* is reached. From our graphical solution we know that this is indeed the situation and therefore corner *d* will be reached.

Again it will be determined if it would be profitable to proceed from corner *d* to the next corner. From our knowledge of the solution to this problem, we expect the simplex method to tell us *no*, and we would therefore expect the simplex method to identify corner *d* as the solution.

This has been an intuitive overview of the simplex method.

We now turn to the details.

THE AUGMENTED FORM OF THE LINEAR-PROGRAMMING PROBLEM

The first step toward a mathematical solution of the linear-programming problem is to convert the inequality form of the problem into the augmented form. The inequality form of the Jerry Company problem is repeated below.

$$\begin{aligned} \text{Max } P = 20C &+ 30M \\ 2C &+ \ 1M \le 8000 \\ 2C &+ \ 4M \le 16{,}000 \\ C & \qquad \ \ \ge 0 \\ & \quad\ \ M \ge 0 \end{aligned}$$

This is called the inequality form because the constraints are stated as inequalities. It is essential, however, that we convert these inequalities to equalities. We will then be working with equations.

Slack Variables Introduced in Each Constraint

To accomplish this conversion, a slack variable is introduced in each inequality. Consider the first inequality

$$2C + 1M \le 8000$$

To convert this to an equality, a slack variable S_1 is introduced in the

following way:

$$2C + 1M + 1S_1 = 8000$$

Whenever the sum $2C + 1M$ is less than 8000, S_1 will take up the slack. For example, if $C = 1000$ and $M = 1000$, then $S_1 = 5000$ and the equality is maintained.

$$\begin{aligned} 2(1000) + 1(1000) + 1S_1 &= 8000 \\ 2000 + 1000 + 5000 &= 8000 \\ 8000 &= 8000 \end{aligned}$$

We can interpret this slack variable as the unused number of worker-hours in the fabric-cutting department.

Since each inequality must be converted to an equality, we proceed in a similar fashion to introduce another slack variable S_2 in the second inequality,

$$2C + 4M + 1S_2 = 16{,}000$$

We can interpret S_2 as the unused capacity in the sewing department.

Objective Function

The final step is to introduce these new variables S_1 and S_2 into the objective function. This can be accomplished in the following way:

$$\text{Max } P = 20C + 30M + 0S_1 + 0S_2$$

These new variables have a coefficient of zero since the profitability associated with a unit of unused resource is zero. Because these new variables have a coefficient of zero in the objective function, they are often omitted.

Summarizing the Problem in Augmented Form

We can now summarize the problem in the following way:

$$\begin{aligned} \text{Max } P = 20C + 30M + 0S_1 + 0S_2 & \\ 2C + 1M + 1S_1 \qquad &= 8000 \\ 2C + 4M \qquad + 1S_2 &= 16{,}000 \end{aligned}$$

This is called the augmented form of the linear-programming problem.

THE FIRST BASIS

The first basis is the mathematical representation of the problem at the origin. It can be written by simply moving the right-hand side of the objective

function to the left-hand side. The constraints remain unchanged. This step is shown in Table 9-1.

Table 9-1 First Basis

Row 1:	$P - 20C - 30M$	$= 0$
Row 2:	$2C + 1M + 1S_1$	$= 8000$
Row 3:	$2C + 4M \qquad + 1S_2$	$= 16{,}000$

At the origin recall that

$$C = 0$$
$$M = 0$$

That is, nothing is produced. Consequently the basis which we have just derived can be evaluated in the following way:

$$\begin{aligned} P - 20(0) - 30(0) \qquad\qquad &= 0 \\ 2(0) + 1(0) + 1S_1 \qquad &= 8000 \\ 2(0) + 4(0) \qquad + 1S_2 &= 16{,}000 \end{aligned}$$

or

$$P = 0$$
$$S_1 = 8000$$
$$S_2 = 16{,}000$$

We see that at this point $C = 0$, $M = 0$, profit ($P = 0$) is zero, unused capacity in the fabric-cutting shop is 8000 worker-hours, and unused capacity in the sewing shop is 16,000 worker-hours.

Basic Variables

Variables P, S_1, and S_2, found in Table 9-1, are called *basic variables*. They represent the level of profit and the products or activities which are at nonzero (positive) levels. Basic variables have the characteristics that their coefficient in the basis is + 1, that they occur only once per column, and that there is one basic variable per row. These three conditions are met by variables P, S_1, and S_2. All other variables—in this case, C and M—are considered *nonbasic variables*; their *coefficients* in the basis can take on any value, but the variables themselves are always equal to zero. In our first basis we have

$$C = 0$$
$$M = 0$$

Therefore no camping or mountain tents are produced. To ensure that you see

the difference between basic and nonbasic variables, we turn to another example.

EXAMPLE

Inequality form

$$\begin{aligned} \text{Max } P &= 3X_1 + 2X_2 \\ 1X_1 + 1X_2 &\le 10 \\ 2X_1 + 3X_2 &\le 24 \\ X_1, X_2 &\ge 0 \end{aligned}$$

Augmented form

$$\begin{aligned} \text{Max } P &= 3X_1 + 2X_2 \\ 1X_1 + 1X_2 + 1S_1 \qquad &= 10 \\ 2X_1 + 3X_2 \qquad + 1S_2 &= 24 \end{aligned}$$

First basis

$$\begin{aligned} P - 3X_1 - 2X_2 \qquad\qquad &= 0 \\ 1X_1 + 1X_2 + 1S_1 \qquad &= 10 \\ 2X_1 + 3X_2 \qquad + 1S_2 &= 24 \end{aligned}$$

Basic variables	*Nonbasic variables*
$P = 0$	$X_1 = 0$
$S_1 = 10$	$X_2 = 0$
$S_2 = 24$	

TOWARD A SECOND BASIS

The first basis represents a profit position of zero. Surely this can be improved.

Which Variable to Enter

To accomplish this improvement, we look at the coefficients of the variables in row 1 of the first basis. Each *negative* coefficient represents the amount by which the profit function will *increase* upon the introduction of 1 unit of that variable. In the Jerry Company example, we can see that each unit of C introduced will increase profit by \$20 and each unit of M will increase profit by \$30. Only one of these variables can be introduced. It will be the one with the highest per unit profit contribution. In this case M is that variable.

Each basis in the simplex method is a mathematical representation of a corner point. And as we proceed from one point to the next, a basic variable

will be added to the new basis and one will be dropped. In our example we have already determined that the variable to be added in moving to the next basis is M. Now we will determine exactly how much of M should be introduced and which variable to drop.

How Much of the New Variable Enters and Which Variable Leaves the Basis

From an intuitive point of view it should be clear that as more and more mountain tents are introduced, less and less unused capacity will be left in each department. In fact, at some point, we will exhaust all the capacity in each department. Let's find these points first for the fabric-cutting department and then for the sewing department.

Since there are 8000 worker-hours of capacity in the fabric-cutting department and every mountain tent requires 1 hour of this capacity, at most

$$\frac{8000}{1} = 8000$$

mountain tents can be processed before the capacity of this shop is exhausted.

In the sewing department each mountain tent requires 4 worker-hours of the total 16,000 worker-hours available. Therefore the capacity of that shop will be exhausted when

$$\frac{16{,}000}{4} = 4000$$

mountain tents are processed.

Since at most 8000 mountain tents can be processed through the fabric-cutting department and 4000 through the sewing department, it seems only reasonable to schedule 4000 mountain tents through *both*. If more than 4000 are scheduled, the excess over 4000 cannot be sewn.

We can then conclude that 4000 mountain tents will be introduced and that in the next basis the slack for the sewing department will drop to zero ($S_2 = 0$ in the next basis). Our new basic variable in the new basis will therefore be M (since it will take on a positive value), and S_2 will become a nonbasic variable (since the value it will take on will be zero). Said another way, M replaces S_2 as a basic variable.

Once again let's look at the process for determining the basic variable which leaves the basis. This time, however, we will consider the process from a very mechanical point of view. First we take the ratios of the right-hand side of the basis to the coefficients of the entering variable in that row. Then we identify the departing variable as the one associated with the smallest ratio.[1]

[1] On occasion this ratio will be negative. If this is the case, it should be ignored. Can you explain why this is so?

right-hand side of first constraint

$$\frac{8000}{1} = 8000$$

coefficient of
entering variable
in first constraint

right-hand side of second constraint

$$\frac{16{,}000}{4} = 4000$$

smallest of the positive ratios
identifies the max that can be
introduced

coefficient of
entering variable
in second constraint

Generating the Pivot Row

Once we have determined both the new basic variable and the variable which becomes nonbasic, a new basis can be derived. We have already determined that M will be the new basic variable, and from our definition of a basic variable it *must* have a coefficient of 1, and must *not* appear elsewhere in its own column.

To derive our new basis, we return to the first basis in Table 9-1. The first row on which we will operate is row 3. The reason for this is that row 3 contains the *current basic variable* S_2, which will become *nonbasic* in the next basis.

The next step is to force the coefficient of M in row 3 to 1. How can this be done? By dividing the equation through by 4. We will therefore have

New row 3 (pivot row): $\frac{1}{2}C + 1M \qquad\qquad + \frac{1}{4}S_2 = 4000$

and this becomes our new row 3. This new row will be called a *pivot row* and will be used to generate the other rows in the new basis.

Completing the Second Basis

The next steps are aimed at generating new rows 1 and 2 such that M is assured the status of a basic variable. This means that the coefficient of M in both of these rows must be zero—M cannot appear in either of them. How can row 1 be modified to accomplish this? By multiplying the pivot row (new row 3) by +30 and adding it to row 1. This will eliminate M from row 1. Multiplying new row 3 by +30, we have

$$15C + 30M \qquad\qquad + \tfrac{30}{4}\,S_2 = 120{,}000$$

and adding it to row 1 we have

$$\begin{array}{lll} & P - 20C - 30M & = 0 \\ & \quad 15C + 30M & + {}^{30}\!/_{4}\, S_2 = 120{,}000 \\ \hline \text{New row 1:} & P - \ \ 5C & + {}^{30}\!/_{4}\, S_2 = 120{,}000 \end{array}$$

This now becomes our new row 1.

Next we turn to the computation of a new row 2. Again our purpose is to eliminate M from this row. This can be accomplished by multiplying the pivot row (new row 3) by -1 and adding it to row 2 of the first basis. Multiplying by -1, we have

$$-\tfrac{1}{2}C - 1M \qquad -\tfrac{1}{4}S_2 = -4000$$

Adding to row 2 we have

$$\begin{array}{lll} & 2C + 1M + 1S_1 & = 8000 \\ & -\tfrac{1}{2}C - 1M & -\tfrac{1}{4}S_2 = -4000 \\ \hline \text{New row 2:} & \tfrac{3}{2}C & + 1S_1 - \tfrac{1}{4}S_2 = 4000 \end{array}$$

and this becomes new row 2.

The Second Basis

The new basis is summarized in Table 9-2 where P, M, and S_1 are basic variables and C and S_2 are nonbasic variables. Since nonbasic variables all take on a value of zero, we have

$$C = 0$$
$$S_2 = 0$$

and we can therefore read the intensity of the basic variables directly.

$$P = 120{,}000$$
$$M = 4000$$
$$S_1 = 4000$$

At this stage in the problem we are producing 4000 mountain tents, no camping tents, have no slack in the sewing department ($S_2 = 0$), have 4000 worker-hours

Table 9-2 Second Basis

Row 1:	$P - 5C$		$+ {}^{30}\!/_{4}\, S_2 = 120{,}000$
Row 2:	$\tfrac{3}{2}C$	$+ 1S_1$	$- \tfrac{1}{4}\, S_2 = 4000$
Row 3:	$\tfrac{1}{2}C + 1M$		$+ \tfrac{1}{4}\, S_2 = 4000$

of slack or unused capacity in the fabric-cutting department ($S_1 = 4000$), and are making a profit of \$120,000. In reference to Figure 9-1, we are now at corner *f*.

SUMMARIZING THE STEPS IN THE SIMPLEX METHOD

Before we move from this basis to the third one, it might be useful to review the steps which must be undertaken in going from one basis to the next. Would you believe that there is very little more to learn about the simplex method? From here on it is just the repeated application of these steps directed toward the generation of new bases until the solution is found. Therefore let's carefully review these steps to make sure they are fully understood.

1 Identification of the variable which becomes the new basic variable (*the variable in row 1 which has the largest negative coefficient)*

2 Identification of the basic variable which will become a nonbasic variable in the next basis (the smallest of the positive ratios of the right-hand side to the coefficient of the entering basic variable in that row)

3 Generation of pivot row

4 Completion of new basis

5 Repetition of steps 1 to 4 until a solution is reached

THIRD BASIS

Continuing from the second basis, we now proceed to apply steps 1 to 4 until the next basis is generated.

Starting with step 1, we see that in row 1 it is variable C which has the largest negative coefficient. It therefore becomes the variable which enters the new basis.

Application of step 2 results in the following ratios.

$$\frac{4000}{3/2} = 8000/3$$

$$\frac{4000}{1/2} = 8000$$

The smallest of these positive ratios is $8000/3$ and the basic variable associated with this ratio is S_1. We can therefore conclude that S_1 is the current basic variable which will become a nonbasic variable in the next basis and that row 2 will become the new pivot row.

Our next step is to force the coefficient of the new basic variable C in row 2 to be 1. This can be accomplished by dividing row 2 by $3/2$.

New row 2 (pivot row): $1C + 2/3 S_1 - 2/12 S_2 = 8000/3$

This becomes our new pivot row.

We must now force the coefficient of C in the other rows—row 1 and row

3—to zero. To accomplish this for row 1, we multiply the new pivot row by 5

$$5C + {}^{10}\!/_3\, S_1 - {}^{10}\!/_{12} S_2 = {}^{40,000}\!/_3$$

and add it to row 1 of the second basis

$$
\begin{array}{lr}
 & P - 5C \qquad\qquad + {}^{30}\!/_4\, S_2 = 120{,}000 \\
 & 5C + {}^{10}\!/_3\, S_1 - {}^{10}\!/_{12} S_2 = {}^{40,000}\!/_3 \\
\hline
\text{New row 1:} & P + {}^{10}\!/_3\, S_1 + {}^{80}\!/_{12} S_2 = {}^{400,000}\!/_3
\end{array}
$$

and this becomes our new row 1.

Now we turn to row 3. In order to make C a basic variable in this row, the pivot row must first be multiplied by $-{}^1\!/_2$.

$$-{}^1\!/_2 C \qquad -{}^1\!/_3 S_1 + {}^1\!/_{12} S_2 = -{}^{4000}\!/_3$$

This in turn is added to row 3 of the second basis:

$$
\begin{array}{lr}
 & {}^1\!/_2 C + 1M \qquad + {}^1\!/_4 S_2 = 4000 \\
 & -{}^1\!/_2 C \qquad -{}^1\!/_3 S_1 + {}^1\!/_{12} S_2 = -{}^{4000}\!/_3 \\
\hline
\text{New row 3:} & 1M - {}^1\!/_3 S_1 + {}^1\!/_3 S_2 = {}^{8000}\!/_3
\end{array}
$$

We have now completed the generation of the third basis. It is summarized in Table 9-3.

Table 9-3 Third Basis

Row 1:	P			$+ {}^{10}\!/_3\, S_1 + {}^{80}\!/_{12} S_2$	$= {}^{400,000}\!/_3$
Row 2:		$1C$		$+ {}^2\!/_3\, S_1 - {}^2\!/_{12} S_2$	$= {}^{8000}\!/_3$
Row 3:			$1M$	$- {}^1\!/_3\, S_1 + {}^1\!/_3 S_2$	$= {}^{8000}\!/_3$

From this basis we can conclude that the nonbasic variables are S_1 and S_2. Consequently no slack exists in either department.

$$S_1 = 0$$
$$S_2 = 0$$

The basic variables are P, C, and M.

$$P = {}^{400,000}\!/_3 = 133{,}333$$
$$C = {}^{8000}\!/_3 \; = 2666$$
$$M = {}^{8000}\!/_3 \; = 2666$$

Profit is \$133,333, and this is generated by producing 2666 camping and 2666 mountain models.[1] With reference to Figure 9-1, we are at corner *d*.

SOLUTION

Now we are ready for a repeated application of steps 1 through 4. Again we search row 1 for the largest negative coefficient. This time, however, the new basis has no negative coefficients at all. With only positive coefficients we can conclude that the introduction of any of the nonbasic variables would result in a *reduction* of profit. We have therefore reached the solution. There is no better solution than that represented by the third basis.

To maximize profit, Mr. Grover should therefore produce 2666 camping tents and 2666 mountain tents. And from this combination his profit will be \$133,333.

EXAMPLE

Now that you have seen a linear-programming problem solved by the simplex method, perhaps it would be helpful if you solved the following example yourself. After trying this problem, you may check its solution, which is given below.

Problem

$$\begin{aligned} \text{Max } P &= 2A + 1B + 3C \\ 2A + 1B + 1C &\le 20 \\ 3A + 4B + 2C &\le 24 \\ 2A + 5B + 6C &\le 30 \\ A, B, C &\ge 0 \end{aligned}$$

Solution

First Basis

Row 1:	$P - 2A - 1B - 3C$				$= 0$
Row 2:	$2A + 1B + 1C$	$+ 1S_1$			$= 20$
Row 3:	$3A + 4B + 2C$		$+ 1S_2$		$= 24$
Row 4:	$2A + 5B + 6C$			$+ 1S_3$	$= 30$

Nonbasic variables	Basic variables
	$P = 0$
$A = 0$	$S_1 = 20$
$B = 0$	$S_2 = 24$
$C = 0$	$S_3 = 30$

[1] It is necessary to round the result to the nearest integer since in linear programming there is no mechanism for ensuring integer solutions. The advanced technique of integer programming, however, will lead to an integer solution.

Step 1: Enter variable C.
Step 2: $^{20}/_1 = 20$

$^{24}/_2 = 12$

$^{30}/_6 = 5 \leftarrow$

Step 3: Row 4 becomes the pivot row, and since C is the new basic variable, the row is divided by 6.

New row 4 (pivot row): $^2/_6A + {}^5/_6B + 1C \qquad + {}^1/_6S_3 = 5$

Step 4: To get new row 1, multiply pivot row by 3 and add to old row 1.

$$\begin{array}{lllll} & P - 2A - 1B & -3C & & = 0 \\ & A + {}^{15}/_6\,B & +3C & + {}^1/_2S_3 & = 15 \\ \hline \text{New row 1:} & P - 1A + {}^9/_6B & & + {}^1/_2S_3 & = 15 \end{array}$$

To get new row 2, multiply pivot row by -1 and add to old row 2.

$$\begin{array}{lllll} & 2A + 1B & +1C + 1S_1 & & = 20 \\ & -{}^2/_6A - {}^5/_6B & -1C & -{}^1/_6S_3 & = -5 \\ \hline \text{New row 2:} & {}^{10}/_6\,A + {}^1/_6B & +1S_1 & -{}^1/_6S_3 & = 15 \end{array}$$

To get new row 3, multiply pivot row by -2 and add to old row 3.

$$\begin{array}{llll} & 3A + 4B + 2C + 1S_2 & & = 24 \\ & -{}^4/_6A - {}^{10}/_6\,B - 2C & -{}^2/_6S_3 & = -10 \\ \hline \text{New row 3:} & {}^{14}/_6\,A + {}^{14}/_6\,B & +1S_2 - {}^2/_6S_3 & = 14 \end{array}$$

We can now write our second basis:

Second Basis

Row 1:	$P - 1A + {}^9/_6B$		$+ {}^1/_2S_3$	$= 15$
Row 2:	${}^{10}/_6\,A + {}^1/_6B$	$+1S_1$	$-{}^1/_6S_3$	$= 15$
Row 3:	${}^{14}/_6\,A + {}^{14}/_6\,B$	$+1S_2$	$-{}^2/_6S_3$	$= 14$
Row 4:	${}^2/_6A + {}^5/_6B + 1C$		$+ {}^1/_6S_3$	$= 5$

Step 1: Enter variable A.

Step 2: $\dfrac{15}{10/6} = 9$

$\dfrac{14}{14/6} = 6 \leftarrow$

$\dfrac{5}{2/6} = 15$

Step 3: Row 3 becomes the new pivot row, and since A is the new basic variable, the row is divided by $^{14}/_{6}$.

New row 3 (pivot row): $1A + 1B \quad + {}^{6}/_{14}S_2 - {}^{1}/_{7}S_3 = 6$

Step 4: To get new row 1, multiply pivot row by 1 and add to old row 1.

$$\begin{array}{lrrrrl} & P - 1A + & {}^{9}/_{6}B & & + {}^{1}/_{2}S_3 & = 15 \\ & 1A + & 1B & + {}^{6}/_{14}S_2 & - {}^{1}/_{7}S_3 & = 6 \\ \hline \text{New row 1:} & P \quad + & {}^{15}/_{6}B & + {}^{6}/_{14}S_2 & + {}^{5}/_{14}S_3 & = 21 \end{array}$$

To get new row 2, multiply pivot row by $-{}^{10}/_{6}$ and add to old row 2.

$$\begin{array}{lrrrrl} & {}^{10}/_{6}A + {}^{1}/_{6}B & + 1S_1 & & - {}^{1}/_{6}S_3 & = 15 \\ & -{}^{10}/_{6}A - {}^{10}/_{6}B & & - {}^{10}/_{14}S_2 & + {}^{10}/_{42}S_3 & = -10 \\ \hline \text{New row 2:} & -{}^{9}/_{6}B & + 1S_1 & - {}^{10}/_{14}S_2 & + {}^{3}/_{42}S_3 & = 5 \end{array}$$

To get new row 4, multiply pivot row by $-{}^{2}/_{6}$ and add to old row 4.

$$\begin{array}{lrrrl} & {}^{2}/_{6}A + {}^{5}/_{6}B + 1C & & + {}^{1}/_{6}S_3 & = 5 \\ & -{}^{2}/_{6}A - {}^{2}/_{6}B & - {}^{1}/_{7}S_2 & + {}^{2}/_{42}S_3 & = -2 \\ \hline \text{New row 4:} & {}^{3}/_{6}B + 1C & - {}^{1}/_{7}S_2 & + {}^{9}/_{42}S_3 & = 3 \end{array}$$

We now have our third basis:

Third Basis

Row 1:	$P \quad + {}^{15}/_{6}B$	$+ {}^{6}/_{14}S_2 + {}^{5}/_{14}S_3 = 21$
Row 2:	$- {}^{9}/_{6}B$	$+ 1S_1 - {}^{10}/_{14}S_2 + {}^{3}/_{42}S_3 = 5$
Row 3:	$1A + 1B$	$+ {}^{6}/_{14}S_2 - {}^{1}/_{7}S_3 = 6$
Row 4:	${}^{3}/_{6}B + 1C$	$- {}^{1}/_{7}S_2 + {}^{9}/_{42}S_3 = 3$

Since all the coefficients in row 1 are positive, we can conclude that we have

reached the solution. We therefore produce 6 units of A and 3 units of C. Profit with this combination will be 21. No B is produced—it is a nonbasic variable—and no slack exists in department 2 or 3. There is slack, however, in department 1, since S_1 is a basic variable. In fact there are exactly 5 units of unused capacity in this department.

SENSITIVITY ANALYSIS

Suppose we are offered additional resources; suppose production times can be reduced by hiring additional workers; suppose production costs increase; or suppose the market price of one product is lowered. What would be the consequence of these changes on the outcome of the linear-programming model? These questions can be answered by changing one or more of the data and observing the effect that this change will have on the outcome of the model. This is called sensitivity analysis.

Returning to the Jerry Company case, let's assume that management is contemplating a change in the market price of camping tents. This will lead to an increase in profits from \$20 per unit to \$22 per unit. The question that must be answered is this: Should it still schedule 2666 camping tents and 2666 mountain tents?

One method for answering this question is to solve the linear-programming model once again, but this time with a revised profit of \$22 for camping tents. Suppose this were done and the results still showed that production should be 2666 camping tents and 2666 mountain tents. We would conclude that the results were fairly insensitive to changes in the profitability of camping tents. Suppose, on the other hand, the results of this new solution showed production quantities that were widely different from our initial solution. We would conclude that the results were indeed very sensitive to these data.

This process is called sensitivity analysis, and its purpose is to explore the consequence of changes in the data. One of the many benefits obtained from sensitivity analysis is the clearer understanding of the problem and its many interrelationships.

SHADOW PRICES

The final basis of a linear-programming problem offers some especially useful information in the form of shadow prices.

A shadow price is the additional gross profit that could be generated if an additional unit of a scarce resource is obtained. In the Jerry Company case the fabric-cutting department is a scarce resource since it is used to capacity ($S_1 = 0$). Those constraints which are not used to capacity are not considered to be scarce resources. Additional units of fabric-cutting capacity would allow output to be increased beyond 2666 camping tents and 2666 mountains tents, and this additional output would result in higher profits.

The shadow price associated with the fabric-cutting department will tell us exactly what the increase in gross profit would be for a unit increase in this scarce resource.

The shadow price can be read from row 1 of the final basis. It is the coefficient of the slack variable which is associated with the scarce resource (constraint). For example, suppose we are interested in the shadow price associated with the fabric-cutting constraint. The slack variable for the constraint is S_1. Its coefficient in row 1 of the final basis (Table 9-3) is $^{10}/_3$, or 3.33. The shadow price for the fabric-cutting department is therefore 3.33. This means that if an additional unit of scarce resource (an additional worker-hour) is obtained, gross profit will increase by \$3.33.

The shadow price represents gross profit. The per unit cost of obtaining the additional unit of scarce resource must be subtracted from it to give the net profit. For example, if additional worker-hours of fabric-cutting capacity could be added for \$3.50 per hour, it would be unprofitable for management to acquire any additional units.

The shadow price for the sewing department is the coefficient of S_2 in row 1. The gross profit of an additional unit of sewing capacity is $^{80}/_{12}$, or \$6.66. That is, an additional unit of capacity in the sewing department will generate a gross profit of \$6.66.

Suppose additional worker-hours of capacity in the sewing department could be added for \$2.75 per hour. Would it be profitable?

SUMMARY

The simplex method is a set of five steps which are used to move from one basis to the next until a solution is reached. This method can be used regardless of the number of variables or constraints.

When the linear-programming problem contains more than four or five variables it can take quite a while before this solution is reached. In those situations it is best to solve the problem on a computer. Several such computer programs have been written and are available on most computer systems.

QUESTIONS

1 What role is played by the corner points in the simplex method?
2 Define the following:
 a Basic variable
 b Nonbasic variable
 c Slack variable
3 Suppose a nonbasic variable in row 1 of the solution basis had a coefficient of zero. What would this imply?
4 Why is the shadow price of a resource which is not used to capacity equal to zero?
5 What are the benefits of sensitivity analysis to the decision maker?

PROBLEMS

9-1 Solve problem 7-1 first graphically and then by the simplex method.
9-2 Solve problem 7-2 first graphically and then by the simplex method.
9-3 Solve problem 8-4 first graphically and then by the simplex method.
9-4 Solve problem 8-5 first graphically and then by the simplex method.
9-5 Solve problem 8-6 first graphically and then by the simplex method.
9-6 Solve by the simplex method:

$$\begin{aligned} \text{Max } P &= 2A + 3B + 1C \\ 1A + 1B + 1C &\le 10 \\ 2A + 1B + 2C &\le 8 \\ A, B, C &\ge 0 \end{aligned}$$

9-7 Solve by the simplex method:

$$\begin{aligned} \text{Max } P &= 4X_1 + 2X_2 + 6X_3 \\ 1X_1 + 1X_2 + 4X_3 &\le 8 \\ 3X_1 + 3X_2 + 3X_3 &\le 6 \\ 1X_1 + 3X_2 + 1X_3 &\le 9 \\ X_1, X_2, X_3 &\ge 0 \end{aligned}$$

9-8 Solve by the simplex method:

$$\begin{aligned} \text{Max } P &= 10X_1 + 5X_2 + 1X_3 \\ 1X_1 + 1X_2 + 1X_3 &\le 10 \\ 2X_1 + 1X_2 + 4X_3 &\le 12 \\ X_1, X_2, X_3 &\ge 0 \end{aligned}$$

9-9 A manufacturer of three products, A, B, and C, must decide how many of each to produce. The marketing manager has assured the production department that whatever it makes can be sold. Limiting the output are three production departments—fabrication *F*, assembly *S*, and painting *P*. In particular, each department has 1000 worker-hours of capacity available.

In the manufacture of product A, 2 units of *F*, 3 units of *S*, and 1 unit of *P* are consumed; for B, 3 units of *F*, 1 unit of *S*, and 1 unit of *P*; for C, 4 units of *F*, 3 units of *S*, and 4 units of *P*. If products A and C generate $10 profit each and B generates $20, how many of each should be produced?

9-10 Phil Saunders, production manager for the Allied Production Company, must schedule his production for the next two periods. Twenty-two units have been sold for delivery in period 1 and 40 have been sold for delivery in period 2. Regular production capacity is 25 units per time period, while overtime capacity is 10 units per time period. Overtime costs are $2 per unit, while regular-time production costs are $1.50 per unit. Units can also be produced in one period and sold in the next. The storage cost for this strategy is $1 per unit per period.

Set this up as a linear-programming problem.

Hint: Let X_{ij} = amount produced in period i for sale in period j on regular time

Let X_{ioj} = amount produced in period i on overtime for sale in period j

If a linear-programming computer code is available, solve for the best production schedule.

CASE STUDY: Western Life Insurance Company

The Western Life Insurance Company, located in Wichita, Kansas, writes life insurance in six Western states. Founded in 1953, it has enjoyed year after year of continuous prosperity.

Life insurance companies receive revenues in the form of premiums which they in turn invest in stocks, bonds, cash, and other investment media. To a large extent it is this "investment portfolio" which determines the profitability of the company. Consequently the effective management of this portfolio is extremely important.

In the past Western's investment group has made these investment decisions on the basis of seasoned judgment. Most of the analysts have been with the company for at least 10 years, and they tend to be quite conservative in their investment strategy.

At the present time they are preparing a plan for the reinvestment of $300,000 in government bonds which mature in 1 week.

Several stocks, corporate bonds, and government bonds that are being considered are listed below; also listed is their projected rate of return:.

Investment	Rate of return,[1] percent
Common stocks:	
Texaco	7
IBM	12
General Motors	9
Chrysler	8
Corporate bonds:	
Westinghouse	8
Du Pont	6
Government bonds	6

[1] Rate of return for common stocks includes their expected rate of market price increase plus dividend rate.

The problem faced by the investment department is to determine the quantity of funds to be allocated between these alternatives.

Federal and state laws as well as company policy limit the choice of alternatives. First, no more than 30 percent of the funds can be invested in common stocks. Second, no single industry group can represent more than 40 percent of the common stock total. Third, government bonds must be at least 25 percent of the common stock investment. Fourth, investment in high-risk stocks, which include IBM, must not represent more than 20 percent of the common stock investment. Fifth, at least 5 percent of the funds allocated to common stocks must be kept in a cash reserve. Sixth, at least 1 percent of the funds invested in corporate bonds must also be kept in a cash reserve.

There is some pressure from the more senior and conservative members of the group to take the $300,000 and reinvest it once again in government bonds at 6 percent. The advantage of this is that no cash reserves need to be held and there is little or no risk associated with the yield.

QUESTIONS

1 Would you agree with the reinvestment of the $300,000 in government bonds? Justify your answer.
2 Formulate this problem as a linear-programming model.
3 If a linear-programming code is available, solve the model. Which stocks and bonds should be chosen and how much cash should be allocated to each alternative? What is the overall rate of return for the portfolio?
4 Suppose the expected rate of return on IBM was revised downward to 10.5 percent. What effect would this have on the decision?
5 Suppose the restriction which limits the maximum invested in common stock to $90,000 were lifted. By how much would this improve the yield of the portfolio?
6 Suppose that $50,000 cash will be needed in 2 months. Which investments should be sold? After the stocks are sold, what will be the new yield of the portfolio?
7 Discuss the use of linear programming for portfolio decisions. What are the advantages? Disadvantages?

CASE STUDY: Vista Properties, Inc.

Vista Properties is a real estate development company currently facing several problems in the design and construction of a large shopping center. The company already owns 140,000 square feet of land on which this center will be built, and it has an option to buy an additional 20,000 square feet of adjoining land. One problem is whether this option should be exercised. The second problem concerns the use of the available space. What kind of stores should be included in the shopping center, and how much of this available space should be allocated to each of them?

Company Background

Vista Properties was founded in 1963 by Ted Wasser. Although the company emphasizes primarily the development of shopping centers, it has occasionally built condominiums.

Vista has had an enviable record. With the exception of one condominium project, it has earned substantial profits on all its investments. Mr. Wasser credits this success to his team of qualified managers, who carefully screen an average of 40 projects before one is undertaken.

Especially useful during the screening phase is the demographic information provided by Data Profile, Inc. For $130, a complete printout of demographic data is supplied from Data Profile's computer. These data are

based on the most recent census information. Included are the number of people within a given radius of the proposed site, growth rates, annual incomes, number of children, whether or not they own a house, education, and so on. From these data Vista's management can proceed to make several key estimates for the proposed site.

After the site is chosen, Vista architects develop plans for the outside and inside of the complex. In the last 5 years considerable effort has been directed at blending the buildings and parking facilities with their environment.

Once the project is begun, tight cost controls are imposed. Every effort is made to complete the project within the allocated budget.

After the project is completed and after the tenants have moved into their stores, Vista's management job still continues. It coordinates promotional campaigns for its tenants including advertising, dollar days, fairs, and concerts.

Vista's interest is to make the property more valuable by building up the sales level of its tenants.

After this phase of rapid growth is over, Vista usually sells the property at a considerable profit. In general the shopping center is sold 7 years after it is opened.

Midvale Shopping Center

The Midvale Shopping Center project passed the screening phase some 12 months ago. At that time 140,000 square feet of land were purchased.

The shopping center is located 25 miles west of a major metropolitan area and is 5 miles from its closest competitive center. Demographic data show this to be a rapidly growing area.

Construction is scheduled to be started in 6 months and will cost $6 per square foot. This figure does not include interior finishing work but does include the cost of parking space which is required for each square foot of interior space.

Of the 140,000 square feet of land available no more than 45,000 square feet of floor space can be used according to local zoning restrictions. The rest must be used for parking and aesthetic purposes.

The problem which the company now faces is to determine how this 45,000 square feet of space is to be divided. There are 12 possible types of stores which can be included. The list is given in Exhibit A. Since certain types of stores are considered essential, the list is divided into groups and each group has a minimum number of square feet. In group A, however, it is considered essential to have both a supermarket and a discount department store. Each of these must have at least 10,000 square feet. To prevent unreasonably large stores, maximum sizes are also given in Exhibit A.

Once it is determined how the space is to be divided, it is very unlikely that any difficulty will be encountered in obtaining tenants. Since there is a broad market for shopping center space within a reasonable distance, the

Exhibit A Shopping Center Data

Group		Type of store	(1) Cost of interior improvements[1]	(2) Present value[1]	(3) Guarantee rent[1]	(4) Group minimum[2]	(5) Store maximum[2]
A	1	Supermarket	9	60	3.2	20	20.0
	2	Discount store	13	80	4.1		20.0
B	3	Fabric	12	45	3.0	2	0.9
	4	Women's specialty	8	50	3.2		3.0
	5	Men's specialty	7	48	3.2		2.0
C	6	Hardware	6	50	3.0	2	4.0
	7	Drug	7	46	3.1		1.6
D	8	Gift	8	35	2.5	2	3.0
	9	Bakery (sale only, no baking)	9	50	2.4		1.3
	10	Ice cream/Sandwich	10	40	2.6		1.5
	11	Music and hi-fi	7	46	2.3		1.5
	12	Barber	11	35	3.0		1.0

[1] Dollars per square foot.
[2] Thousands of square feet.

rental fees are market-determined and will be readily accepted by the tenants. It is expected, therefore, that there will be no negotiation on price.

The criterion by which a shopping center is judged is the present value of its after-tax flows. These flows are shown in column 2 of Exhibit A. They represent the present value of rent revenues less fixed charges, depreciation, and taxes over the 7-year life of the project. To this is added the projected sale price of the property when it is sold. The figures are given on a per square foot basis and do not include interior improvements. Mr. Wasser has argued, however, that the net of these two figures should be used. The cost of interior improvements is given in column 1 of Exhibit A.

Vista's available capital for financing interior improvement is limited to \$450,000. It is unlikely that any additional funds could be acquired.

An essential financial consideration for these centers is that the guarantee rent must cover the fixed charges including interest charges on the debt. Each tenant must sign a lease which guarantees Vista a fixed rental payment each year. In addition, if the tenants' sales revenue exceeds a certain level, a percentage (6 percent) of this excess is paid to Vista. The guarantee rents for each type of store are given in column 3 of Exhibit A. Fixed charges are estimated to be \$125,000.

The Option

Vista has an option to purchase 20,000 square feet of land adjacent to its present parcel. Only 6000 square feet can be used for stores. The option will expire in 3 months and carries a price of \$10 per square foot of land.

Capital outlay for this additional parcel has been computed in the following way:

Land costs: \$10 per square foot × 20,000 square feet	= \$200,000
Construction costs: \$6 per square foot × 6000 square feet	= 36,000
Interior improvements (average): \$10 per square foot × 6000 square feet	= 60,000
Total capital required	\$296,000

To raise this capital, Mr. Wasser would borrow as much as possible and then issue common stock for the remainder. This would add \$29,000 per year to his fixed charges, assuming an average capital cost of slightly less than 10 percent.

QUESTIONS

1 What stores should be included in the complex?
2 Should the option be exercised?
3 Suppose Mr. Wasser is unable to obtain the capital needed for the option through normal financial channels. Someone, however, does offer him the capital at a much

higher interest rate. This would increase his fixed charges from $29,000 to $58,000. How would this affect the investment?

4 Does the assumption of linearity raise any doubts concerning the use of this model to solve this problem?

5 Suppose that it is decided to remove the constraints which limit the maximum number of square feet which a store may occupy. Does this affect the solution?

CASE STUDY: Electricité de France

Introduction

In France the railroad, coal, power, and gas industries are all nationalized. Each industry is under a separate directorship, but major decisions including operating and investment plans must be submitted to government officials and then Parliament for their approval.

Electricité de France generates and distributes electric power throughout the country. At present it is preparing a 10-year investment plan which will eventually be submitted to the government.

Several different kinds of generating plants are currently in use. These include steam, hydroelectric, and nuclear facilities. The steam plants are fired by either oil or coal and at present outnumber the other plants by 2 to 1. Hydroelectric plants rely upon the flow of water to generate power. There are four types of plants in this category. The first is a small hydroelectric plant with a small reservoir of water. The second is a medium-sized hydroelectric plant with a medium reservoir of water. The third is a large hydroelectric plant with a large reservoir of water. The fourth type of hydroelectric plant harnesses the flow of water from a moving river and therefore has no reservoir of water. Nuclear plants are relatively new but will undoubtedly represent a growing source of energy in the future.

The problem which Electricité de France faces is this: How much additional capacity should be recommended in each of these categories if the increase in demand over the next 10 years is to be met?

Demand

All plants generate a homogeneous output which is called a megawatt hour of electricity.

It is important that total output from the new sources meet three kinds of demand. First, the total *annual power* output must meet a certain minimum level. Second, a minimum level of *peak power* must be met where peak power is defined as the highest average hourly consumption of electricity during the 4 days of the year when demand is highest. Third, a minimum level of

guaranteed power must be met, where guaranteed power is defined as the average hourly power demanded in midwinter.

The French Parliament has set the following 10-year goals:

A (annual power)	=	7,200,000 MW
B (peak power)	=	2,307 MW
C (guaranteed power)	=	1,692 MW

All figures are in megawatts (MW) and represent the expected increase in demand by the end of this 10-year period.

Budget

Parliament has also set aside 80 billion francs for investment in this project, and it is very unlikely that additional funds will be appropriated (see Exhibit A). Operating expenditures are considered quite separate and are therefore not included in this budget.

Exhibit A

TO: Electricité de France

FROM: Subcommittee on Power Resources—Parliament

Thank you very much for participating in our last meeting. Your recommendations resulted in the reconsideration of the power resources budget. We now feel that this budget will be 80 billion francs. Unfortunately, your original request of 150 billion francs could not be granted, but you will notice that the increase from our very first proposal has been substantial.

The committee would like to ask that this budget allocation be treated as an upper limit in your 10-year plan. The cost of acquiring capital funds is high from both a political and a financial point of view. Any effort that can be made in minimizing the necessary investment will be greatly appreciated.

Output Capacities

The output capacity of any plant is measured in three ways—by its guaranteed, peak, and annual power. Plants differ in their ability to perform in each of these categories.

For comparative purposes the output of a plant is described by the guaranteed power it can deliver. The peak and annual power for that plant can then be described as multiples of the guaranteed power. For example, the peak power output of a steam plant is 1.15 times its guaranteed power and its annual power is 7000 times its guaranteed power. The multiples for all six generating plants are given in the accompanying table.

		Annual power	Peak power	Guaranteed power
1	Steam plant	7,000	1.15	1.0
2	Hydroelectric (small)	1,300	1.10	1.0
3	Hydroelectric (medium)	4,600	1.20	1.0
4	Hydroelectric (large)	7,350	3.00	1.0
5	Hydroelectric (river)	12,600	1.10	1.0
6	Nuclear	5,100	2.50	1.0

The Cost of Output

The cost of generating a megawatt of guaranteed power can be broken down into two components: investment and operating expenses.

Investment costs are those costs which are incurred for the construction of the generating plant and the purchase of its equipment.

Operating expenses include fuel, labor, and maintenance. In some cases, such as a hydroelectric plant, the investment costs are very high, whereas the operating expenses are relatively low. For steam plants, just the opposite is true.

		Investment*	Present value of operating expenses
1	Steam plant	35	62
2	Hydroelectric (small)	150	15
3	Hydroelectric (medium)	300	20
4	Hydroelectric (large)	370	40
5	Hydroelectric (river)	360	60
6	Nuclear	100	41

*Millions of French francs.

The accompanying table shows the investment and operating expenses for the six plants. The figures are given in millions of francs per megawatt of guaranteed power. Operating expenses are incurred each year over the life of a facility; therefore the figure shown is the present value of this expense stream. All plants are assumed to have a life of 20 years.

Objective

Within Electricité de France there is some controversy over the objective. Some feel that it should be the minimization of investment costs, while others feel that it should be the minimization of operating expenses.

Those favoring the minimization of operating expenses feel that the investment budget is only a constraint and should probably be used in full. They argue that the full use of this budget will permit the investment in

hydroelectric plants which have relatively high investment costs but low operating costs. With low operating costs the operational problems of the new power system will be smaller and the costs to the consumer will be less.

The group favoring the minimization of investment has been influenced by a recent memorandum received from a government subcommittee on power. See Exhibit A. It feels that the power objectives should be met at the lowest investment cost possible.

QUESTIONS

1 Formulate the objective. Should it be the minimization of investment costs or the minimization of operating expenses? Should any other objective be considered?

2 Formulate a linear-programming model which can determine the power output that should be generated in each of the three categories.

3 Formulate and solve the models for each objective. What is the difference between their solutions? Do they both use the budget fully? Which formulation and solution would you recommend?

4 Suppose the cost of oil and coal is expected to increase substantially over the 10-year planning period. If this increases the present value of the operating expenses for this category from 62 to 150, what effect would this have on the strategy taken?

APPENDIX A: Simplex Method for Minimization Problems

INTRODUCTION

Until now the focus has been on maximization problems. They have been stated in such a way that the objective function is to be maximized subject to a set of constraints.

Often, however, the problem is to minimize the objective function subject to a set of constraints. Examples might include the minimization of labor costs, the minimization of production costs, the minimization of machine downtime, the minimization of processing time, the minimization of distance traveled, the minimization of project length, or the minimization of advertising expenditures.

AN EXAMPLE IN MINIMIZATION: MENU PLANNING

Recently there has been considerable success using linear-programming methods for institutional menu planning. At first glance, this might seem a rather sterile approach to such a topic, but results have shown that the menu-planning models have reduced food costs while providing a more varied, more tasteful, and lower-cost menu.

The objective in menu planning is to minimize the total costs of all the ingredients. There are usually several constraints. First, there are nutritional constraints. That is, the combination of ingredients used during any day must meet the minimum daily nutritional requirements. Second, there are recipe constraints. They specify the combination of ingredients necessary to make certain dishes. And third, there are constraints which limit the frequency of some items. For example, if lamb is served, it cannot be served again for 6 days.

A realistic menu-planning problem with ingredients listed for breakfast, lunch, and dinner over several days is an exceedingly large problem. Here we will consider but a simple one. It is hoped it will give you the flavor of the full-blown problem.

Objective Function

Suppose that two ingredients A (vegetables) and B (meat) are used in a stew. The cost of ingredient A is \$1 per pound and that of B is \$2 per pound. We can write the objective function in the following way:

$$\text{Min } C = 1A + 2B$$

where A is the number of pounds of vegetables and B is the number of pounds of meat used in the stew. Now we turn to the constraints.

Constraints

In order to feed everyone satisfactorily, at least 50 pounds of stew must be served. Therefore, the first constraint becomes

$$A + B \geq 50$$

The second constraint is the nutritional constraint. At least 300 units of vitamins must be served during that meal. Every pound of A contains 4 units of vitamins, and every pound of B contains 20 units. The nutritional constraint can, therefore, be written as

$$4A + 20B \geq 300$$

The last constraint specifies that at most 20 pounds of meat can be purchased from the source of supply today. This constraint can be written as

$$B \leq 20$$

Summarizing the problem we have:

$$\begin{aligned} \text{Min } C = 1A + \;& 2B \\ A + \;& B \geq 50 \\ & B \leq 20 \\ 4A + 20&B \geq 300 \\ A, &B \geq 0 \end{aligned}$$

Graphical Solution

The graphical solution is shown in Figure 9A-1. When the number of variables exceeds two, this method is no longer appropriate. Next we will turn to the simplex method for solving this and larger-sized minimization problems.

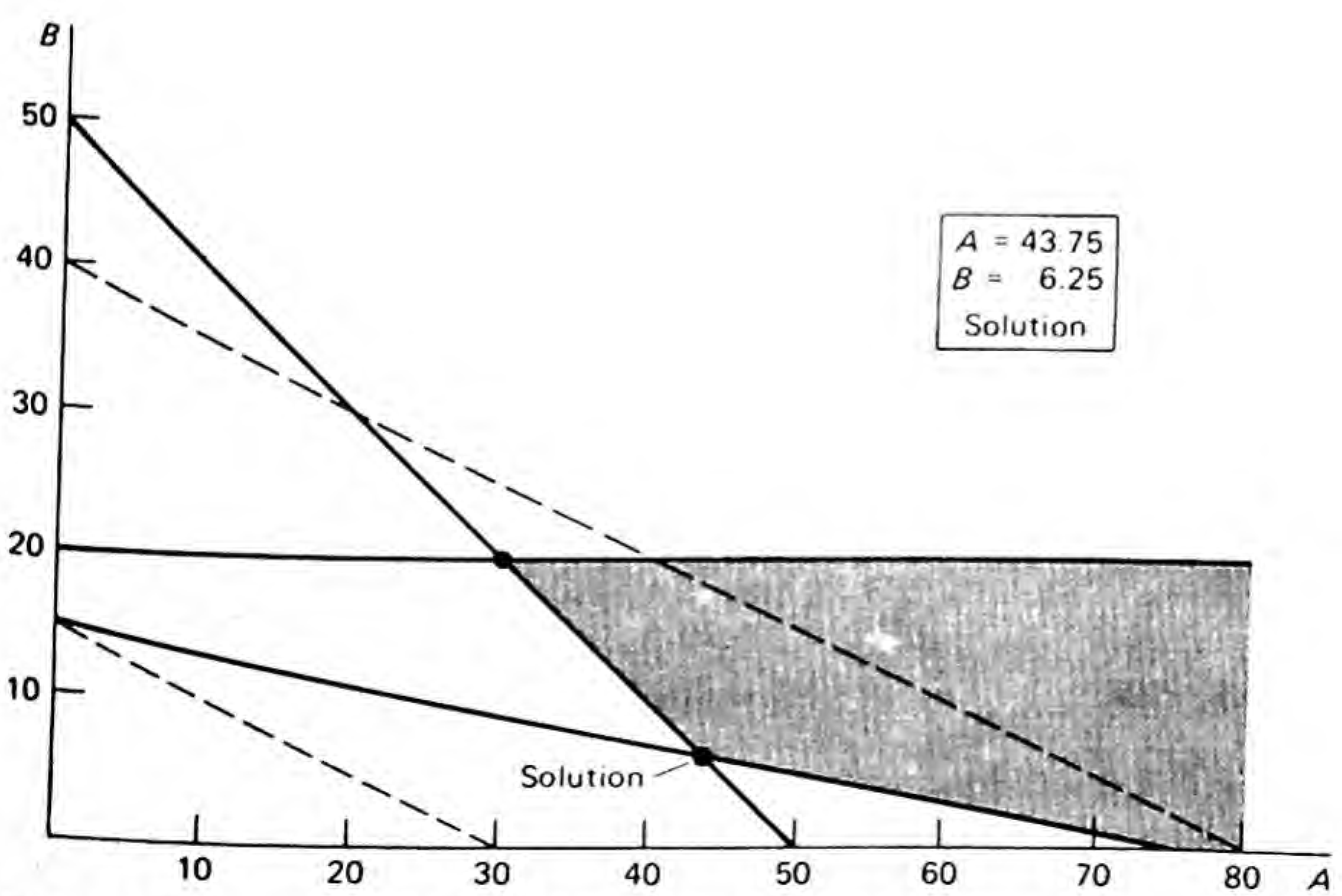


Figure 9A-1 Graphical solution to menu-planning problem.

SIMPLEX METHOD FOR MINIMIZATION PROBLEMS

Surplus Variables

The first step is to write this problem in the augmented form. The only difference in the constraint set between the maximization examples previously covered and this one is the presence of "greater than or equal to" signs.

According to the first constraint,

$$A + B \geq 50$$

the values chosen for the left-hand side must sum to 50 or more. To convert this to an equality, a *surplus* variable S_1 must be introduced in the following way:

$$A + B - 1S_1 = 50$$

Whenever $A + B$ exceeds 50, S_1 will take on the amount of this excess. For example, if $A = 40$ and $B = 20$, S_1 must equal 10.

$$\begin{aligned} A + B - 1S_1 &= 50 \\ 40 + 20 - 10 &= 50 \\ 50 &= 50 \end{aligned}$$

Augmented Form

We can now write the problem in its augmented form.

$$\begin{array}{lllllll} \text{Min } C = 1A + 2B & & & & & & \\ & 1A + & 1B & - 1S_1 & & & = 50 \\ & & 1B & & + 1S_2 & & = 20 \\ & 4A + & 20B & & & - 1S_3 & = 300 \end{array}$$

where S_2 is a slack variable since the second constraint has a "less than or equal to" sign and S_3 is a surplus variable.

Toward the First Basis

Our next step is to construct the first basis. Following the procedure used for maximization problems, we would have:

$$\begin{array}{llllllll} \text{Row 1:} & C - 1A - & 2B & & & & = 0 \\ \text{Row 2:} & 1A + & 1B & - 1S_1 & & & = 50 \\ \text{Row 3:} & & 1B & & + 1S_2 & & = 20 \\ \text{Row 4:} & 4A + & 20B & & & - 1S_3 & = 300 \end{array}$$

The difficulty with this formulation, however, is that a *feasible* basis does not exist. Why? Because we do not have a basic variable in every row. Recall that a basic variable must have a +1 as a coefficient and in addition must have a zero coefficient in

other rows. Notice that S_1 and S_3 *would* qualify as basic variables *if* their coefficients were +1. Some might ask: Why not multiply row 2 and row 4 by −1? Then the coefficient of S_1 and S_3 will be positive. But this would change the sign of the right-hand sides to −50 and −300. And when negative values appear in the right-hand side, we do not have a feasible basis.

Artificial Variables

We must, therefore, artificially contrive a feasible basis by introducing what we will call *artificial variables* into rows 2 and 4.

$$\begin{array}{llr}
\text{Row 2:} & 1A + 1B - 1S_1 \qquad\qquad + 1S_4 & = 50 \\
\text{Row 4:} & 4A + 20B \qquad\qquad - 1S_3 \qquad + 1S_5 & = 300
\end{array}$$

These artificial variables will be used only to get us off to a feasible start. We ensure that they do not appear in the optimal solution by giving them a high cost in the objective function. With these high costs a cost minimization approach will quickly eliminate them from the basis. By a high cost we generally mean 10 times as high as any cost in the objective function. Since the highest cost coefficient is 2, we will let the cost of the artificial variables be 20.

The first basis can now be written in the following way:

$$\begin{array}{llr}
\text{Row 1:} & C - 1A - 2B \qquad\qquad - 20S_4 - 20S_5 & = 0 \\
\text{Row 2:} & 1A + 1B - 1S_1 \qquad\qquad + 1S_4 & = 50 \\
\text{Row 3:} & 1B + 1S_2 & = 20 \\
\text{Row 4:} & 4A + 20B \qquad - 1S_3 \qquad + 1S_5 & = 300
\end{array}$$

Finally . . . the First Feasible Basis

One more step remains, however, before S_4 and S_5 become basic variables. The coefficients of S_4 and S_5 in row 1 must be forced to zero. First we focus on S_4. To force the coefficient of S_4 in row 1 to zero, we simply multiply row 2 by +20 and add it to row 1.

$$\begin{array}{llr}
 & C - 1A - 2B \qquad\qquad - 20S_4 - 20S_5 & = 0 \\
 & 20A + 20B - 20S_1 \qquad + 20S_4 & = 1000 \\
\hline
\text{Revised row 1:} & C + 19A + 18B - 20S_1 \qquad\qquad - 20S_5 & = 1000
\end{array}$$

Rewriting the first basis, we have:

$$\begin{array}{llr}
\text{Row 1:} & C + 19A + 18B - 20S_1 \qquad\qquad - 20S_5 & = 1000 \\
\text{Row 2:} & 1A + 1B - 1S_1 \qquad\qquad + 1S_4 & = 50 \\
\text{Row 3:} & 1B + 1S_2 & = 20 \\
\text{Row 4:} & 4A + 20B \qquad - 1S_3 \qquad + 1S_5 & = 300
\end{array}$$

Next we must force the coefficient of S_5 in row 1 to zero. We can accomplish this by

multiplying row 4 by +20 and adding it to row 1.

$$
\begin{array}{lllr}
C + 19A + 18B - 20S_1 & & & -20S_5 = 1000 \\
80A + 400B & & -20S_3 & +20S_5 = 6000 \\
\hline
C + 99A + 418B - 20S_1 & & -20S_3 & = 7000
\end{array}
$$

Finally the first feasible basis can be written. It is presented in Table 9A-1. The basic variables are S_2, S_4, and S_5, while the nonbasic variables are A, B, S_1, and S_3. Now we are ready to proceed to solve the problem.

Table 9A-1 First Feasible Basis

Row 1:	$C + 99A + 418B - 20S_1$		$-20S_3$		$= 7000$
Row 2:	$1A + 1B - 1S_1$			$+1S_4$	$= 50$
Row 3:	$1B$	$+1S_2$			$= 20$
Row 4:	$4A + 20B$		$-1S_3$		$+1S_5 = 300$

The Simplex Method

To determine the variable which enters the next basis, we choose the row 1 variable with the largest *positive* coefficient. Variable B will then be introduced into the next basis as a basic variable. From this point we proceed exactly as we did in the maximization case.

Step 1: Enter variable B.

Step 2: $^{50}/_1 = 50$

$^{20}/_1 = 20$

$^{300}/_{20} = 15 \leftarrow$

Step 3: Row 4 becomes the pivot row, and since B becomes the new basic variable, row 4 is divided by 20.

New row 4 (pivot row): $\quad {}^1/_5A + B \qquad - {}^1/_{20}S_3 \qquad + {}^1/_{20}S_5 = 15$

Step 4: To get new row 1, we multiply the pivot row by -418 and add.

$$
\begin{array}{llllr}
 & C + 99A + 418B - 20S_1 & & -20S_3 & = 7000 \\
 & - {}^{418}/_5A - 418B & & + {}^{418}/_{20}\,S_3 & - {}^{418}/_{20}\,S_5 = 6270 \\
\hline
\text{New row 1:} & C + {}^{77}/_5\,A & -20S_1 & + {}^{18}/_{20}S_3 & - {}^{418}/_{20}\,S_5 = 730
\end{array}
$$

To get new row 2, we multiply the pivot row by -1 and add.

$$
\begin{array}{lllllr}
 & 1A + 1B - 1S_1 & & +1S_4 & & = 50 \\
 & -{}^1/_5A - 1B & + {}^1/_{20}S_3 & & - {}^1/_{20}S_5 & = -15 \\
\hline
\text{New row 2:} & {}^4/_5A \qquad -1S_1 & + {}^1/_{20}S_3 & +1S_4 & - {}^1/_{20}S_5 & = 35
\end{array}
$$

To get new row 3, we multiply the pivot row by -1 and add.

$$
\begin{array}{rl}
 & 1B + 1S_2 \qquad\qquad\qquad\qquad = 20 \\
 & -\tfrac{1}{5}A - 1B \qquad + \tfrac{1}{20}S_3 \qquad - \tfrac{1}{20}S_5 = -15 \\
\hline
\text{New row 3:} & -\tfrac{1}{5}A \qquad + 1S_2 + \tfrac{1}{20}S_3 \qquad - \tfrac{1}{20}S_5 = 5
\end{array}
$$

We can now summarize the second basis. It appears in Table 9A-2.

Table 9A-2 Second Basis

Row 1:	$C + {}^{77}\!/_5 A$		$-20S_1 + {}^{18}\!/_{20}S_3$		$- {}^{418}\!/_{20}S_5$	$= 730$
Row 2:	${}^{4}\!/_5 A$		$-1S_1 + {}^{1}\!/_{20}S_3$	$+ 1S_4$	$- {}^{1}\!/_{20}S_5$	$= 35$
Row 3:	$-{}^{1}\!/_5 A$		$+1S_2 + {}^{1}\!/_{20}S_3$		$- {}^{1}\!/_{20}S_5$	$= 5$
Row 4:	${}^{1}\!/_5 A$	$+1B$	$- {}^{1}\!/_{20}S_3$		$+ {}^{1}\!/_{20}S_5$	$= 15$

The basic variables are B, S_2, and S_4, and the nonbasic variables are A, S_1, S_3, and S_5. Cost at this point is \$730. Notice that the cost at this second basis (730) is considerably lower than the cost at the first basis (7000). We are heading in the right direction.

Moving toward the next basis, we search row 1 for the largest positive coefficient. Variable A will, therefore, be the new entering variable. Continuing in what now should be a very familiar pattern, we have:

Step 1: Enter variable A.

Step 2: $\dfrac{35}{4/5} = 43.75 \leftarrow$

$\dfrac{5}{-1/5} = -25$

$\dfrac{15}{1/5} = 75$

Step 3: Row 2 becomes the pivot row, and since A becomes the new basic variable, row 2 is divided by $4/5$.

New row 2 (pivot row): $A \quad - {}^{5}\!/_4 S_1 \quad + {}^{1}\!/_{16}S_3 + {}^{5}\!/_4 S_4 - {}^{1}\!/_{16}S_5 = {}^{175}\!/_4$

Step 4: To get the new row 1, the pivot row is multiplied by $-{}^{77}\!/_5$ and added to the old row 1.

$$
\begin{array}{rl}
 & C + {}^{77}\!/_5 A \quad - 20S_1 \quad + {}^{18}\!/_{20}S_3 \qquad\quad - {}^{418}\!/_{20} S_5 = 730 \\
 & -{}^{77}\!/_5 A \quad + {}^{385}\!/_{20} S_1 \quad - {}^{77}\!/_{80}S_3 - {}^{77}\!/_4 S_4 + {}^{77}\!/_{80}S_5 = -{}^{2695}\!/_4 \\
\hline
\text{New row 1:} & C \qquad - {}^{15}\!/_{20}S_1 \quad - {}^{5}\!/_{80}S_3 - {}^{77}\!/_4 S_4 - {}^{1595}\!/_{80} S_5 = {}^{225}\!/_4
\end{array}
$$

To get new row 3, the pivot row is multiplied by $1/5$ and added to the old row 3.

$$
\begin{array}{rl}
 & -{}^{1}\!/_5 A \qquad + 1S_2 + {}^{1}\!/_{20}S_3 \qquad - {}^{1}\!/_{20}S_5 = 5 \\
 & {}^{1}\!/_5 A - {}^{1}\!/_4 S_1 \qquad + {}^{1}\!/_{80}S_3 + {}^{1}\!/_4 S_4 - {}^{1}\!/_{80}S_5 = {}^{35}\!/_4 \\
\hline
\text{New row 3:} & -{}^{1}\!/_4 S_1 + 1S_2 + {}^{5}\!/_{80}S_3 + {}^{1}\!/_4 S_4 - {}^{5}\!/_{80}S_5 = {}^{55}\!/_4
\end{array}
$$

To get new row 4, we multiply the pivot row by $-1/5$ and add to the old row 4.

$$
\begin{array}{lcccl}
 & \tfrac{1}{5}A + 1B & & -\tfrac{1}{20}S_3 & + \tfrac{1}{20}S_5 = 15 \\
 & -\tfrac{1}{5}A & + \tfrac{1}{4}S_1 & -\tfrac{1}{80}S_3 - \tfrac{1}{4}S_4 & + \tfrac{1}{80}S_5 = -\tfrac{35}{4} \\
\hline
\text{New row 4:} & 1B & + \tfrac{1}{4}S_1 & -\tfrac{5}{80}S_3 - \tfrac{1}{4}S_4 & + \tfrac{5}{80}S_5 = \tfrac{25}{4}
\end{array}
$$

We can now summarize the third basis. It appears in Table 9A-3.

Table 9A-3 Third Basis

Row 1:	$1C$	$-\tfrac{15}{20}S_1$		$-\tfrac{5}{80}S_3 - \tfrac{77}{4}S_4 - \tfrac{1595}{80}S_5$	$= \tfrac{225}{4}$
Row 2:	$1A$	$-\tfrac{5}{4}S_1$		$+\tfrac{1}{16}S_3 + \tfrac{5}{4}S_4 - \tfrac{1}{16}S_5$	$= \tfrac{175}{4}$
Row 3:		$-\tfrac{1}{4}S_1$	$+1S_2$	$+\tfrac{5}{80}S_3 + \tfrac{1}{4}S_4 - \tfrac{5}{80}S_5$	$= \tfrac{55}{4}$
Row 4:	$1B$	$+\tfrac{1}{4}S_1$		$-\tfrac{5}{80}S_3 - \tfrac{1}{4}S_4 + \tfrac{5}{80}S_5$,	$= \tfrac{25}{4}$

Solution

Since there are no positive coefficients in row 1 of the third basis, it is also the solution basis and the following strategy should be taken to minimize costs.

$$A = 43.75$$
$$B = 6.25$$

The cost of this combination will be \$43.75. Notice also that the two artificial variables S_4 and S_5 are nonbasic variables or

$$S_4 = 0$$
$$S_5 = 0$$

and there are $S_2 = \tfrac{55}{4}$, or 13.75, unpurchased pounds of meat.

In summary, then, once the problem is in its augmented form and a feasible first basis is determined, the only difference between the solutions of a minimization problem and a maximization problem is that in the minimization case the variable in row 1 that is entered into the next basis is whichever has the largest *positive* value. In maximization problems, the variable which has the largest negative coefficient is entered.

PROBLEMS

9A-1 Solve the following problem using the simplex method:

$$
\begin{aligned}
\text{Min } C &= 4X_1 + 2X_2 \\
1X_1 + 3X_2 &\geq 6 \\
3X_1 + 1X_2 &\geq 6 \\
X_1, X_2 &\geq 0
\end{aligned}
$$

9A-2 Solve the following problem using the simplex method:

$$\begin{aligned} \text{Min } C = 5X_1 + 2X_2 + 1X_3 \\ 1X_1 + 1X_2 + 1X_3 \geq 5 \\ 5X_1 + 3X_2 + 4X_3 \geq 6 \\ X_1, X_2, X_3 \geq 0 \end{aligned}$$

9A-3 Solve the following problem using the simplex method and verify your answer graphically:

$$\begin{aligned} \text{Min } C = 2X + 3Y \\ 1X + 2Y \geq 4 \\ 2X + 1Y \geq 4 \\ X, Y \geq 0 \end{aligned}$$

9A-4 Solve the following problem using the simplex method and verify your answer graphically:

$$\begin{aligned} \text{Min } C = 2X_1 + 1X_2 \\ 1X_1 + 3X_2 \geq 6 \\ 3X_1 + 1X_2 \geq 6 \\ X_1, X_2 \geq 0 \end{aligned}$$

9A-5 The manager of marketing for the Worldwide Frisbee Company is once again preparing his annual advertising strategy. His objective is to minimize the total advertising expenditure while achieving at least some minimal level of exposure to each of four audiences. The audiences include teen-agers, college students, singles between 25 and 35, and newlyweds. To reach these audiences, the media he can use are radio, TV, newspapers, magazines, and billboards.

Let X_j = expenditure on medium j
E_i = minimum exposure desired for audience i
e_{ij} = effectiveness of jth advertising medium on ith audience

Write a linear-programming model for this problem. Do the assumptions of LP (linearity and certainty) raise any doubts about this model's relevance?

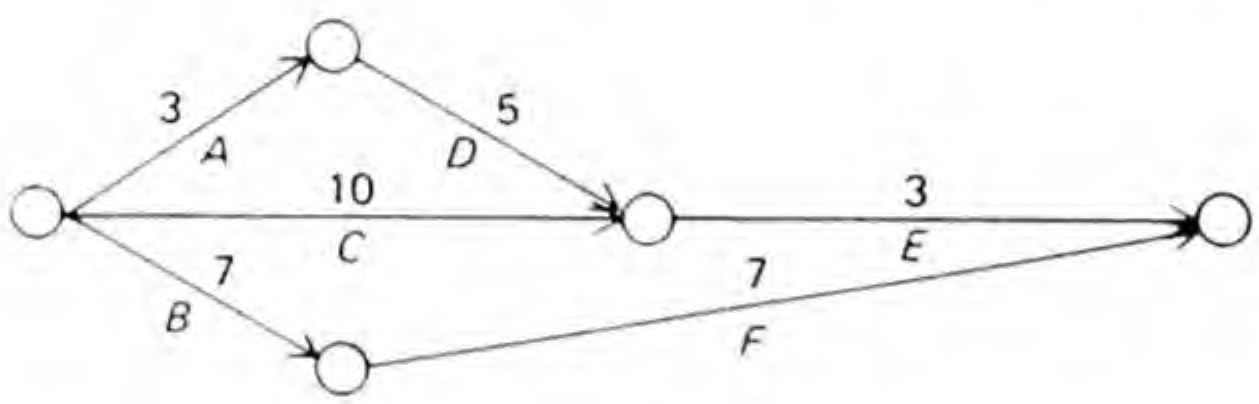
9A-6 The Bruckner Company processes several thousand jobs through their job shop each year. At present they must schedule two jobs A and B. Their objective in scheduling these two jobs is to choose that quantity of each which minimizes their costs subject to demand and supply constraints. The demand requirement is such that at least 100 units of both are required to keep the assembly department in full operation. The shop has at most 150 worker-hours to allocate to these jobs. Job A requires 2 worker-hours per unit, and job B requires 1 worker-hour per unit. If the cost of producing a unit of A is \$5 and the cost of producing a unit of B is \$2, how many of each should be made? (Solve by the simplex method and verify your answer graphically.)

9A-7 The Ace Construction Company must complete a job in as short a time as possible. The job contains activities, and some of these activities must be

completed before others. The activities and their precedence relationships are shown below:

Activity	Precedence
A	—
B	—
C	—
D	A
E	C and D
F	B

The network of jobs can be drawn in the following way:



The figure above each activity represents the average time necessary to complete it.

Let X_i represent the earliest completion of an activity where $i = A, B, C, D, E, F$. Set this up in linear-programming format.

9A-8 The Sanford Company is one of the largest manufacturers of industrial insulation. It maintains two factories and three regional warehouses. The cost of producing a roll of insulation at each of the factories and of shipping it to one of the regional warehouses is given below:

To / From	Warehouse 1	Warehouse 2	Warehouse 3	Capacity
Factory A	.25	.65	.30	1500
Factory B	.40	.55	.28	2000

The capacity at factory A is 1500 units, and the capacity at factory B is 2000 units. The demand at warehouses 1, 2, and 3 is 1000, 800, and 1700 units, respectively.

Let X_{ij} represent the quantity manufactured in factory i and shipped to warehouse j. For example, $X_{B,3}$ would be the quantity manufactured in factory B and shipped to warehouse 3.

Given that the company wishes to minimize its production and shipping costs, set this problem up in linear-programming format.

APPENDIX B: "Equal to" Signs in the Inequality Form

Often a problem may have "equal to" signs in the inequality form of the problem. In this appendix, the procedure for handling such an occurrence will be covered. Consider the example:

$$\text{Min } C = 1A + 2B$$
$$2A + 3B \geq 5$$
$$3A + 1B = 6$$
$$A,\ B \geq 0$$

The example is portrayed graphically in Figure 9B-1. The feasible region consists of the line segment RS, and the optimal solution occurs at point S.

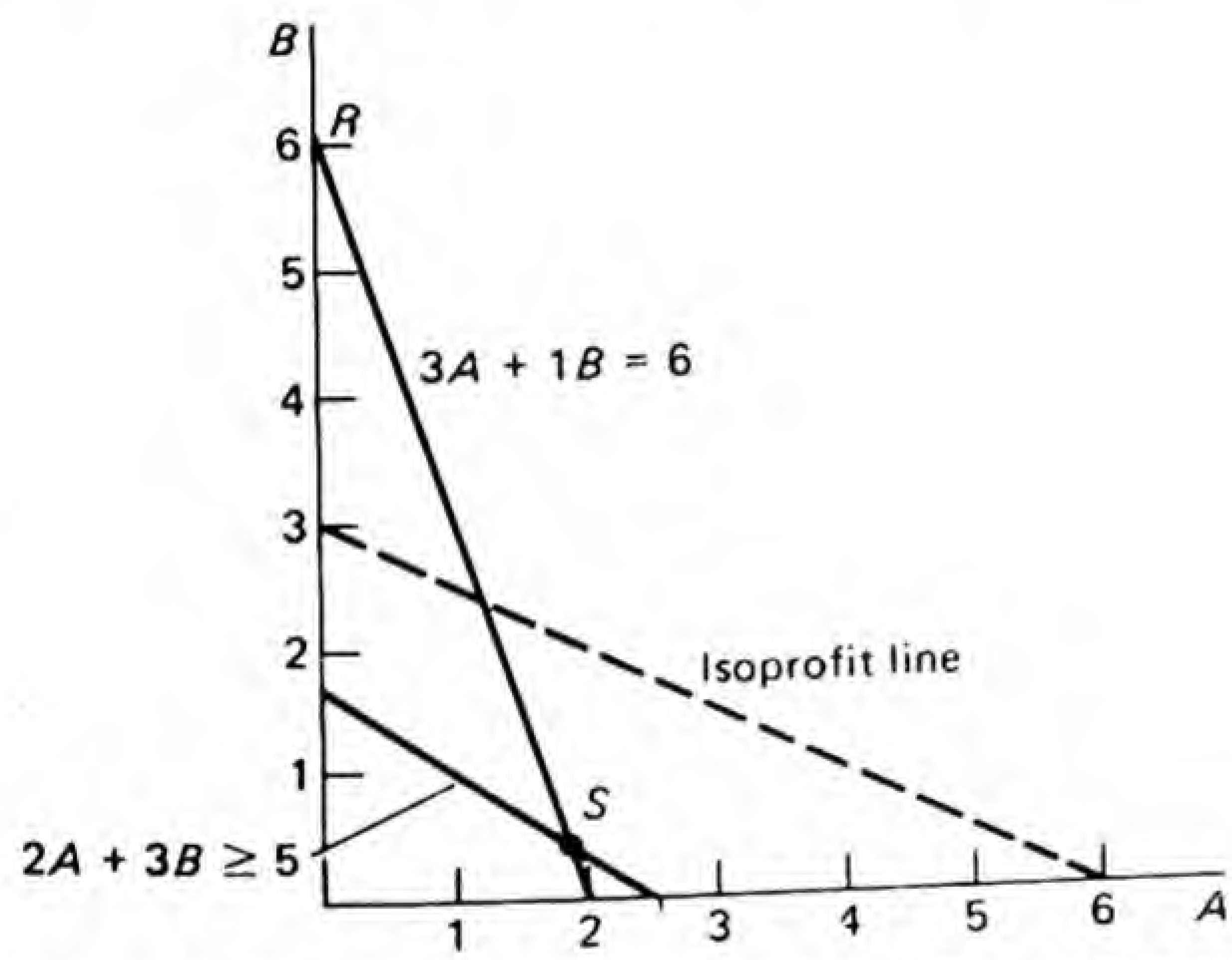


Figure 9B-1 Graphical solution to LP problem with "equal to" signs.

To solve by the simplex method, the first step is to arrange the problem in augmented form.

$$\text{Min } C = 1A + 2B$$
$$2A + 3B - 1S_1 = 5$$
$$3A + 1B \qquad = 6$$

Moving toward our first basis, we have:

$$C - 1A - 2B \qquad = 0$$
$$2A + 3B - 1S_1 = 5$$
$$3A + 1B \qquad = 6$$

We do not, however, have a feasible basis since two basic variables are required and we do not have even one. Two artificial variables must, therefore, be introduced in

the following way:

$$\begin{array}{lllll}
\text{Row 1:} & C - 1A - 2B & & -20S_2 - 20S_3 & = 0 \\
\text{Row 2:} & & 2A + 3B - 1S_1 + & 1S_2 & = 5 \\
\text{Row 3:} & & 3A + 1B & + \ 1S_3 & = 6
\end{array}$$

Our last step in arriving at an initial basis is to force the coefficients of S_2 and S_3 in row 1 to zero. To accomplish this, we multiply row 2 by $+20$ and add to row 1. Then we multiply row 3 by $+20$ and add to row 1. The result is shown in Table 9B-1, and it is clear that we have an initial feasible basis.

Table 9B-1 First Basis

Row 1:	$C + 99A + 78B - 20S_1 \qquad = 220$
Row 2:	$2A + 3B - 1S_1 + 1S_2 \qquad = 5$
Row 3:	$3A + 1B \qquad + 1S_3 = 6$

Since this is a minimization problem, the largest positive coefficient in row 1 will identify the variable to be entered.

Now we proceed to solve this minimization problem in exactly the same way we solved the minimization problem in Appendix A. Choosing the largest positive coefficient in row 1, we determine that variable A will become a basic variable in the second basis. Continuing, we have:

Step 1: Enter variable A.

Step 2: $5/2 = 2.5$

$6/3 = 2 \leftarrow$

Step 3: Row 3 becomes the pivot row, and since A is the new basic variable, the row is divided by 3.

New row 3 (pivot row): $A + 1/3B \qquad + 1/3S_3 = 2$

Step 4: To get new row 2, multiply the pivot row by -2 and add to old row 2.

$$\begin{array}{rl}
2A + 3B - 1S_1 + 1S_2 & = 5 \\
-2A - 2/3B \qquad\qquad - 2/3S_3 & = -4 \\
\hline
\text{New row 2:} \quad 7/3B - 1S_1 + 1S_2 - 2/3S_3 & = 1
\end{array}$$

To get new row 1, multiply the pivot row by -99 and add to old row 1.

$$\begin{array}{rl}
C + 99A + 78B - 20S_1 \qquad\qquad & = 220 \\
-99A - 33B \qquad\qquad - 33S_3 & = -198 \\
\hline
\text{New row 1:} \quad C \qquad + 45B - 20S_1 \qquad - 33S_3 & = 22
\end{array}$$

We can now write the second basis. It is presented in Table 9B-2. To continue in the same fashion, the third basis is computed. It is presented in Table 9B-3.

Table 9B-2 Second Basis

Row 1:	$C + 45B - 20S_1 \qquad - 33S_3 = 22$
Row 2:	$\tfrac{7}{3}B - 1S_1 + 1S_2 - \tfrac{2}{3}S_3 = 1$
Row 3:	$A + \tfrac{1}{3}B \qquad + \tfrac{1}{3}S_3 = 2$

Table 9B-3 Third Basis

Row 1:	$C \quad - \tfrac{5}{7}S_1 - \tfrac{135}{7}S_2 - \tfrac{141}{7}S_3 = \tfrac{19}{7}$
Row 2:	$1B - \tfrac{3}{7}S_1 + \tfrac{3}{7}S_2 - \tfrac{2}{7}S_3 = \tfrac{3}{7}$
Row 3:	$1A \quad + \tfrac{1}{7}S_1 - \tfrac{1}{7}S_2 + \tfrac{3}{7}S_3 = \tfrac{13}{7}$

Since there are no positive coefficients in row 1, we can conclude that an optimal solution has been reached and that the minimum cost can be achieved when

$$B = \tfrac{3}{7}$$
$$A = \tfrac{13}{7}$$

PROBLEMS

9B-1 Solve by the simplex method:

$$\begin{aligned} \text{Min } C &= 4X + 3Y \\ X + 2Y &\geq 2 \\ 8X + 6Y &= 12 \\ X, Y &\geq 0 \end{aligned}$$

9B-2 Solve by the simplex method and verify your answer graphically:

$$\begin{aligned} \text{Min } C &= 5X + 2Y \\ X + Y &\geq 5 \\ X &= 3 \\ X, Y &\geq 0 \end{aligned}$$

9B-3 Solve by the simplex method and verify graphically:

$$\begin{aligned} \text{Max } P &= 20X + 2Y \\ X + Y &\geq 5 \\ X &\leq 5 \\ Y &\leq 5 \\ X + Y &= 8 \\ X, Y &\geq 0 \end{aligned}$$

9B-4 Solve by the simplex method:

$$\begin{aligned} \text{Max } P &= 5A + 1C \\ A + \ C &\leq 5 \\ A \qquad &= 2 \\ A, C &\geq 0 \end{aligned}$$

APPENDIX C: Sensitivity Analysis

INTRODUCTION

The concept of sensitivity analysis was introduced earlier in the chapter. The purpose of this appendix is to explore this concept on a more formal basis.

First we will analyze the sensitivity of a problem's solution to changes in the original objective function coefficients, and then sensitivity will be analyzed with respect to changes in the original right-hand sides of the constraint equations.

COEFFICIENTS OF THE OBJECTIVE FUNCTION

How would a change in a coefficient of the objective function affect the solution? Within what range of values will the coefficient of an objective function leave the solution unaffected? To answer these questions, we must first differentiate according to whether the coefficient in question is associated with a basic or nonbasic variable in the *solution* basis.

Nonbasic Variables

To be more specific, the following question will be asked about the problem depicted in Table 9C-1. How would a change in the coefficient of variable Y found in the objective function affect the solution?

According to Table 9C-1, each unit of Y produced yields a \$10 profit contribution. Suppose it is suspected that the profitability of Y is actually less than this. Would the solution given in Table 9C-1 remain optimal?

The answer to this can be found by turning to the solution basis. In it we find that Y is a nonbasic variable, and is not produced at all. If it were even less profitable, it still would not be produced. Therefore, any *reduction* in the row 1 coefficients of those variables which are *nonbasic* in the solution basis do not affect the solution to the problem.

However, an increase in the per unit profit contribution of Y might make it profitable enough to be produced. How large would the increase have to be before this would happen? This question is slightly more complex to answer and requires the introduction of new notation. Instead of the per unit profitability of Y being 10, let us denote it as $10 + \Delta$ or ten plus some change, delta. Rewriting row 1 of the first basis, we have:

$$\text{Row 1:} \quad P - 30X - (10 + \Delta)Y = 0$$

Table 9C-1 Waverly Company Problem

	Product *X*, per unit	Product *Y*, per unit	Capacity, hours
Profit contribution	\$30	\$10	
Department A	2 hours	1 hour	10
Department B	1 hour	2 hours	10

$$\text{Max profit} = 30X + 10Y$$
$$2X + 1Y \le 10$$
$$1X + 2Y \le 10$$
$$X, Y \ge 0$$

$$\left.\begin{aligned}&\text{Row 1:}\quad P - 30X - 10Y &&= 0\\ &\text{Row 2:}\quad 2X + 1Y + 1S_1 &&= 10\\ &\text{Row 3:}\quad 1X + 2Y \qquad + 1S_2 &&= 10\end{aligned}\right\}\ \text{First basis}$$

$$\left.\begin{aligned}&\text{Row 1:}\quad P \qquad + 5Y + 15S_1 &&= 150\\ &\text{Row 2:}\quad 1X + \tfrac{1}{2}Y + \tfrac{1}{2}S_1 &&= 5\\ &\text{Row 3:}\quad \tfrac{3}{2}Y - \tfrac{1}{2}S_1 + 1S_2 &&= 5\end{aligned}\right\}\ \text{Second or solution basis}$$

$$\text{Solution:}\quad X = 5 \qquad Y = 0$$
$$S_2 = 5 \qquad S_1 = 0$$

or

$$P - 30X - 10Y - \Delta Y = 0$$

This modified problem is solved in Table 9C-2. Some interesting observations can be made by comparing the solution of this problem with that of the original problem in Table 9C-1. First we find that for each basis, rows 2 and 3 in the modified problem are exactly the same as rows 2 and 3 in the original problem. In fact, whenever sensitivity analysis is undertaken on the objective function, the constraint rows in all bases including the solution basis will be unaffected.

The second observation concerns the objective function. The $-\Delta Y$ introduced into the objective function of the initial basis appears unchanged in the solution basis. Again, this will always hold true. A change $-\Delta Y$ introduced into the objective function of the initial basis will remain unchanged in *each* and *every* basis including the solution basis.

The modified solution basis is repeated in Table 9C-3.

We know that as long as the coefficients of the row 1 variables remain positive, the solution remains optimal. If any coefficient becomes negative, further bases must be generated and the solution will change. Returning to row 1 in Table 9C-3, we must ask: When will all coefficients remain positive? The answer is that they will remain positive as long as $(5 - \Delta)$ is greater than zero or:

$$5 - \Delta > 0$$
$$-\Delta > -5$$

Table 9C-2 Solution of the Modified Problem

Row 1:	$P - 30X - 10Y - \Delta Y$	$= 0$
Row 2:	$2X + 1Y \quad + 1S_1$	$= 10$
Row 3:	$1X + 2Y \quad + 1S_2$	$= 10$
New row 2 (pivot row):	$X + \frac{1}{2}Y \quad + \frac{1}{2}S_1$	$= 5$
To get new row 1:		
Old row 1:	$P - 30X - 10Y - \Delta Y$	$= 0$
+ 30 (pivot row):	$30X + 15Y \quad + 15S_1$	$= 150$
New row 1:	$P \quad + 5Y - \Delta Y + 15S_1$	$= 150$
To get new row 3:		
Old row 3:	$1X + 2Y \quad + 1S_2$	$= 10$
− 1 (pivot row):	$-1X - \frac{1}{2}Y \quad - \frac{1}{2}S_1$	$= -5$
New row 3:	$\frac{3}{2}Y \quad - \frac{1}{2}S_1 + 1S_2$	$= 5$
Row 1:	$P \quad + 5Y - \Delta Y + 15S_1$	$= 150$
Row 2:	$1X + \frac{1}{2}Y \quad + \frac{1}{2}S_1$	$= 5$
Row 3:	$\frac{3}{2}Y \quad - \frac{1}{2}S_1 + 1S_2$	$= 5$

Note that rows 2 and 3 have not changed at all.

Table 9C-3 Modified Solution Basis for Investigating a Change in the Coefficient of Variable *Y*

The $-\Delta Y$ comes through *all* bases unchanged.

Row 1:	$P \quad + (5 - \Delta)Y + 15S_1$	$= 150$
Row 2:	$1X + \quad \frac{1}{2}Y + \frac{1}{2}S_1$	$= 5$
Row 3:	$\frac{3}{2}Y - \frac{1}{2}S_1 + 1S_2$	$= 5$

which can be simplified to[1]

$$\boxed{\Delta < 5}$$

We can conclude that as long as the change is less than \$5, the solution to the problem is unchanged. If, on the other hand, the change is greater than \$5, row 1 in our final

[1] Multiplying an inequality through by -1 changes the direction of the

basis will have a negative coefficient and additional bases will be necessary to find the new solution.

As an example, consider that you are unsure of the profit contribution associated with product Y. You feel that it is probably \$10 per unit (the original coefficient in the objective function) but might be as high as \$13 per unit. The question is: Will this increase of \$3 ($\Delta = +3$) change the solution which we have already found in Table 9C-1? No, because the change of $\Delta = +3$ is within the range ($\Delta < 5$) which we computed to have no effect on the solution.

Basic Variables

Returning to Table 9C-1, let's ask what would happen if the profitability of just X changed. Rewriting row 1, we have the following:

$$\text{Row 1:}\quad P - (30 + \Delta)X - 10Y \qquad = 0$$

Just as it did in the previous section when we were dealing with nonbasic variables, the $-\Delta X$ comes through each basis unchanged. It appears as $-\Delta X$ in the modified solution basis of Table 9C-4.

Table 9C-4 Modified Solution Basis for Investigating a Change in Variable *X*

Row 1:	$P - \Delta X + 5Y + 15S_1 \qquad = 150$
Row 2:	$1X + \tfrac{1}{2}Y + \tfrac{1}{2}S_1 \qquad = 5$
Row 3:	$\tfrac{3}{2}Y - \tfrac{1}{2}S_1 + 1S_2 = 5$

We can see from Table 9C-1 that X is a basic variable in the solution basis. But in Table 9C-4 the introduction of a $-\Delta X$ in row 1 violates the definition of a basic variable. It will therefore be necessary to manipulate the equations in Table 9C-4 until variable X is restored to basic status. To accomplish this, we multiply row 2 in Table 9C-4 by Δ and add it to row 1.

$$\begin{array}{llccc}
\text{Row 1:} & P - \Delta X + & 5Y + & 15S_1 = 150 \\
\text{Row } 2 \times \Delta: & \quad\;\; \Delta X + & \tfrac{1}{2}\Delta Y + & \tfrac{1}{2}\Delta S_1 = 5\Delta \\
\hline
\text{New row 1:} & P & + (5 + \tfrac{1}{2}\Delta)Y & + (15 + \tfrac{1}{2}\Delta)S_1 = 150 + 5\Delta
\end{array}$$

The new modified solution basis is shown in Table 9C-5.

Again, the present solution will remain optimal if all coefficients in the objective function of the final basis remain positive. This will occur as long as both $(5 + \tfrac{1}{2}\Delta)$ and $(15 + \tfrac{1}{2}\Delta)$ are greater than zero. To examine them one at a time, $(5 + \tfrac{1}{2}\Delta)$ will be positive as long as

$$5 + \tfrac{1}{2}\Delta > 0$$
$$\tfrac{1}{2}\Delta > -5$$
$$\Delta > -10$$

Table 9C-5 New Modified Solution Basis with Variable X Restored to Basic Variable Status

Row 1:	P		$+(5+\frac{1}{2}\Delta)Y+$	$(15+\frac{1}{2}\Delta)S_1$	$=150+5\Delta$
Row 2:		$1X+$	$\frac{1}{2}Y+$	$\frac{1}{2}S_1$	$=5$
Row 3:			$\frac{3}{2}Y-$	$\frac{1}{2}S_1+1S_2$	$=5$

Therefore the coefficient of Y in this basis will remain positive as long as Δ is greater than -10. This region is illustrated in Figure 9C-1.

The coefficient of S_1 will remain positive as long as

$$15+\tfrac{1}{2}\Delta>0$$
$$\tfrac{1}{2}\Delta>-15$$
$$\Delta>-30$$

This region also is illustrated in Figure 9C-1.

Combining both regions in Figure 9C-1, we can conclude that if Δ is greater than -10, it will *also* be greater than -30. Therefore, as long as Δ is greater than -10, row 1 will remain positive and the solution will remain optimal.

$$\boxed{\Delta>-10}$$

Let's test this result. Suppose the profitability of X was suspected to be \$8 per unit less than its originally estimated profitability of \$30 per unit. We would therefore have $\Delta=-8$, that is, a negative change of \$8. Since -8 is greater than -10, the solution should remain optimal. Checking row 1 in Table 9C-5, we have

$$\text{Row 1:}\quad P+[5+\tfrac{1}{2}(-8)]Y+[15+\tfrac{1}{2}(-8)]S_1=150+5(-8)$$

This can be simplified to

$$P+1Y \qquad +11S_1 \qquad =110$$

and we conclude that indeed row 1 does stay positive. Notice, however, that the

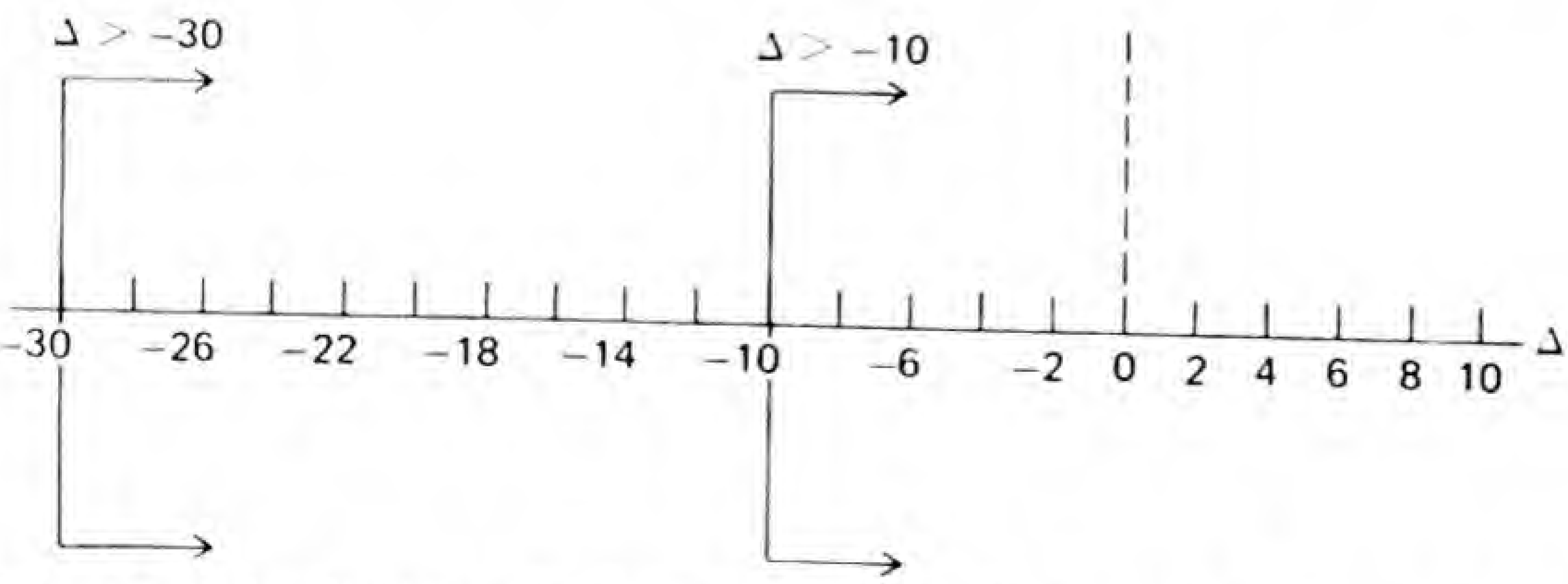


Figure 9C-1 Region within which the coefficient of Y remains positive in the objective function of the final basis.

profitability drops. Profit associated with this optimal solution drops from 150 to 110. The reason for this is that the per unit profitability of the product has dropped. The quantity produced of this product, however, remains the same.

RIGHT-HAND-SIDE CONSTRAINTS

Now we turn to the right-hand side of the constraint equations and determine the consequence, if any, of a change in these constraint values. For example, we might like to determine the consequence of a decrease in capacity from 10 to 8 units in department B. That is, if the capacity in department B were reduced by 2 units, would the solution depicted in Table 9C-1 remain optimal?

In evaluating the consequence of these changes, the same general approach taken in the last section will be taken here. This time, however, we will differentiate our method depending upon whether the slack variable corresponding to the right-hand side of interest is basic or nonbasic in the solution basis. We first turn to the analysis of right-hand sides when the corresponding slack variable is a basic variable in the solution basis.

When the Slack Variable in the Corresponding Row Is a Basic Variable in the Solution Basis

The first basis is rewritten in Table 9C-6 and incorporates a delta in the right-hand side of row 3.

Notice that the delta is similar to the basic variable S_2 in the same row. Both have a coefficient of 1 and do not appear elsewhere in their individual columns. Therefore, whatever happens to S_2 in each basis also happens to Δ. Since S_2 in the solution

Table 9C-6 Waverly Company Problem Analyzed for Changes in Department B's Constraint

Row 1:	$P - 30X - 10Y \qquad = 0$
Row 2:	$2X + 1Y + 1S_1 \qquad = 10$
Row 3:	$1X + 2Y \qquad + 1S_2 = 10 + 1\Delta$

Table 9C-7 Solution Basis Modified to Illustrate the Influence of a Change in the Constraint Capacity Associated with Department B

Row 1:	$P \qquad + 5Y + 15S_1 \qquad = 150$
Row 2:	$1X + \frac{1}{2}Y + \frac{1}{2}S_1 \qquad = 5$
Row 3:	$\frac{3}{2}Y - \frac{1}{2}S_1 + 1S_2 = 5 + 1\Delta$

Note that the slack variable associated with this department, S_2, is a basic variable.

basis of Table 9C-2 retains its coefficient of 1 (it is a basic variable), we can conclude Δ will also appear and retain its coefficient of 1. The solution basis is rewritten in Table 9C-7 to reflect this.

It should be apparent that a change in the right-hand side of the initial problem does not affect the left-hand side of row 1 in the solution basis. Since the left-hand side of row 1 is not affected, it remains positive. However, in order to have a *feasible* solution basis, all of the right-hand values must remain positive. Remember the nonnegativity conditions? It makes no sense to produce -15 units of product X. We must therefore ensure that the right-hand side remains positive. If so, then the solution basis of Table 9C-7 remains optimal.

The right-hand side will remain positive as long as the following holds true:

$$(5+\Delta) > 0$$

or

$$\boxed{\Delta > -5}$$

We can therefore conclude that the solution shown in Table 9C-7 will remain optimal, provided that the change associated with department B's capacity is greater than -5.

For example, if the capacity of department B were decreased by 2 units, we would have $\Delta = -2$, and since -2 is indeed greater than -5, the present solution would remain optimal.

When the Slack Variable in the Corresponding Row Is a Nonbasic Variable in the Solution Basis

Next, we will analyze the consequence of a change in the capacity of department A. In Table 9C-8 a delta is introduced in the right-hand side of row 2. Again, it is similar to the S_1 in the same row. Both of them have a coefficient of 1 and appear nowhere else in their columns. Consequently, whatever happens to S_1 from one basis to the next must also happen to Δ.

Moving down to the solution basis in Table 9C-8, we see that S_1 is a nonbasic variable and has a coefficient of $+15$ in row 1, $+1/2$ in row 2, and $-1/2$ in row 3. We can therefore conclude that the coefficient of Δ must also be $+15$ in row 1, $+1/2$ in row 2, and $-1/2$ in row 3.

Since the solution basis in Table 9C-8 remains optimal as long as the right-hand values remain feasible, we must now determine for what values of Δ these right-hand sides remain positive. Looking at the first right-hand side, we can conclude that it remains positive as long as the following holds:

$$(150 + 15\Delta) > 0$$

or

$$\begin{aligned} 150 + 15\Delta &> 0 \\ 15\Delta &> -150 \\ \Delta &> -10 \end{aligned}$$

Table 9C-8 Sensitivity Analysis Performed on the First Right-Hand-Side Constraint

	Initial basis	
Row 1:	$P - 30X - 10Y$	$= 0$
Row 2:	$2X + 1Y + 1S_1$	$= 10 + 1\Delta$
Row 3:	$1X + 2Y \qquad + 1S_2$	$= 10$
	Solution basis	
Row 1:	$P \qquad + 5Y + 15S_1$	$= 150 + 15\Delta$
Row 2:	$1X + \frac{1}{2}Y + \frac{1}{2}S_1$	$= 5 + \frac{1}{2}\Delta$
Row 3:	$\frac{3}{2}Y - \frac{1}{2}S_1 + 1S_2$	$= 5 - \frac{1}{2}\Delta$

This region is illustrated in Figure 9C-2. Turning to the next constraint, we have the following:

$$(5 + \tfrac{1}{2}\Delta) > 0$$

or

$$\begin{aligned} 5 + \tfrac{1}{2}\Delta &> 0 \\ \tfrac{1}{2}\Delta &> -5 \\ \Delta &> -10 \end{aligned}$$

Since this is a duplication of the region just identified, we can move on to the last right-hand-side constraint.

$$\begin{aligned} (5 - \tfrac{1}{2}\Delta) &> 0 \\ -\tfrac{1}{2}\Delta &> -5 \\ -\Delta &> -10 \\ \Delta &< 10 \end{aligned}$$

Again, we illustrate this region in Figure 9C-2.

Now if all the regions illustrated in Figure 9C-2 are combined, it can be concluded that as long as the *change* is greater than -10 but less than $+10$, the right-hand side

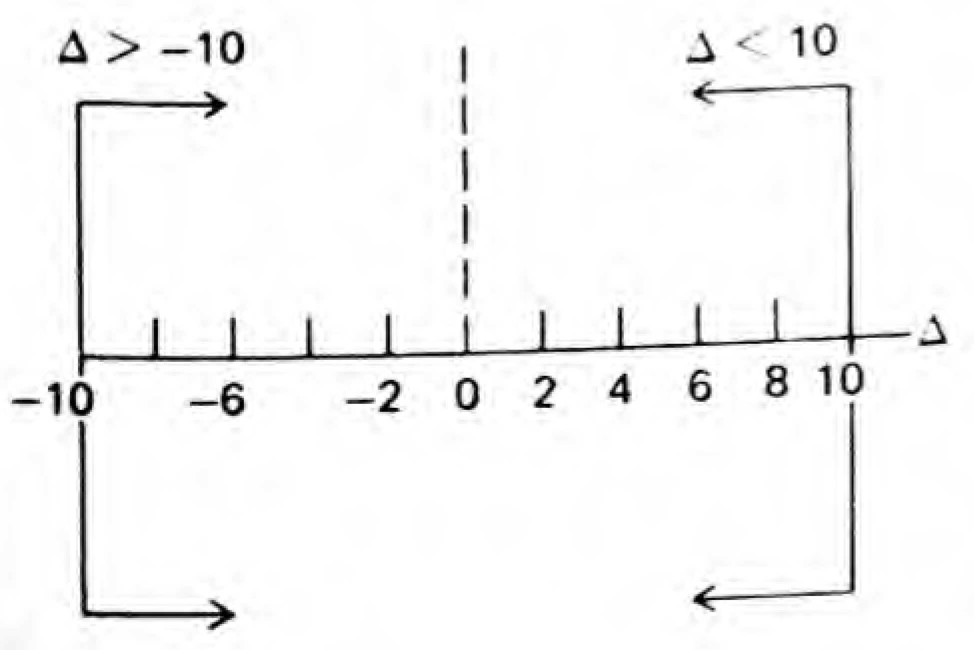


Figure 9C-2 Region within which the right-hand side of the solution basis remains feasible for a change in the first right-hand side constraint.

of the solution basis will remain feasible and the solution will remain optimal. This interval can be summarized in the following way:

$$\boxed{-10 < \Delta < +10}$$

As an example, consider an increase of 3 units of capacity in the first constraint. In other words, $\Delta = +3$. We can conclude that the solution basis depicted in Table 9C-8 is still optimal since a delta of $+3$ is indeed within the interval.

Returning to the final basis of Table 9C-8, we can also see that an increase of 3 units of capacity in the first constraint will affect the value of the objective function. That is, the right-hand side of row 1 will now become:

$$150 + 15\Delta$$

or

$$150 + 15(3) = 195$$

Profit therefore increases by \$45 to \$195.

PROBLEMS

9C-1 Suppose that the objective function in Table 9C-1 were changed to the following. Would the solution basis shown in that same table remain optimal?

a Max profit $= 30X + 5Y$
b Max profit $= 30X + 12Y$
c Max profit $= 45X + 10Y$
d Max profit $= 50X + 3Y$
e Max profit $= 19X + 10Y$
f Max profit $= 22X + 10Y$

9C-2 Suppose that the right-hand-side values in Table 9C-1 were changed to the following. Would the solution basis shown in that same table remain optimal?

a 10, 50
b 10, 6
c 19, 10
d 9, 10

9C-3 The first and final bases of a linear-programming problem are given below:

$$\begin{array}{llr}
\text{Row 1:} & P - 5T - 7B & = 0 \\
\text{Row 2:} & 2T + 1B + 1S_1 & = 10{,}000 \\
\text{Row 3:} & 1T + 2B \qquad + 1S_2 & = 12{,}000
\end{array}$$

$$\begin{array}{ll}
\text{Row 1:} & P \qquad + 1S_1 + 3S_2 = 46{,}000 \\
\text{Row 2:} & 1T \qquad + \tfrac{2}{3}S_1 - \tfrac{1}{3}S_2 = {}^{8000}/_{3} \\
\text{Row 3:} & 1B - \tfrac{1}{3}S_1 + \tfrac{2}{3}S_2 = {}^{28{,}000}/_{6}
\end{array}$$

- **a** What is the solution to this problem? Identify each variable, its level of intensity, and the profit.
- **b** For what changes in the profitability of T will the final solution remain optimal?
- **c** For what changes in the capacity of the first constraint will the solution given above remain optimal?

9C-4 Refer to the first and final bases of problem 9C-3.

- **a** For what changes in the profitability of B will the final solution remain optimal?
- **b** For what changes in the capacity of the second constraint will the solution remain optimal?

9C-5 Refer to the first and final bases of problem 9C-3.

- **a** Give an interval for the coefficient of T such that the solution remains optimal.
- **b** Give an interval for the first constraint such that the solution remains optimal.

9C-6 The first and final bases of a linear-programming problem are given below:

$$\begin{array}{lllll}
\text{Row 1:} & P - 3A - \;\;4B & & & = 0 \\
\text{Row 2:} & \quad 2A + \;\;1B + 1S_1 & & & = 16 \\
\text{Row 3:} & \quad 4A + 10B & & + \;1S_2 & = 20
\end{array}$$

$$\begin{array}{lllll}
\text{Row 1:} & P \quad + \tfrac{7}{2}B & & + \tfrac{3}{4}S_2 & = 15 \\
\text{Row 2:} & \quad\quad - \;\;4B + 1S_1 & & - \tfrac{1}{2}S_2 & = 6 \\
\text{Row 3:} & \quad 1A + \tfrac{5}{2}B & & + \tfrac{1}{4}S_2 & = 5
\end{array}$$

- **a** What is the solution to this problem? Identify each variable, its level of intensity, and profit.
- **b** For what changes in the profitability of A will the final solution remain optimal?
- **c** For what changes in the capacity of the first constraint will the solution remain optimal?

9C-7 Refer to the first and final bases of problem 9C-6.

- **a** For what changes in the profitability of B will the final solution remain optimal?
- **b** For what changes in the capacity of the second constraint will the solution remain optimal

9C-8 Refer to the first and final bases of problem 9C-6.

- **a** Give an interval for the coefficient of A such that the solution remains optimal.
- **b** Give an interval for the second constraint such that the solution remains optimal.

APPENDIX D: Duality

INTRODUCTION

Every linear-programming problem can be expressed in another form, called the *dual*. Formulating and solving the dual provides us with new insights into linear programming as well as providing useful management information.

All the linear-programming problems covered in this and previous chapters can be designated as *primal*. And any primal problem can be converted into its dual by following just a few steps. First we will look at these steps and then we will explore the reasons behind them.

THE MECHANICS OF THE PRIMAL-DUAL RELATIONSHIP

The Jerry Company problem depicted in Table 7-1 is repeated in Table 9D-1. It is labeled the primal problem. By following the rules given below it can be converted into a dual problem.

Rule 1 If the objective function in the primal is maximized, the objective function in the dual is minimized.

Rule 2 The right-hand-side values of the primal become the coefficients of the objective function in the dual. The first right-hand-side value becomes the first coefficient in the objective function. The second right-hand-side value becomes the second coefficient in the objective function, and so on.

Rule 3 The coefficients of the objective function in the primal become the right-hand-side values in the dual.

Rule 4 The coefficients in the rows of the primal constraints become the coefficients

Table 9D-1 Primal and Dual Problems

	Camping tents, per unit	Mountain tents, per unit	Capacity
Profit contribution	\$20	\$30	
Fabric cutting	2 worker-hours	1 worker-hour	8,000 worker-hours
Sewing	2 worker-hours	4 worker-hours	16,000 worker-hours

$$\left.\begin{aligned} \text{Max } P &= 20C + 30M \\ 2C + 1M &\le 8000 \\ 2C + 4M &\le 16{,}000 \\ C, M &\ge 0 \end{aligned}\right\}\text{Primal}$$

$$\left.\begin{aligned} \text{Min } E &= 8000Y + 16{,}000Z \\ 2Y + 2Z &\ge 20 \\ 1Y + 4Z &\ge 30 \\ Y, Z &\ge 0 \end{aligned}\right\}\text{Dual}$$

of the columns in the dual constraints. The coefficients of the first row in the primal become the coefficients of the first column in the dual, and so on.

Rule 5 The direction of the inequality found in the primal is reversed. If the primal has "less than or equal to" constraints, they will become "greater than or equal to" in the dual.

These rules have been followed in developing the dual problem presented in Table 9D-1. Both Y and Z are called dual variables, whereas C and M are called primal variables. Utilizing the simplex solution method, it is possible to solve this dual problem and obtain values for these dual variables. If indeed this were done, you would find that the value of the objective function for the primal solution would equal the value of the objective function for the dual solution. This is no accident, and comes from the fact that there is an interesting correspondence between the primal and dual formulations. We now leave the mechanical aspects of primal-dual relationships and shift our focus to an economic analysis and interpretation of the correspondence between them.

THE ECONOMIC MEANING OF THE DUAL

In analyzing the economic meaning of the dual, we will first look at the variables, then the constraints, and finally the objective function.

The Variables

In the primal problem, the variables M and C represent the level of an activity. For example, in the Jerry Company case they represent the quantity of camping and mountain tents produced. In the dual problem the variables Y and Z represent the accounting value of a unit of scarce resource. In the dual problem depicted in Table 9D-1, the first dual variable Y represents the accounting value to the firm of a unit of capacity in the fabric-cutting department. Similarly, Z represents the accounting value to the firm of a unit of capacity in the second constraint department, sewing. Another interpretation which can be given to these dual variables is that they represent how much each unit of additional resource is worth to the firm.

The Constraints

The first constraint in the dual problem was written in the following way:

$$2Y + 2Z \geq 20$$

The coefficients on the left-hand side of the inequality represent the per unit inputs necessary to produce 1 unit of output. That is, each camping tent requires 2 units of fabric-cutting capacity and 2 units of sewing capacity. Multiplying these coefficients by the per unit accounting value of each of the scarce resources gives us the *total accounting value* of the inputs necessary to produce 1 unit of output, camping tents.

The right-hand side of the inequality represents the per unit profit of camping tents. The purpose of this inequality, then, is to assure that the accounting values Y and Z fully allocate the per unit profit associated with a camping tent to the scarce resources. Said another way, we are trying to find what portion of the per unit profit we "owe" to each of the scarce resources.

The second constraint can be interpreted in the same way. The values of Y and Z which are found must *also* fully allocate the per unit profit made from each mountain tent.

Objective Function

The objective function for our dual problem was written in the following way:

$$\text{Min } E = 8000Y + 16{,}000Z$$

It represents the total accounting value of all the firm's scarce resources. Our objective, then, is to find the minimum accounting value of Y and Z which, according to our constraints, completely allocates the per unit profits to each of the resources.

SOLUTION OF THE DUAL

The dual problem which is presented in Table 9D-1 is solved in Table 9D-2. Since this is a problem in minimization, the method described in Appendix A must be followed.

In the solution basis we see that the value of the dual objective function is 13,333. This is precisely the value found for the primal objective function. It is characteristic of primal-dual problems that the optimal value of the primal objective function will always equal the optimal value of the dual objective function.

We can also see from this solution basis that the accounting value of a unit of Y is $^{10}/_{3}$, or \$3.33, and the accounting value of a unit of Z is $^{80}/_{12}$, or \$6.66. Since Y represents the accounting value of a unit of resource in the fabric-cutting department,

Table 9D-2 Solution of the Dual Problem

Row 1: $E - 8000Y - 16{,}000Z - 160{,}000L_3 - 160{,}000L_4 = 0$
Row 2: $2Y + 2Z - 1L_1 + 1L_3 = 20$
Row 3: $1Y + 4Z - 1L_2 + 1L_4 = 30$

Row 1: $E + 472{,}000Y + 944{,}000Z - 160{,}000L_1 - 160{,}000L_2 = 8{,}000{,}000$
Row 2: $2Y + 2Z - 1L_1 + 1L_3 = 20$
Row 3: $1Y + 4Z - 1L_2 + 1L_4 = 30$

Row 1: $E + 236{,}000Y - 160{,}000L_1 + 76{,}000L_2 - 236{,}000L_4 = 920{,}000$
Row 2: $\tfrac{3}{2}Y - 1L_1 + \tfrac{1}{2}L_2 + 1L_3 - \tfrac{1}{2}L_4 = 5$
Row 3: $\tfrac{1}{4}Y + 1Z - \tfrac{1}{4}L_2 + \tfrac{1}{4}L_4 = \tfrac{30}{4}$

Row 1: $E - 2666L_1 - 2666L_2 - 157{,}333L_3 - 157{,}333L_4 = 133{,}333$
Row 2: $1Y - \tfrac{2}{3}L_1 + \tfrac{1}{3}L_2 + \tfrac{2}{3}L_3 - \tfrac{1}{3}L_4 = \tfrac{10}{3}$
Row 3: $1Z + \tfrac{1}{6}L_1 + \tfrac{1}{6}L_2 - \tfrac{1}{6}L_3 + \tfrac{2}{6}L_4 = \tfrac{80}{12}$

Solution: $E = 133{,}333$ $L_1 = 0$
$Y = \tfrac{10}{3}$ $L_2 = 0$
$Z = \tfrac{80}{12}$

Table 9D-3 Primal Solution Basis to Jerry Company Problem Taken from Table 9-3

Row 1:	P	$\frac{10}{3} S_1 + \frac{80}{12} S_2 = 133{,}333$
Row 2:	C	$\frac{2}{3} S_1 - \frac{2}{12} S_2 = \frac{8000}{3}$
Row 3:	M	$- \frac{1}{3} S_1 + \frac{1}{3} S_2 = \frac{8000}{3}$

we can conclude that the decision maker should be willing to pay up to \$3.33 for additional units of this scarce resource. Likewise, the decision maker should be willing to pay up to \$6.66 for additional units of capacity in the sewing department.

The values of these dual variables are also called shadow prices. Some mention of them was made earlier in the chapter, where we learned that shadow prices can read from the coefficients of the slack variables in row 1 of the final basis in the primal problem.

To illustrate this, the final basis of the primal problem depicted in Table 9D-1 is presented in Table 9D-3. The shadow price for the first scarce resource can be read from the coefficient of the first slack variable in row 1 of the solution basis. The shadow price for the second scarce resource can be read from the coefficient of the second slack variable in row 1 of the solution basis. From Table 9D-3 we can see that the first shadow price is $\frac{10}{3}$ and the second one is $\frac{80}{12}$. This exactly corresponds to the values of the two dual variables found in Table 9D-2.

ADDITIONAL OBSERVATIONS ON THE PRIMAL-DUAL RELATIONSHIP

Dual Surplus Variables and Primal Variables

The augmented form of the dual problem is shown in Table 9D-4. The surplus variables L_1 and L_2 have a special interpretation. They are referred to as accounting losses. Variable L_1 represents the accounting loss associated with the first dual constraint. Repeating this constraint, we have the following:

$$2Y + 2Z - 1L_1 = 20$$

Solving for L_1 and introducing parentheses for convenience, we have:

$$L_1 = (2Y + 2Z) - 20$$

where the sum inside the parentheses represents the total accounting value of resources used in producing 1 unit of camping tents. From this total value is subtracted the per unit profit of camping tents.

Table 9D-4 Augmented Form of the Dual Problem

$$\begin{aligned} \text{Min } E = 8000Y + 16{,}000Z & \\ 2Y + 2Z - 1L_1 &= 20 \\ 1Y + 4Z - 1L_2 &= 30 \end{aligned}$$

$$L_1 = (2Y + 2Z) - 20$$

Accounting loss	Total accounting value of resources used per unit of camping tents	Per unit profit

A positive value for the accounting loss would mean that the value of the resources, if used in this way, would be greater than the profit generated by a camping tent. If this is the case, we would not expect to see the product scheduled; the resources would be used elsewhere.

If, on the other hand, the accounting loss is zero, these resources would be used to produce camping tents. They would not be worth more ($L_1 = 0$) if used elsewhere.

Returning to our solution of the dual in Table 9D-2 we find that L_1 does indeed equal zero. We can then draw the conclusion that our resources *will* be used to produce camping tents. Returning to the primal solution found in Table 9D-3, we find that camping tents are, in fact, produced.

In every primal-dual problem, we can conclude that if an accounting loss is zero, its associated primal variable will be positive. Correspondingly, if an accounting loss is positive, its associated primal variable will be zero.

Verify that this relationship holds for L_2 and M.

Primal Slack Variables and Dual Variables

Another interesting correspondence between the primal and dual has to do with the primal slack variable and the dual variables.

When a primal slack variable is greater than zero, excess capacity exists in the resource. You would therefore expect that additional units of this resource have no value. Since the dual variable represents the value of additional units of this scarce resource, you would also expect it to be zero.

If, on the other hand, the value of a primal slack variable is zero, no excess capacity would exist and you would expect the value of its associated dual variable to be positive. That is, additional units of the resource would have value.

To return to our example, primal slack variable S_2 is equal to zero in the final basis. Its corresponding dual variable is Z and it has a positive value in the final basis of the dual.

THE PRIMAL-DUAL RELATIONSHIPS SUMMARIZED

There is, then, a close relationship between the primal and dual formulations. First, the optimal objective function value of the primal is exactly equal to the optimal objective function value of the dual. Second, the dual surplus variables and primal variables are related in such a way that when one is zero, the other is positive, or when one is positive, the other is zero. This same relationship holds for the primal slack variables and their associated dual variables.

PROBLEMS

9D-1 Given the following primal problem, write its dual:

$$\begin{aligned} \text{Max } P = 6A + 12B + 14C \\ 1A + \ 2B + \ 5C \le 12 \\ 2A + \ 1B + \ 4C \le 10 \\ A, B, C \ge 0 \end{aligned}$$

9D-2 Take your answer to problem 9D-1 and write *its* dual. That is, write the dual of the dual. Does it look familiar?

9D-3 Refer to the first and final bases given in problem 9C-3.

a Write the dual problem in inequality form. Let Y and Z be the dual variables.

b Write the dual problem in augmented form.

c What are the optimal values of the dual variables Y and Z?

d Compute the optimal value of the dual objective function. Is it the same as the primal?

e Compute the accounting losses for each constraint in the dual problem.

f Show that if the accounting loss is zero, its associated activity or product is scheduled (its level of intensity is positive).

g Show that if the value of the dual variable is positive, its associated primal slack variable will be equal to zero.

9D-4 Refer to the first and final bases given in problem 9C-6.

a Write the dual problem in inequality form. Let Y and Z be the dual variables.

b Write the dual problem in augmented form.

c What are the optimal values of the dual variables?

d Compute the optimal value of the dual objective function. Is it the same as the primal?

e Compute the accounting losses for each constraint in the dual problem.

f Show that if the accounting loss is zero, its associated activity or product is scheduled (its level of intensity is positive).

g Show that if the accounting loss is positive, its associated activity or product is not scheduled (its level of intensity is zero).

h Show that if the value of the dual variable is positive, its associated primal slack variable will be equal to zero.

i Show that if the value of the dual variable is zero, its associated primal slack variable will be positive.

Chapter 10

Linear Programming: Transportation Method

INTRODUCTION

There is a category of problems that can be analyzed by either the linear-programming or the transportation method. When this is the case, the choice is always made in favor of the transportation method. It is much more efficient.

Traditionally the transportation method has been used to analyze transportation or logistical problems. In these problems a company might have several production facilities—or sources—all of which have the capability of producing a certain homogeneous product. After the product has been manufactured it must be shipped to one of several distribution centers or destinations. The problem is to determine the quantity of output that should be shipped from each production facility to each distribution center so that transportation costs are minimized.

Today, the transportation method is used to solve a wide variety of problems. But all these problems share a certain format that permit analysis by this method.

Both this format and the transportation method itself will be presented in the case which follows.

CASE STUDY: Sudsdown Beer Company (Part 1)

The Sudsdown Beer Company with headquarters in Cleveland, Ohio, operates breweries in Cleveland and Denver. The company was founded over 75 years ago and has established a reputation for selling one of the best beers brewed in the United States.

The company sells its product in two major market areas, which include Cleveland and Denver, and three secondary market areas, which are served from distribution centers located in San Francisco, Chicago, and Atlanta. These distribution centers receive their beer by truck from the breweries in Cleveland and Denver.

Market Trends

In the last several years there has been a change in the company's growth pattern. This change has suggested that its future lies in the Southern market.

Recently it was decided to consider the construction of a new brewery on a parcel of land which the company has the option to purchase in Dallas, Texas. It was expected that the output from this new brewery would not only meet the demand from the Dallas market area but also have the capacity to supply beer to the three distribution centers.

Factors Influencing the Investment

Several factors are being considered in the evaluation of this site. They include the demand for the product, cost of land, construction costs, availability and quality of water, availability and cost of power, labor supply, proximity of transportation facilities including highways and railroads, local environmental regulations, and transportation costs.

Transportation Costs

Transportation costs represent a sizable portion of the cost of producing beer. Raw materials including rice, barley, and hops are shipped into the brewery from the Midwest and Northwest.

Much of the finished product is also shipped a considerable distance. At the present time the breweries at Denver and Cleveland ship thousands of cases weekly to the distribution centers in San Francisco, Chicago, and Atlanta.

The cost of shipping beer to these distribution centers has gone up dramatically in the last several years. A new brewery strategically located in the South should reduce some of these costs substantially.

The management team studying the feasibility of the new brewery recently turned its attention to these transportation costs. In order to estimate the cost of transporting the finished product to various distribution centers the team agreed that it must first determine the new shipping pattern between breweries and these distribution centers.

It was clear to those on the management team that the Atlanta dis-

tribution center would now receive much of its beer from the new Dallas plant, but they were not sure as to the exact portion of Atlanta's demand that would be met from the Dallas facility. They also wondered if it would be economical for the Dallas plant to supply some of its output to the San Francisco distribution center. In fact, it became clear to them that this new plant could change the entire shipping pattern of the Sudsdown Company.

The management team decided to call in Bob Gruber from the company's management science department and ask him to recommend a shipping pattern between the three breweries and the three distribution centers and then to determine the transportation costs associated with this strategy.

Bob began his study by collecting some data.

The Data

Transportation Costs Bob collected the following data on transportation costs, demand, and supply capacities. In Table 10-1 the transportation costs between all possible breweries and distribution centers are given on a per case basis. For example, if a case were shipped from the brewery at Denver to the distribution center at Chicago, the cost would be 40 cents.

Demand The demand data are presented in Table 10-2. It can be seen that the estimated average monthly demand for the San Francisco distribution center is 70,000 cases; the estimated average monthly demand for the Chicago center is 40,000 cases; and the estimated average monthly demand for the Atlanta center is 65,000 cases.

	To distribution center		
From brewery	San Francisco	Chicago	Atlanta
Denver	$ .40	.40	.40
Cleveland	1.00	.30	.50
Dallas	.80	.60	.35

Table 10-1 Transportation costs on a per case basis between breweries and distribution centers.

Distribution center	Demand
San Francisco	70
Chicago	40
Atlanta	65
Total demand	175

Table 10-2 Average monthly demand data for the distribution centers in thousands of cases.

Brewery	Supply capacity
Denver	30
Cleveland	90
Dallas	55
Total supply capacity	175

Table 10-3 Average monthly supply capacities available for shipment to the distribution centers.

Supply The supply capacities shown in Table 10-3 represent the supply of finished product available for shipment to the distribution centers. The finished product which will be used to meet demand within the immediate market area of the three breweries has already been subtracted from total production output.

The figures shown represent the maximum efficient output levels for all three plants. Plant capacities are 30,000 cases at Denver, 90,000 at Cleveland, and 55,000 at Dallas.[1]

A TRANSPORTATION PROBLEM

The Sudsdown Beer Company case can be structured as a typical transportation problem. This can be accomplished by summarizing all the data presented in Tables 10-1 through 10-3 in one table or *matrix*.

In Table 10-4 the breweries—or sources—are listed down the left and the distribution centers—or destinations—are listed across the top. A matrix of three rows and three columns is then constructed, and in each of these boxes or *cells* is entered the transportation cost for that particular source-destination combination. Along the bottom we include the demand figures below

[1] At present the output at the Cleveland and Denver plants is 135,000 and 40,000 cases. But this is at a cost of substantial inefficiencies. Neither plant can be increased in physical size.

From source	To destination: San Francisco	Chicago	Atlanta	Supply capacity in thousands of cases
Denver	.40	.40	.40	30
Cleveland	1.00	.30	.50	90
Dallas	.80	.60	.35	55
Demand in thousands of cases	70	40	65	175

Table 10-4 The Sudsdown problem in transportation format.

their respective destinations, and to the right of the last row in the matrix we include the capacity of each source.

Notice that the total demand and total supply are equal. That is, exactly 175,000 cases are demanded and 175,000 cases can be supplied. Purposefully the complexity of the problem has been kept as low as possible until the basics are covered. Later in the chapter there will be a problem in which demand does not equal supply.

By looking at this structured problem, it should become clear that our objective is to assign demand to these cells in such a way as to remain within the supply constraint for each source, meet the demand at each destination, and above all accomplish this at lowest cost.

THE TRANSPORTATION METHOD

In structuring the Sudsdown Company problem, we have arrived at a typical transportation format. All transportation problems can be depicted in this way: a matrix with the rows representing sources, columns representing destinations, some measure of effectiveness such as cost in the upper right-hand corner of each cell, demand constraints, and supply constraints.

When a problem falls into this format, it can be solved quite efficiently by using the transportation method, to which we will now turn.

Initial Allocation by the Northwest Corner Rule

There are countless ways to solve transportation-type problems; some are more efficient than others and some easier to understand than others. The method presented here is easier to understand and more intuitive than most, but it may not always be as efficient. Most complex problems are solved with the help of a computer, and consequently an intuitive understanding of the method is more important than the knowledge of the most efficient one. The method that will be used here is known as the northwest corner method.

In the northwest corner method the first step is to make a trial allocation in the following way. As many units (cases) as possible are entered in the upper left-hand corner (northwest) of the matrix. As we move toward the opposite corner of the matrix, allocations continue until the opposite corner (lower right-hand side) is reached. To illustrate this we return to the example.

In Table 10-5 it can be seen that at most 30 (thousand) cases can be assigned to the Denver–San Francisco cell. To allocate more than this would violate the supply constraint for that brewery. Continuing to allocate, we move to the Cleveland–San Francisco cell. (Do you know why we didn't move to the Denver–Chicago or the Denver–Atlanta cell?) At most 40 cases can be allocated to the Cleveland–San Francisco cell because any more than this would result in more than 70 cases being shipped to San Francisco and only 70 are needed. Continuing to the Cleveland–Chicago cell, we find at most 40 cases can be assigned there. Why? Because the demand at Chicago is only 40 cases. Ten cases are assigned to the Cleveland–Atlanta cell, and finally 55 cases are assigned to the Dallas–Atlanta cell.

From source	To destination: San Francisco	Chicago	Atlanta	Supply in thousands of cases
Denver	.40 30	.40	.40	30
Cleveland	1.00 40	.30 40	.50 10	90
Dallas	.80	.60	.35 55	55
Demand in thousands of cases	70	40	65	175

Note: Entries in cells are in thousands of cases.

Table 10-5 An initial allocation by the northwest corner rule.

This initial allocation may not be a very good one, but at least it is a start. And from this start we will continually make improvements until a solution is reached.

Before we make our first improvement, we can compute the cost of this initial allocation. This is done in Table 10-6. The total cost of this allocation is therefore 88.25 thousand dollars.

Can an Improvement Be Made?

We return to Table 10-5 and ask: Can an improvement be made? Or saying it another way: Is it possible to shift allocations into empty cells and thereby effect an overall cost reduction? For example, can cases be reallocated to the Denver–Chicago cell with a resultant decrease in overall costs? This is precisely what we will now do. We will evaluate the *net* savings associated with shifting assignments from assigned cells to unassigned cells.

To simplify this presentation, the breweries are relabeled A, B, and C and the distribution centers are relabeled 1, 2, and 3. Turning to Table 10-7, we

Table 10-6 The Total Cost of the Initial Allocation

From source	To destination	(1) Cost per case	(2) Number of cases (in thousands)	(1) × (2)
Denver	San Francisco	$0.40	30	$12.00
Denver	Chicago	0.40		
Denver	Atlanta	0.40		
Cleveland	San Francisco	1.00	40	40.00
Cleveland	Chicago	0.30	40	12.00
Cleveland	Atlanta	0.50	10	5.00
Dallas	San Francisco	0.80		
Dallas	Chicago	0.60		
Dallas	Atlanta	0.35	55	19.25
Total cost in thousands of dollars				$88.25

To destination

	From source	1 San Francisco	2 Chicago	3 Atlanta	Supply in thousands of cases
A	Denver	.40 −1 30	.40 +1 +.70	.40	30
B	Cleveland	1.00 +1 40	.30 −1 40	.50 10	90
C	Dallas	.80	.60	.35 55	55
	Demand in thousands of cases	70	40	65	175

Diamond shows consequence of moves

Circles show pattern of moves

Table 10-7 Evaluating empty cell A2 (Denver-Chicago).

will now evaluate the consequence of reallocating cases to each of the empty cells.

Step by Step toward Finding an Improvement

The following steps will be taken in search of a possible reallocation:

Step 1 Each empty cell is evaluated for possible reallocation one at a time and independently of the other empty cells.

Step 2 For each empty cell a *pattern* of moves must be established such that a unit or case is added to that cell and the appropriate adjustments made to other cells so as not to violate demand and supply constraints.

Step 3 Then the cost consequence of that pattern of moves is evaluated.

Step 4 The process continues with the repetition of steps 2 and 3 until the cost consequence associated with each empty cell has been calculated.

Step 5 Next a comparison is made between the cost consequences associated with all the empty cells. That empty cell which exhibits the greatest per unit cost savings is used as the basis for reallocation.

Step 6 Finally, a reallocation is made of as many units as possible, and a new table is developed. The new table with its new allocation then serves as the basis for an identical round of steps until it is found that further cost savings are not possible. At that point, the optimal solution has been reached. Now to apply these steps.

Evaluating the Empty Cells

According to step 1, we must evaluate each empty cell. Turning to Table 10-7, we will start cell A2.

Cell A2 According to step 2, a pattern of moves must be established for cell A2 such that a unit is added to that cell and the appropriate adjustments are made to other cells so as not to violate the demand and supply constraints. Let's see how this works.

First we add a unit (or, in this situation, a case) to cell A2 (see the circles in Table 10-7). Then we must remove a unit from cell A1 or the quantity supplied by the Denver brewery would have been increased to 31, a possibility which is not allowed since its capacity is only 30. Now since we took a unit out of A1, we must add one to cell B1. The purpose of this is to maintain the quantity shipped to San Francisco at 70 units. Why didn't we add the unit to C1 instead of B1? Because a basic rule of the transportation method is that when an empty cell is evaluated, adjustments to *other* cells can be made only to those with previous allocations and cell C1 does not have a previous allocation. Finally we must remove a unit from cell B2 to meet the Cleveland supply constraint. This completes step 2, and we have developed a pattern of moves which has resulted in a unit addition to cell A2 and a series of unit adjustments to assure that the demand-supply constraints along the "rim" of the table are unaffected.

In step 3 we determine the cost consequence of this pattern of per unit moves. In moving 1 unit into cell A2, costs have increased by .40; by moving a unit out of cell A1, costs have decreased by .40; by moving a unit into cell B1, costs have increased by 1.00; and by moving a unit out of B2, costs have decreased by .30. The net cost consequence of this pattern of moves is the following:

$$\begin{array}{r} +\ .40 \\ -\ .40 \\ +1.00 \\ \underline{-\ .30} \\ +\ .70 \end{array}$$

That is, if a unit is reallocated into A2, the net cost consequence would be to increase costs by .70 for each unit so reallocated. This figure is entered in the diamond within cell A2.

Cell A3 Step 4 tells us that we must now move on to evaluate another empty cell. Suppose we choose A3. In Table 10-8, the pattern of moves which

	From source	To destination 1 San Francisco	2 Chicago	3 Atlanta	Supply in thousands of cases
A	Denver	.40 (-1) 30	.40 +.70	.40 +.50 (+1)	30
B	Cleveland	1.00 (+1) 40	.30 40	.50 10 (-1)	90
C	Dallas	.80	.60	.35 55	55
	Demand in thousands of cases	70	40	65	

Table 10-8 Evaluating empty cell A3.

meets all our requirements is shown in the circles. A unit is added to cell A3, removed from cell A1, added to cell B1, and removed from cell B3. Notice that there is no other possible way to arrange this reallocation. For example, a unit cannot be removed from cell A2 because our basic rule was that with the exception of the empty cell being evaluated, all adjustments can be made only to previously assigned cells and A2 is not one of them. The per unit cost consequence of these moves is

$$\begin{array}{r} +\ .40 \\ -\ .40 \\ +1.00 \\ -\ .50 \\ \hline +\ .50 \end{array}$$

and it is entered in the diamond within cell A3 of Table 10-8.

Cell C1 Continuing, we must evaluate the consequence of reallocating a unit to cell C1. The pattern of moves includes adding to C1, removing from C3, adding to B3, and removing from B1. The cost consequence of this is

$$\begin{array}{r} +\ .80 \\ -\ .35 \\ +\ .50 \\ -1.00 \\ \hline -\ .05 \end{array}$$

and we can conclude that by introducing a unit to cell C1, the net consequence is a saving of .05 per unit reallocated. We enter this in Table 10-9.

Cell C2 Finally we turn to cell C2. The pattern of moves is adding to C2,

From source	To destination: 1 San Francisco	2 Chicago	3 Atlanta	Supply in thousands of cases
A Denver	[.40] 30	[.40] ⟨+.70⟩	[.40] ⟨+.50⟩	30
B Cleveland	[1.00] 40	[.30] 40	[.50] 10	90
C Dallas	[.80] ⟨−.05⟩	[.60] ⟨+.45⟩	[.35] 55	55
Demand in thousands of cases	70	40	65	

Table 10-9 All empty cells evaluated.

removing from C3, adding to B3, and removing from B2. The cost consequence of this is

$$\begin{array}{r} +.60 \\ -.35 \\ +.50 \\ \underline{-.30} \\ +.45 \end{array}$$

and we can conclude that the net consequence of reallocating a unit to cell C2 would be to increase costs by .45. Again we enter this in Table 10-9.

Identifying the Largest Per Unit Saving

Having completed step 4, we now move to step 5 and compare the evaluations associated with each empty cell. The cell with the largest net per unit saving becomes the basis for reallocation.

In our example, cell C1 with a saving of 5 cents for every unit reallocated to that cell becomes our basis for reallocation.

Now that we have determined to reallocate units to cell C1, the natural question we turn to is: How many units should be reallocated?

Making the Improvement

We have decided that an improvement can be made by reallocating to cell C1. It would therefore be to our advantage to reallocate as many units as possible to that cell and save .05 for each unit reallocated. But we must be careful to remember that any unit added to C1 must be removed from C3, added to B3, and removed from B1. And when removing units from C3 and B1, we must not let the assigned units in these cells fall below zero. Therefore the most that can be reallocated to cell C1 is 40 units. Why? Because 40 units must be removed from cell C3, 40 units added to cell B3, and 40 units removed from cell B1. What would have happened if you reallocated 50 units to C1? You should conclude that this would have resulted in −10 units being assigned to cell B1, and how you ship a negative quantity from the Cleveland brewery to the San Francisco distribution center is a problem which no one can solve.

We can therefore conclude that when determining the number of units which can be reallocated, the focus is directed to those cells in the pattern of moves from which units are removed. The most that can be reallocated depends upon the lowest number of units that have previously been allocated to these cells.

Making the improvement to C1, we add 40 units to C1, remove 40 units from C3, add 40 units to B3, and remove 40 units from B1. A new table is drawn in Table 10-10, in which this reallocation has been made. The cost of this new allocation is computed in Table 10-11. Comparing this new cost of 86.25 thousand dollars with the cost of the initial allocation, 88.25 thousand dollars (Table 10-6), we see that an improvement has been made.

	To destination 1 San Francisco	2 Chicago	3 Atlanta	Supply in thousands of cases
A Denver	30 [.40]	[.40]	[.40]	30
B Cleveland	[1.00]	40 [.30]	50 [.50]	90
C Dallas	40 [.80]	[.60]	15 [.35]	55
Demand in thousands of cases	70	40	65	175

Table 10-10 First reallocation.

Table 10-11 Total Cost of the First Reallocation

Cell	(1) Cost per case	(2) Number of cases (in thousands)	(1) × (2)
A1	.40	30	12.00
B2	.30	40	12.00
B3	.50	50	25.00
C1	.80	40	32.00
C3	.35	15	5.25
Total cost in thousands of dollars			$86.25

Can Further Improvement Be Made?

The allocation depicted in Table 10-10 is clearly better than the original allocation in Table 10-5. The question which we must now answer is: Can a further improvement be made?

To answer this question, we start with Table 10-10 and proceed to evaluate all the empty cells in order to determine whether a further reallocation will result in a saving. And our method for evaluating these empty cells is exactly as it was before.

The Steps Applied Again Starting fresh from Table 10-10, let's turn our attention to the evaluation of cell B1 for possible reallocation. If we add a unit to B1, a unit must be removed from C1, a unit added to C3, and a unit removed from B3. The per unit cost consequence of this pattern of moves is

$$\begin{array}{r} +1.00 \\ -\ .80 \\ +\ .35 \\ \underline{-\ .50} \\ +\ .05 \end{array}$$

This figure is recorded in Table 10-12.

From source	To destination 1 San Francisco	2 Chicago	3 Atlanta	Supply in thousands of cases
A Denver	.40 30	.40 +.65	.40 +.45	30
B Cleveland	1.00 +.05	.30 40	.50 50	90
C Dallas	.80 40	.60 +.45	.35 15	55
Demand in thousands of cases	70	40	65	

Table 10-12 Evaluation of empty cells for second reallocation.

When we turn to cell A2, the pattern of moves is slightly more complex. Can you identify the pattern? Add a unit to A2, remove one from A1, add one to C1, remove one from C3, add one to B3, and remove one from B2. The per unit cost consequence of this pattern is

$$
\begin{array}{r}
+.40 \\
-.40 \\
+.80 \\
-.35 \\
+.50 \\
\underline{-.30} \\
+.65
\end{array}
$$

and this is also entered in Table 10-12.

The remaining patterns of moves for evaluating empty cells A3 and C2 are given in Table 10-13. Their per unit cost consequences are entered in Table 10-12.

Solution

Our next step is to look at Table 10-12 and see if there is a unit cost savings associated with any of the empty cells. But none exists. It would therefore *not* be profitable to carry our reallocation process any further. We have reached the optimal solution!

It can therefore be concluded that whenever the evaluation of empty

Table 10-13 Evaluating Empty Cells A3 and C2

Empty cell	Pattern of moves	Cost consequence
A3	+ A3 − A1 + C1 − C3	+ .40 − .40 + .80 − .35 = + .45
C2	+ C2 − C3 + B3 − B2	+ .60 − .35 + .50 − .30 = + .45

cells reaches a point where there are no net cost savings to be gained from reallocations, the problem is solved. In our example the best allocation is therefore 30 units shipped from A to 1, 40 units shipped from B to 2, 50 units shipped from B to 3, 40 units shipped from C to 1, and 15 units shipped from C to 3. The total cost of this solution has already been computed in Table 10-11; it is 86.25 thousand dollars. There is no other solution that will yield a lower total cost.

EXAMPLE

To give you more experience in solving transportation problems, another example is included in this section. After the example you will find its solution.

	A	B	C	Supply
1	[12]	[7]	[6]	10
2	[10]	[12]	[8]	12
3	[5]	[3]	[2]	15
Demand	11	9	17	37

Solution

	A	B	C	Supply
1	[12] 10	[7] (−7)	[6] (−4)	10
2	[10] 1	[12] 9	[8] 2	12
3	[5] (+1)	[3] (−3)	[2] 15	15
Demand	11	9	17	

	A	B	C	Supply
1	[12] 1	[7] 9	[6] (−4)	10
2	[10] 10	[12] (+7)	[8] 2	12
3	[5] (+1)	[3] (+4)	[2] 15	15
Demand	11	9	17	

	A	B	C	Supply
1	[12] (+4)	[7] 9	[6] 1	10
2	[10] 11	[12] (+3)	[8] 1	12
3	[5] (+1)	[3] (0)	[2] 15	15
Demand	11	9	17	

Therefore the previous step represents the optimal solution. Can you compute the total cost of this solution?

DEMAND AND SUPPLY UNEQUAL

In the previous sections it was assumed that total demand always equaled total supply. For example, in the Sudsdown case total demand was 175 cases and total available supply was 175 cases. In the more common situation demand might not equal supply.

When this happens, two situations can exist. First, demand may be greater than supply and there will consequently be some unsatisfied demand. Second, supply may be greater than demand and on this occasion there will be unused capacity. First we turn to the case of demand greater than supply.

Demand Greater than Supply

In Table 10-14, demand (33 units) is greater than supply (30 units). We would therefore expect to find 3 units of unsatisfied demand in the solution of the problem.

The analysis of *any* transportation problem requires that in matrix form the demand is *made* equal to the supply. In this situation, where demand is greater than supply, this can be accomplished by introducing a "dummy" supply row. The dummy supply row is introduced in Table 10-15, and the supply associated with this *row* is 3 units. Now demand equals supply.

Zero costs are introduced into the cells of the dummy supply row. Therefore if any demand is allocated to that row, it will have a zero cost associated with it since in fact it will not be filled.

	A	B	C	Supply	Demand	Supply
1	[3]	[8]	[6]	7	5	7
2	[4]	[7]	[10]	9	15	9
3	[5]	[10]	[6]	14	13	14
Demand	5	15	13		33	30

Table 10-14 Demand greater than supply.

	A	B	C	Supply
1	$\boxed{3}$	$\boxed{8}$	$\boxed{6}$	7
2	$\boxed{4}$	$\boxed{7}$	$\boxed{10}$	9
3	$\boxed{5}$	$\boxed{10}$	$\boxed{6}$	14
Dummy	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	3
Demand	5	15	13	33

Table 10-15 Introduction of dummy supply row for problem where demand is greater than supply.

	A	B	C	Supply
1	$\boxed{3}$ 5	$\boxed{8}$ 2	$\boxed{6}$ +2	7
2	$\boxed{4}$ +2	$\boxed{7}$ 9	$\boxed{10}$ +7	9
3	$\boxed{5}$ 0	$\boxed{10}$ 4	$\boxed{6}$ 10	14
Dummy	$\boxed{0}$ +1	$\boxed{0}$ −4	$\boxed{0}$ 3	3
Demand	5	15	13	33

	A	B	C	Supply
1	$\boxed{3}$ 5	$\boxed{8}$ 2	$\boxed{6}$ +2	7
2	$\boxed{4}$ +2	$\boxed{7}$ 9	$\boxed{10}$ +7	9
3	$\boxed{5}$ 0	$\boxed{10}$ 1	$\boxed{6}$ 13	14
Dummy	$\boxed{0}$ +5	$\boxed{0}$ 3	$\boxed{0}$ +4	3
Demand	5	15	13	33

The above matrix is the solution to the problem because no further improvement can be made

Table 10-16 Solution to problem with dummy supply row.

After the dummy supply row is introduced and the zero costs are entered in the cells, the solution of the problem proceeds just as in the last example. The complete solution is given in Table 10-16.

In this solution we find that demand center B is the one whose total demand cannot be satisfied. In fact, it will be short exactly 3 units. The other demand centers will receive their full amounts.

Supply Greater than Demand

When supply is greater than demand, it will be necessary to introduce a dummy demand column. Again the costs in this dummy demand column will be zero. Once these adjustments are made, the solution process continues exactly as in the example of demand greater than supply.

SUMMARY

The transportation method is a set of well-defined steps which can be used to find the optimal solution to certain problems. To use this method, it must be possible to summarize the problem in a matrix with the rows representing sources and the columns representing destinations.

QUESTIONS

1 What characteristics are shared by those problems which can be solved by the transportation method?
2 Do transportation models assume linearity and certainty?
3 How can a transportation problem be solved when demand is greater than supply?
4 Return to Table 10-12. Suppose that shipments must be temporarily suspended between Cleveland and Chicago. How could the model be modified to reflect this change? Do not solve.
5 Return to Table 10-12. Suppose that at least 20 units must be shipped from Dallas to Atlanta. How could the model be modified to reflect this change? Do not solve.

PROBLEMS

10-1 Solve the following transportation problem:

From \ To	1	2	3	Supply
A	.10	.15	.20	23
B	.12	.16	.20	6
C	.10	.20	.30	15
Demand	10	15	19	44

10-2 Solve the following transportation problem:

From \ To	1	2	3	Supply
A	1	3	8	10
B	2	4	3	12
C	6	3	9	13
Demand	16	4	15	35

10-3 Solve the following transportation problem:

From \ To	1	2	3	Supply
A	3	6	7	25
B	4	8	12	13
C	6	9	14	7
Demand	31	10	9	

10-4 Solve the following transportation problem:

From \ To	1	2	3	Supply
A	2	1	5	12
B	6	9	10	25
C	4	2	10	41
Demand	6	11	13	

10-5 The Taylor Company manufactures the same line of toys in three locations and ships to three demand centers. From the data given below, determine the optimal shipping patterns and cost.

Manufacturing costs per unit
Plant 1 $.10
Plant 2 .12
Plant 3 .09

Demand in (000) *units*
Region 1 50
Region 2 40
Region 3 80

Supply in (000) *units*
Plant 1 23
Plant 2 48
Plant 3 99

Transportation costs per unit

	From \ To	Demand center 1	Demand center 2	Demand center 3
Plant	1	.10/unit	.20	.23
	2	.08	.13	.40
	3	.20	.25	.50

CASE STUDY: Sudsdown Beer Company (Part 2)

It was now 2 weeks later and Bob Gruber was meeting with the management team that had been assigned to study the new brewery project. Bob was explaining that he had completed the transportation study and that the following monthly schedule would minimize the cost of shipments to the distribution centers.

	Cases
Denver to San Francisco	30,000
Cleveland to Chicago	40,000
Cleveland to Atlanta	50,000
Dallas to San Francisco	40,000
Dallas to Atlanta	15,000

The cost of this schedule, Bob explained, would be \$86,250 per month, or \$1,035,000 per year.

Joe Guildor, vice president of production, was puzzled by these figures "Why," he asked Bob, "don't you ship more between Dallas and Atlanta and fewer between Dallas and San Francisco? I'm sure that if you did, costs could be reduced even further."

Joe continued: "Another thing that bothers me about your study is that the results don't tell me the extent of the savings that can be attributed to the new brewery. I need these figures before I can justify such a large investment."

"I don't have enough data for this comparison," answered Bob.

"I have some figures here," replied Joe, "that you might find useful. The Cleveland and Denver breweries can continue to meet all the demand but the cost will be high. Cleveland could supply as many as 135,000 cases per month while Denver could supply as many as 40,000 cases. However, since their regular time capacities are 90,000 and 30,000 cases per month, the extra cases would have to be produced on overtime. These extra overtime costs will amount to \$20,000 per month. The new brewery would, of course, eliminate these extra costs."

QUESTIONS

1 Is Joe Guildor right? Should more cases be shipped from Dallas to Atlanta? How would this change in shipping strategy affect costs?

2 Determine the transportation costs associated with the present system. Use the transportation method to make your estimate.

3 Compare the transportation costs associated with the present system of two breweries and the proposed system of three breweries. What is the magnitude of the proposed savings.

CASE STUDY: Sharon Nut and Bolt Company

The Sharon Nut and Bolt Company manufactures a variety of nuts, bolts, fasteners, and screws for both the industrial and consumer marketplace. These items are sold in varying quantities from 100 up to 1 million. Some items which have stable demand are produced at uniform monthly rates, whereas others suffer from fluctuating demand.

According to E. Reubens, company president, a major problem is the scheduling of jobs in the shop. The cost of manufacturing an item depends upon which machine is used. Some machines are faster for certain jobs, some machines require longer setup times, and some machines require constant operator attention.

Up until now the foreman has made all scheduling decisions, but Mr. Reubens wonders whether there is a better way to schedule production. He doubts very much that the foreman is actually achieving the lowest possible costs.

To focus on a specific problem, consider this week's scheduling problem facing department 302. The department operates several screw machines, and the total available hours for each of these machines is as follows:

Machine	Output, parts per hour	Capacity, hours
S-1	50,000	12
S-2	40,000	30
S-3	30,000	40
S-4	40,000	15
S-5	40,000	30
S-6	20,000	50

The production control department delivered this week's production orders late yesterday. According to this schedule, a series of bolts will be machined. The quantities required are as follows:

Item	Quantity
p-101	700,000
p-107	1,000,000
p-109	1,200,000
p-114	350,000
p-130	400,000
p-150	1,500,000

The production costs for these items on each piece of equipment have been estimated by the cost accounting department. The department ac-

complished this by retrieving the job tickets for previous jobs and determining the average length of time that the job took on a particular machine. The time was multiplied by $8, a figure which represents the labor cost including fringe benefits. In addition, tooling and scrap costs on a per unit basis have also been estimated. These manufacturing costs are presented in the following table:

	Cost per thousand pieces, dollars					
	S-1	S-2	S-3	S-4	S-5	S-6
p-101	1.10	1.05	0.90	1.20	3.15	4.40
p-107	1.30	1.75	1.15	2.15	3.80	3.60
p-109	1.60	2.20	2.00	2.50	3.20	2.60
p-114	—	—	3.15	1.90	1.85	2.80
p-130	—	—	—	2.20	3.20	2.30
p-150	—	—	—	2.30	—	4.10

Note: A dash implies that it is not possible to produce the part on that machine.

To get some help in solving this problem, Mr. Reubens called in B. Landy, a management consultant. The consultant said that the problem appeared to be complex and perhaps the transportation method could be used to determine the solution. His only reservation about using this method was that each machine had a setup or "fixed" cost.

Mr. Reubens assured him that although some setup was necessary, it was often negligible. And, in fact, this was especially true for the category of jobs which department 302 faced this week.

QUESTIONS

1 Formulate this as a transportation problem.
2 Why would the transportation format be inappropriate if setup costs were significant?
3 If a transportation code is available on the computer, determine its solution.

CASE STUDY: American Auto Import Company

The American Auto Import Company has been importing Japanese cars for 5 years. These compact cars have been sold in Japan for 15 years and have earned a sound reputation in both countries.

When the cars were first imported, 90 percent of them were sold within 300 miles of San Francisco.

In the last few years management has focused its attention on national

market penetration. Distributorships have been established on the East Coast and more recently in the Midwest. Within the next 5 years dealerships will be established in all major automobile markets in the United States.

Sales over this period have more than tripled, and these cars now rank among the top four imports.

At the present time, however, all these cars are shipped through the port of San Francisco. When they arrive, they are processed and stored in the company's distribution center 5 miles away until sent by truck or train to dealers across the United States.

Recently this shipping-distribution system has come under attack. There are several people who feel that a fresh look at this problem must be taken. Total domestic shipping costs for last year were over \$5 million and a saving as small as 10 percent could save the company a half million dollars.

The Shipping Problem

Cars are shipped from a central point in Japan and do not have to be shipped exclusively to San Francisco. They *could* be shipped to any one of several other United States destinations, such as Portland, Galveston, Baltimore, New York, or Boston.

The shipping charges differ according to destination and in part reflect the different handling charges at each port. For example, dock charges at New York are considerably higher than dock charges at Galveston.

If a new port is used, it will be necessary to build or lease a distribution center near that port. The function of this distribution center will be to process and store incoming cars and then fill orders from dealerships in that region.

The shipping strategy from the new distribution center to dealers is also a problem. It is not obvious what distribution center should ship to what dealer.

To some extent time is also a factor. It generally takes 1 week longer to ship to East Coast points via freighter than by truck or train. In fact with the backlog in New York, shipments through that port in the last few years have taken 2 weeks longer. One of the reasons why time is a factor is that American Auto Imports is billed when the cars leave Japan. An extra week or two in transit has the effect of tying up large sums of money in idle resources. The average car costs \$2000, and with the firm's cost of capital at 20 percent, each extra day costs the firm over \$1.

A Consultant's View

Two months ago Trowbridge House, a nationally recognized consulting firm, was called in by top management to propose a solution method. If the proposal is accepted, the firm will contract to complete the study and help in implementing the solution.

The problem was assigned to Dave Lampert, and after several weeks of intensive study, he developed two alternative solution strategies.

The First Strategy

His first strategy was quite complex and also very costly to execute. In it he would sacrifice little and attempt to model the problem capturing as much of reality as possible.

He would build a model that would consider 30 separate American ports, take into account the fixed and variable costs of establishing these distribution centers, and calculate the fixed and variable costs of shipping from these distribution centers to dealers.

The output of this model would specify the number and location of distribution centers, the dealers to be serviced from these centers, and the number of cars to be shipped into these distribution centers and then to a specific dealer.

There are several difficulties with this approach, however. The first is the magnitude of the data collection problem. For example, with 30 ports and 1000 dealers there are 30,000 freight rates to be established. The second concerns the handling of fixed costs and the type of model that needs to be built. Linear programming or transportation models can deal only in linear or variable costs. To build a model with fixed cost elements would require the use of integer programming techniques, making the model even more difficult to solve.

The Second Strategy

The second strategy simplifies the problem. Only three alternatives would be compared. The first alternative would include the ports of San Francisco, Galveston, and Baltimore; the second alternative would include the ports of San Francisco, Norfolk, and New York; and the third alternative, the ports of San Francisco, Galveston, and Boston. All costs would be assumed to be linear.

QUESTIONS

1 Compare strategy 1 with strategy 2. How do they differ?
2 Is strategy 2 a reasonable compromise?
3 Set up strategy 2 as a transportation model. Explain how you would establish your costs.
4 In general, when is it appropriate to settle for less-sophisticated and quicker solution methods than might otherwise be suggested by the problem under consideration?

CASE STUDY: Harbor Landing[1]

In 1971 Harbor Landing apartments were converted from a rental to a condominium complex. By 1972 about half the units had been sold. The investment firm responsible for managing and selling the remaining rental units had not as yet solved an important problem: the assignment of permanent off-street parking to each unit. In the past parking was on an informal basis, but a city ordinance required that individually *owned* units must have their own permanently assigned off-street parking spaces. Since half the units had already been purchased, any assignment had to meet the approval of current owners.

Background

Harbor Landing is a complex of 116 luxury apartments located 15 miles north of a large city on the Atlantic Ocean. (See map.) It was built during 1964 and received wide acclaim for its imaginative design. In the past it was operated on a rental basis by professional management. In 1971 the owners decided to convert it into a condominium complex.

The characteristics of a condominium include:

1 The division of the total complex into individually owned units with common elements (private beach, social hall, grounds, etc.) to be owned jointly

2 The common responsibility for establishing and participating in an administrative structure which enables the owners to manage the complex (e.g., maintenance and day-to-day operations)

Harbor Landing's transition from a rental into a condominium complex requires that a number of problems be solved. One of them is the assignment of permanent off-street one-car parking facilities to each unit within the complex. This assignment has to be documented in the master deed according to the local bylaws.

Description of the Problem

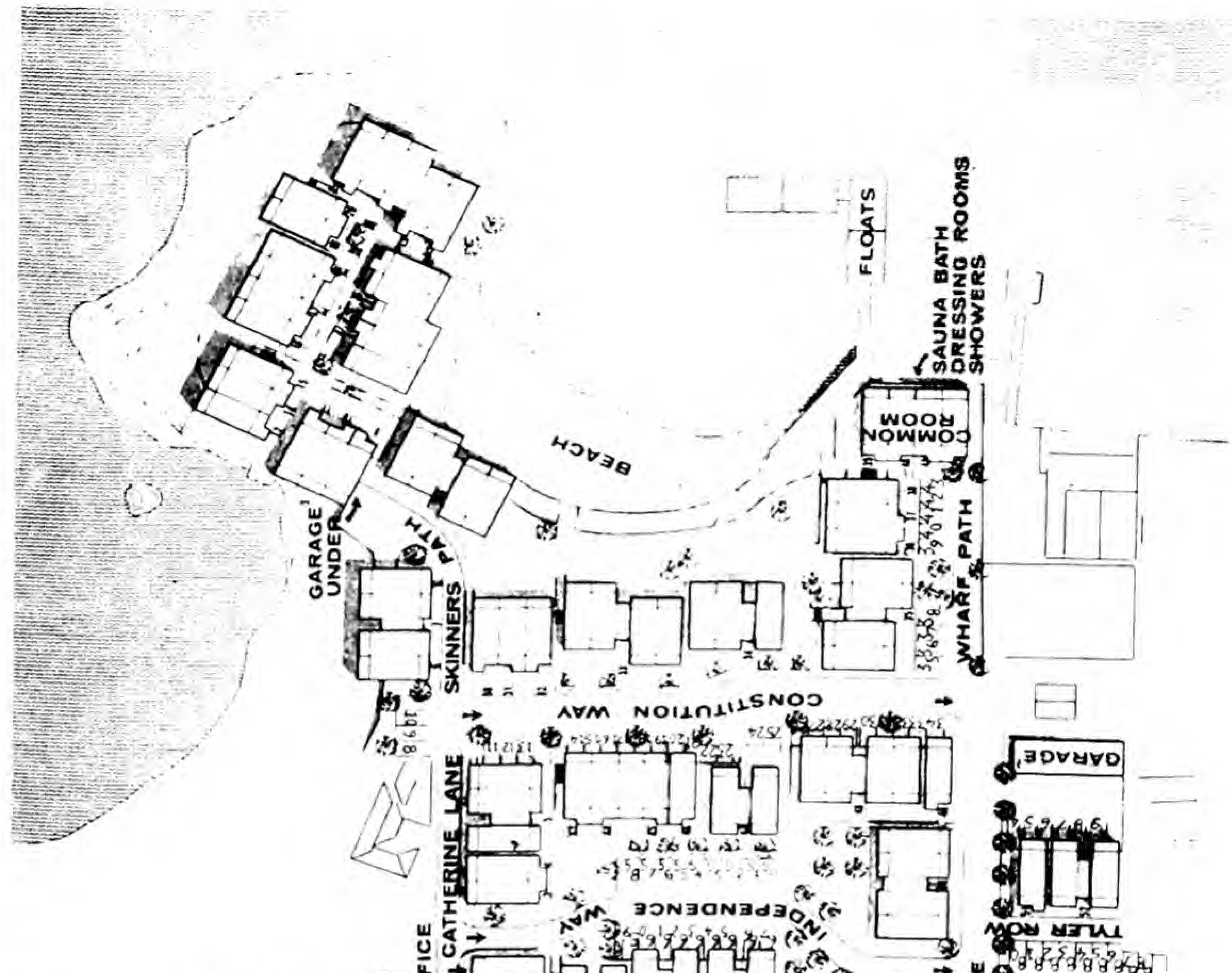
The problem is to determine which space to assign permanently to each apartment. Two types of parking facilities can be utilized.[2] These include:

1 Open-air parking spaces (139) which are scattered throughout the complex (see map).[3] Some of them are close to apartment units, and others are *en bloc* in a lot located alongside Allerton Place.

[1] By Christoph Haehling von Lanzenauer and Barry Shore. Reproduced with the permission of the University of Western Ontario.

[2] Furthermore, there are 16 spaces located in two wooden garages on the complex. These, however, are to be excluded for the purpose of permanent off-street parking assignments.

[3] Unassigned spaces are rented at $10 per month.



BEACH
FLOATS
SAUNA BATH
DRESSING ROOMS
SHOWERS
COMMON ROOM
GARAGE UNDER
SKINNERS PATH
WHARF PATH
CONSTITUTION WAY
CATHERINE LANE
OFFICE
INDEPENDENCE WAY
GARAGE
TYLER ROW
ALLERTON PLACE
BRACKETT PLACE
GARAGE
ENTRANCE
GREGORY ST

2 Underground parking garage with capacity of 20 cars located at the end of Skinners Path below apartments 9, 10, and 11.

Owing to the location of and quality difference in parking facilities, there seems to be no simple and equitable solution to the assignment of spaces. In the past the allocation of parking spaces was controlled by the resident manager. With an average vacancy rate of 10 percent, he was able to keep most tenants happy with their parking assignments. At present, however, the temporary spaces that have been assigned to new owners would leave those yet to purchase the remaining apartments with relatively inferior parking. In short, the status quo is unacceptable.

The common expenses (waterfront, rubbish and snow removal, building repairs, insurance, etc.) in 1972 totaled \$156,000. This was paid by the owners and tenants of the complex in the form of a monthly "common element fee." The proportion of this total which each apartment is assessed is computed on the basis of that apartment's value to the total value of all apartments. These figures range from .44687, for which the monthly fee is \$58.09, to 1.34062, for which the fee is \$174.28.

As long as the garage is retained as paid parking (\$35 per month per space), \$8400 is generated to help offset the \$156,000 in common expenses. In fact this revenue was collected in 1972, and the common element fees did reflect this income. From past experience it is believed that any reasonable price applied to the underground parking garage will have no adverse effect on filling the spaces. If, however, a new plan is devised whereby these underground garages are used as free parking, this revenue will be lost and must be made up by an increase in the common element fee.

The First Step toward Solving the Problem

The board of governors decided that a consultant should be commissioned to develop two or three detailed alternative parking strategies. These in turn could be brought up at the next unit owners' meeting, which was scheduled in 3 months.

During the first meeting with the consultant, the problem was carefully explained and it was finally agreed that the following factors should be considered in making the assignments.

1 The distance between apartment and parking place

2 The quality of the parking facility (underground garage space is considered higher-quality than outside spaces)

3 If a price is to be charged for some or all of the spaces, it should be fair and equitable to all owners

The meeting then closed and the group decided to meet again before the owners' meeting.

QUESTIONS

1 Define the problem confronting the consultant.
2 What criterion should guide the solution of this problem?
3 What alternatives should the consultant consider?
4 Can the problem be modeled?
5 What is the magnitude of the data collection and model solution process?
6 Is it cost-effective to use a large model?
7 Are there other alternative methods for solving the problem?

APPENDIX A: Transportation Method and Linear Programming

The transportation problem summarized in Table 11-4 can also be written in linear-programming format. If we let X_{ij} represent the quantity produced at brewery (source) i and shipped to distribution center (destination) j, then the problem can be summarized in the following way.

Objective function:

$$\text{Min } C = .40X_{A1} + .40X_{A2} + .40X_{A3} + 1.00X_{B1} + .30X_{B2} + .50X_{B3} + .80X_{C1} + .60X_{C2} + .35X_{C3}$$

Demand constraints:

$$1X_{A1} + 1X_{B1} + 1X_{C1} \geq 70$$
$$1X_{A2} + 1X_{B2} + 1X_{C2} \geq 40$$
$$1X_{A3} + 1X_{B3} + 1X_{C3} \geq 65$$

Supply constraints:

$$1X_{A1} + 1X_{A2} + 1X_{A3} \leq 30$$
$$1X_{B1} + 1X_{B2} + 1X_{B3} \leq 90$$
$$1X_{C1} + 1X_{C2} + 1X_{C3} \leq 55$$

$$\text{all } X_{ij} \geq 0$$

It should be clear by looking at this linear-programming formulation of Table 10-4 that indeed the problem could be solved in this way. However, it should also be clear that if you do solve it in this way, it will take much longer than by the transportation method. Therefore if a problem can be formulated as a transportation problem, its solution will be much more efficient.

PROBLEMS

10A-1 Express problem 10-1 in linear-programming format.
10A-2 Express problem 10-2 in linear-programming format.

APPENDIX B: Degeneracy

To use the transportation method as presented in this chapter, it is essential that a certain requirement be met. If it is not met, a modification must be made to the table.

If we let R represent the number of rows and C represent the number of columns in the table, then a transportation problem must have $R + C - 1$ cells allocated at any one time.

Let us look at the first example in this chapter (Table 10-4). It has $R = 3$ rows and $C = 3$ columns. Therefore exactly $R + C - 1$ or $3 + 3 - 1 = 5$ cells should be allocated at all times. Referring to Table 10-5, we can verify that indeed five cells have received allocations. Again in Table 10-10 after the first reallocation has taken place, exactly

Only 4 allocations

	1	2	3	
A	$\boxed{3}$ 5	$\boxed{5}$ 5	$\boxed{8}$	10
B	$\boxed{4}$	$\boxed{6}$ 7	$\boxed{5}$	7
C	$\boxed{11}$	$\boxed{3}$	$\boxed{6}$ 3	3
	5	12	3	20

Table 10B-1 A degenerate allocation.

five cells have received allocations. We can therefore conclude that in this first example this special requirement has been met in all tables.

It is not true, however, that this special requirement will always be met. When fewer than $R + C - 1$ cells have allocations, we have a situation known as *degeneracy*. And when this occurs, a modification must be made to the table.

Consider the problem presented in Table 10B-1. Using the northwest corner rule as the basis for the initial allocation, we have four allocations and according to our basic requirement we *should* have $R + C - 1$ or $3 + 3 - 1 = 5$ allocations. Consequently we have a degenerate table.

The major consequence of degeneracy is that it will be impossible, using the method presented in our first example, to evaluate some empty cells. Try evaluating empty cell C1. You will not be able to find a pattern of moves which will utilize only filled-in cells (other than C1).

To resolve this impasse, we simply add a "*little bit*," d, to one of the empty cells. The cell in which d has been allocated will be treated like all other allocated cells, its only difference being that the quantity d is considered to be only a little little little bit.

In Table 10B-2 a little bit d is allocated to cell A3, and it then becomes possible to evaluate all the empty cells.[1] The evaluations are included in Table 10B-2.

An improvement can be made by reallocating as much as possible to cell B3. Unfortunately cell A3 limits the amount that can be reallocated to cell B3. The most that can be reallocated is d, a very little bit. Undertaking this reallocation, we now have the allocation depicted by Table 10B-3. Perhaps this reallocation deserves some

[1] It is generally possible to get out of a degenerate problem by adding d to any one of several empty cells. We could have added d to cell B3, cell C1 or cell C2.

	1	2	3	
A	$\boxed{3}$ 5	$\boxed{5}$ 5	$\boxed{8}$ d	10
B	$\boxed{4}$ 0	$\boxed{6}$ 7	$\boxed{5}$ −4	7
C	$\boxed{11}$ +10	$\boxed{3}$ 0	$\boxed{6}$ 3	3
	5	12	3	

A little bit d has been allocated to this cell. It is then treated as any other filled-in cell such as A1 or A2.

Table 10B-2 Introduction of d to a degenerate table.

	1	2	3	
A	$\boxed{3}$ 5	$\boxed{5}$ 5	$\boxed{8}$ +4	10
B	$\boxed{4}$ 0	$\boxed{6}$ 7	$\boxed{5}$ d	7
C	$\boxed{11}$ +6	$\boxed{3}$ −4	$\boxed{6}$ 3	3
	5	12	3	

Table 10B-3 First reallocation.

	1	2	3	
A	$\boxed{3}$ 5	$\boxed{5}$ 5	$\boxed{8}$ +4	10
B	$\boxed{4}$ 0	$\boxed{6}$ 4	$\boxed{5}$ 3	7
C	$\boxed{11}$ +10	$\boxed{3}$ 3	$\boxed{6}$ +4	3
	5	12	3	

Table 10B-4 Second reallocation.

explanation In cell A3, $d - d$ is zero. In cell A2, $5 + d$ is 5 since 5 plus a little bit is "close to" 5. In cell B2, $7 + d$ is 7, and in cell B3 we introduce a little bit d.

Again we evaluate the empty cells. See Table 10B-3. A reallocation to cell C2 should be made. The new table is shown in Table 10B-4. Notice that it is not degenerate since it has five allocated cells. Quite often a degenerate table becomes one which is not degenerate, and even the opposite may occur.

By evaluating the empty cells, it becomes clear that we have reached the optimal solution.

PROBLEMS

10B-1 Solve the following transportation problem:

	1	2	3	
A	$\boxed{1}$	$\boxed{4}$	$\boxed{8}$	5
B	$\boxed{2}$	$\boxed{3}$	$\boxed{6}$	6
C	$\boxed{4}$	$\boxed{5}$	$\boxed{14}$	8
	5	11	3	19

10B-2 Solve the following problem:

	1	2	3	
A	4	3	11	12
B	8	9	5	6
C	12	6	9	11
	6	6	17	29

10B-3 When a transportation problem has 1 at the end of every row and column, it is called an assignment problem. Solve the following assignment problem:

	1	2	3	
A	5	4	11	1
B	4	6	15	1
C	15	7	8	1
	1	1	1	3

10B-4 The City of Bound Brook has three different types of plows which it uses to remove snow from three zones within the city limits. The cost of removing the snow in each of these zones depends upon the type of plow which is assigned to it. The costs of removal for each combination of plow and zone are given below:

	Zone		
Plow	A	B	C
1	$2	6	7
2	8	2	1
3	9	2	5

If a plow must be assigned to only one zone, how should the plows be assigned to minimize costs?

10B-5 Three workers must be assigned to three machines. The length of time that it takes to complete the job cycle depends upon the efficiency of the worker

assigned to the machine. Job cycle times for each combination are given below:

Worker	Machine A	B	C
1	2	7	5
2	6	4	2
3	4	3	6

Using the transportation method, determine the most efficient worker-machine combination.

Chapter 11

Inventory Control: Certainty

INTRODUCTION

Many organizations carry large inventories. In fact, inventory may account for as much as one-third of a firm's current assets.

Inventories are necessary, for they provide immediate access to an item when it is needed. If inventories are inadequate, not only might it be impossible to meet demand, but profits may also be eroded and customers lost. If, on the other hand, inventory levels are too high, the cost of carrying this large inventory may be unnecessarily high and profits, once again, may be eroded.

The object in inventory control is to strike the right balance. Inventory levels must be neither too low nor too high. They must be managed efficiently.

Several models will be developed in this chapter which have been successfully used to manage large complex inventory systems. They have proved to be more successful and profitable than any other model developed in this book.

To begin, the analysis will focus upon those inventory systems where demand is known with certainty. In the next chapter the analysis will continue with those systems where demand is not known with certainty but can be described by a probability distribution.

The case which follows will be used to introduce the elements which constitute an inventory system. Later these elements will be combined into a model.

CASE STUDY: Stanley's

Stanley's is a supermarket chain of 30 stores located in California. At the present time it is in the process of installing a new pricing and inventory system in its largest store. If the system is successful, Stanley's intends to have the system operating in all stores within 18 months. The new system operates in the following way.

All packaged food items are now coded with a series of black stripes. Each item has its own pattern. When a customer brings the item to the check-out counter, the item is passed over an optical scanning device. The scanner reads the code and transmits this information to a central computer. The computer instantaneously identifies the current price of the item and returns this new information to the electronic cash register at the check-out counter. This process continues until all the items which the customer has purchased have been priced. After the last item has been priced, the total cost of the customer's order is computed.

Although the system is quite expensive to install, it has several advantages over methods which have been used in the past. First, it is no longer necessary to price each food item on the shelf. Second, the time it takes to process a customer's order at the check-out stand is substantially reduced. Third, it eliminates the costly errors that are frequently made by check-out personnel. Fourth, the information collected by the computer can be used to control inventory levels.

In the past an inventory of all packaged food items was taken once each week. When an item was found to be low, a replenishment order was issued. This system was expensive because it required a large staff of inventory clerks. The system also resulted in frequent stockouts.

The new system is expected to eliminate these problems. Only a few inventory clerks will be needed, and few stockouts should occur.

Instead of taking an inventory on a weekly basis the new system will keep a perpetual record of inventory levels. Whenever a sale is made, the computer inventory record will immediately be updated to include the sale. When the inventory is depleted to a certain level—called the reorder level—a replenishment order will automatically be issued.

Several problems must still be resolved before the design of the new inventory system is completed. First, the reorder levels must be set for each item in inventory. If these levels are set too low, stockouts may occur before the replenishment order arrives.

If, on the other hand, the level is set too high, there may be an

unnecessarily large number of items that must be kept in stock. But few stockouts will occur.

The second problem is the number of items to order. If each order placed is for a large number of items, the cost of storing these items will be large. Imagine what would happen if 6-ounce cans of tuna fish were ordered just once each year. The storage problem would be enormous. The benefit of this strategy, however, would be that only one order would have to be placed. Consider the other extreme, which would place frequent orders for few items. Tuna fish might arrive twice each week. The storage problem would be minimal, but the cost of placing frequent orders would be high.

Both these problems—when to place an order and how many items to order—have yet to be solved for each of the 20,000 products carried by Stanley's. The inventory control department has been asked to study these problems and expects to make a recommendation in 4 weeks.

INVENTORY DECISIONS

In the case just presented the two basic problems of inventory control were identified: *when* to place an order and *how many* items to order. These decision problems occur not only in retail organizations but in manufacturing plants, hospitals, banks, government agencies, airlines, educational institutions, and transportation systems.

Inventory can be broadly defined as an idle resource of any kind that has potential economic value. The cash held by a commercial bank, for example, can be considered as inventory. It must be determined *how much* cash to order from the Federal Reserve to meet customer demands and *when* to place the order. Another example is the inventory of beds maintained by a hospital. *How many* beds are necessary to meet the needs of the community? Still another example is the number of flight attendants needed by an airline to adequately staff its flight schedule.

Inventory problems are widespread, but they share many aspects in common. In the next section a general framework is established within which these problems can be analyzed.

A GENERAL FRAMEWORK FOR INVENTORY ANALYSIS

Inventory systems are complex because they include many interrelated elements. When a system is modeled, first the elements must be identified and then a decision must be made to either include or exclude them from the model.

In this section we will examine those elements common to all inventory problems. They include demand, lead time, procurement source, and costs. We now turn to the first element, demand.

Demand

Inventory decisions are made to accommodate future demand. For example, an order placed in this period for tuna fish will make the product available to customers in some future period. Future demand, then, must be considered in formulating an inventory strategy.

Three states of knowledge can exist about future demand: certainty, risk, and uncertainty. There are some situations in which the future level of demand is known with certainty. For example, in some construction projects the cement, steel, glass, and lumber requirements may be known with certainty well in advance.

Seldom, however, is demand known with certainty. More often there will be some doubt, and it may be possible to express this doubt as a probability distribution of future demand. In the supermarket case, for example, the demand forecast for tuna fish might be expressed in the following way. There is a 10 percent chance that yearly demand will be 10,000 cases, a 20 percent chance that demand will be 15,000, a 40 percent chance that demand will be 20,000, a 20 percent chance that demand will be 25,000, and a 10 percent chance that demand will be 30,000. This probability distribution is illustrated in Figure 11-1. It represents the situation of demand under *risk*.

The third state of knowledge is that nothing is known about future demand. This is called *uncertainty*. In practice this complete lack of knowledge is uncommon.

Lead Time

The second element in this framework is lead time. It is defined as the period between the placing of the order and its receipt in stock. Again, three states of knowledge can exist. Lead time can be known with certainty, risk, or uncertainty.

When lead time is known with certainty, the exact number of days needed for an order to be delivered can be predicted with perfect accuracy. When lead time is known with risk, delivery time can be described by a probability

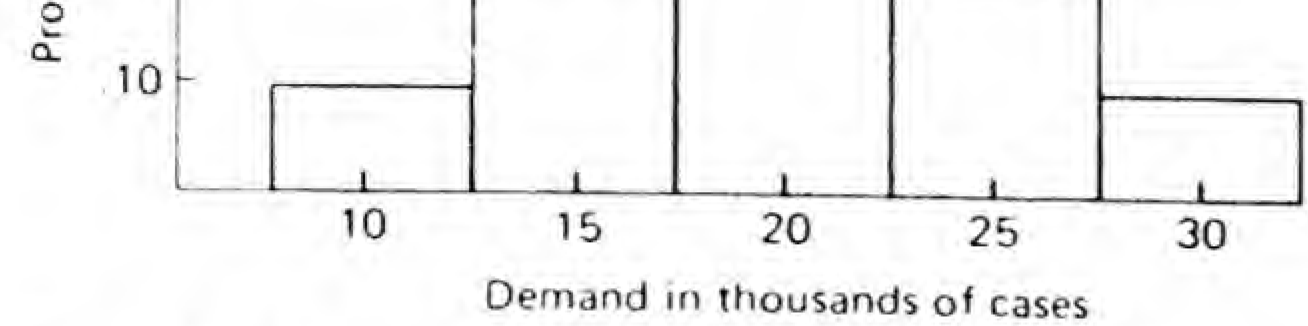


Figure 11-1 Probability distribution of demand.

distribution. Finally, when lead time is uncertain, nothing can be said about when the order will arrive.

Inside or Outside Procurement

The third element in the framework has to do with whether the inventory item under consideration will be purchased from an outside supplier or will be produced within the firm itself.

When the inventory item to be ordered is produced within the firm, special problems arise. Inside orders mean that production schedules, work-force requirements, raw-material orders, and machine availability are all affected. The timing of the order (when) and the order quantity (how much) must take all these secondary effects into consideration. Consequently, inventory decisions for internal procurement are considered quite complex.

When the inventory item is ordered from an outside supplier, it is unnecessary to consider the secondary effects, and the problem is easier to solve.

Costs

A major element in this framework is the cost structure of the system. In some cases price discounts are allowed; in others they are not. Stockouts may be tolerable in one situation and intolerable in others. But in all situations, inventory carrying cost and ordering cost are incurred. In the next section these costs are given a closer examination.

INVENTORY COSTS

Inventory problems require that two decisions be made. The first is *how many* items are to be ordered, and the second is *when* to place the order.

The criterion by which the alternative strategies for these decision problems can be compared is generally the minimization of relevant costs. In this section these costs will be explored. Later, models will be developed whose purpose it will be to determine the order quantity and timing such that these costs are minimized.

Per Piece Procurement Cost

For each unit purchased or manufactured a per piece procurement cost is incurred. In the simplest case the per piece cost remains the same, regardless of quantity purchased; there is no quantity discount. But in the more complex case, a series of price breaks may be offered for purchases of larger and larger quantities.

Fixed Order Costs

Whenever an order is placed, a fixed cost is incurred. This cost includes such components as order preparation cost, order follow-up cost, some incoming

inspection costs, and the clerical cost associated with order payment. Therefore this category includes all those costs which do not vary with the size of the order—those costs which are the same, regardless of how much is ordered.

Inventory Carrying Cost

When inventory is carried in stock from one period to the next, a carrying cost is incurred. This cost, however, will never be found explicitly in the profit and loss statement at year-end. Nonetheless, it is a very real one for decision-making purposes and must be calculated.

The funds which are tied up in inventory could be used for other productive purposes. For example, $100,000 tied up in inventory could be used to invest in a new product or a new plant. Therefore, it seems reasonable to say that an *opportunity cost* is incurred for holding inventory. This cost is at least equal to the lowest return that could have been earned if these funds were invested in other projects. The lowest return that the firm can earn is equal to its cost of acquiring funds or its cost of capital. If the firm earns less than this, it will not be able to meet its financial obligations. We can therefore conclude that the cost of carrying inventory is at the very least this cost of capital.

To this cost of capital other inventory carrying costs must be added. These include storage, obsolescence, insurance, and taxes.

Storage costs are those warehouse-related costs which are incurred for holding the inventory until it is needed.

Many kinds of goods suffer from deterioration or obsolescence. Dry-cell batteries and vegetables, for example, deteriorate with age, but electronic equipment and fashion wear become obsolete slowly.

Insurance is generally purchased to protect the firm's investment in inventory. The larger the inventory, the higher the insurance premiums will be.

Finally, we have inventory taxes. Firms are assessed a tax, based on their level of inventory. The larger the inventory, the higher the tax.

All these costs can be measured as a percentage of the cost of the unit. For example, the yearly storage cost for a particular firm has been computed to be 10 percent of the per unit purchase price of the item. Therefore, if one particular item costs $10 per unit, its storage cost for one year would be the following.

$$\$10 \times .10$$

which is equal to $1.

To obtain the total cost of carrying inventory (in percent), each of these separate costs (in percent) must be added together.

$$I = C + G + K + T$$

where I = total cost of carrying inventory in percent per period
C = opportunity cost (cost of capital) of carrying inventory in percent per period
G = storage cost in percent per period
K = deterioration and obsolescence cost in percent per period
T = insurance and taxes in percent per period

In any structured approach to solving the problems of how much and when to order, an estimate of these carrying costs is essential. Unfortunately, they cannot be read directly from accounting data, and in many cases it takes a considerable amount of effort to obtain them.

Stockout Cost

When demand occurs and no stock is available, a *stockout* is said to exist. This stockout can lead to either of two possible outcomes. The first is that the out-of-stock position can lead to a *back order.* This situation is often found at mail-order houses, where the customer is told that the item is temporarily out of stock and upon replenishment of the stock the item will be sent to the customer. The firm incurs additional costs because of this back order. Generally, the order placed with its supplier must be expedited; this might include telephone calls, letters, and express shipments. In addition to these explicit costs, a pattern of repeated back orders will certainly lead to an erosion of customer loyalty and company goodwill. These last costs, however, are difficult if not impossible to measure.

The second possible outcome of an out-of-stock position is the *lost sale.* Here the cost may be much more severe, but the exact level of the cost is, once again, difficult to measure. In this case, the customer places an order, receives an out-of-stock response, and takes his or her business elsewhere. The cost of this to the firm may be more than the profit lost on the potential sale. Since the customer may never return, the profit on future sales may also be lost.

This completes the development of the framework which will be used as a basis for constructing and comparing the models presented in this chapter. In the next section the first model will be developed.

EOQ MODEL

The EOQ, or economic order quantity, model is the best-known and most widely used of all inventory models. The assumptions that must be made in the development of the model, however, place some limits on its usefulness.

The Assumptions

The first assumption is that demand is known with certainty. Second, it is assumed that lead time is known with certainty. Third, the item is purchased from outside suppliers. Fourth, no quantity discounts are offered. Fifth, no stockouts are allowed.

Inventory Levels

Given these assumptions, the inventory level will behave in a predictable way. This pattern is illustrated in Figure 11-2. Consider a point in time when a new order for Q units arrives in stock. Now in this model not only is demand known with certainty, but demand is the same each and every day. Therefore, inventory is depleted in a steady and predictable pattern as shown in Figure 11-2.

Since lead time L is also known with certainty, it can be determined quite easily when a reorder should be placed. Suppose that lead time is 4 days and demand is 3 units per day. A reorder should be placed when the inventory level falls to

4 units per day × 3 days

or 12 units. This point is referred to as r, the reorder level.

After a reorder is placed, inventory continues to drop during the lead-time period. Just when inventory drops to zero, the replenishment order for Q units arrives and inventory is restored to Q units.

The next and all subsequent inventory cycles are exactly the same as the first one. Initial inventory is Q units, a reorder is placed at r, and when inventory reaches zero, a replenishment order arrives, restoring the inventory level to Q.

As we mentioned earlier, the two problems in inventory control are when to order and how much to order. The method for solving the first problem, given the present set of assumptions, has been presented in this section: an order is placed when inventory levels fall to a lead time's worth of demand. Next we turn to the second problem.

Opposing Costs

Suppose that during 1 year large infrequent orders are placed for a particular unit. As a result of this strategy, average inventory levels would be large and

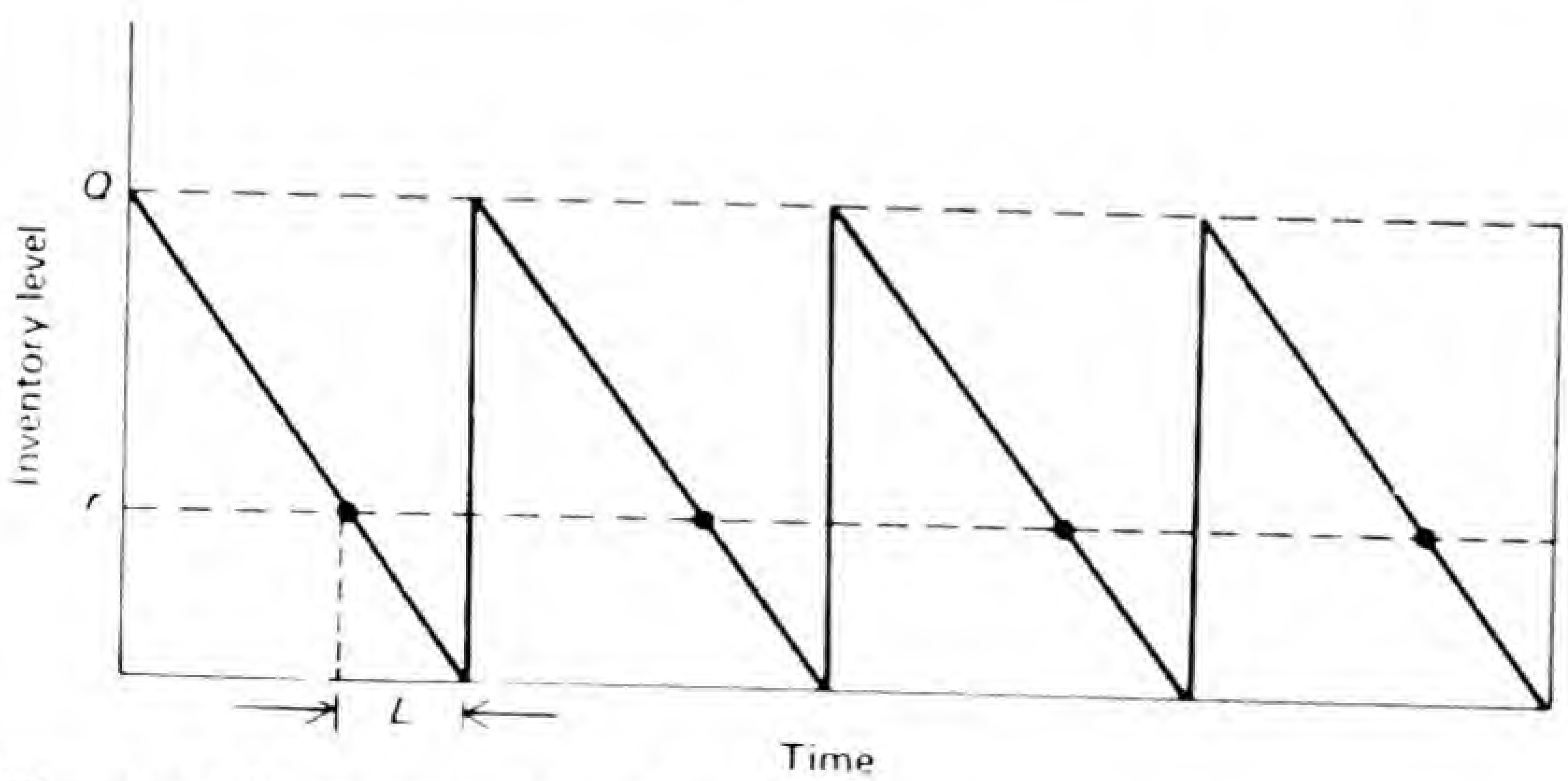


Figure 11-2 Inventory level over time.

yearly inventory carrying costs would therefore be high. The benefit from this strategy, however, would be that yearly fixed order cost would be low because orders would be placed infrequently.

Consider the other extreme, where small orders are frequently placed. The average inventory in this case would be small and the yearly inventory carrying costs would be low. But owing to the frequent placement of orders, fixed order cost per year would be high.

What becomes apparent from these examples is that there are costs which move in opposite directions. They are illustrated in Figure 11-3. As the order quantity associated with any one order increases, the fixed order cost per year goes down since fewer orders are placed. And as the order quantity increases, the yearly carrying cost goes up.

There is one additional cost that must be considered—the total per piece cost. Since no quantity discounts are allowed in this model, the yearly total per piece cost will remain the same, regardless of ordering strategy. Consequently, this cost appears as a horizontal line as shown in Figure 11-4.

The criterion in selecting an order quantity is *total* cost minimization. Therefore, all three component costs must be added together. The result is the total yearly cost curve shown in Figure 11-4. The curve reaches its minimum at TC^*. This occurs when Q^* units are ordered every time an order is placed. If any other quantity is ordered, higher total costs will be incurred.

In the next section a mathematical formula will be developed which can be used to compute Q^*.

EOQ Formula

A formula for determining the optimal ordering quantity Q^* will be developed in two stages. First a mathematical expression for the total costs will be

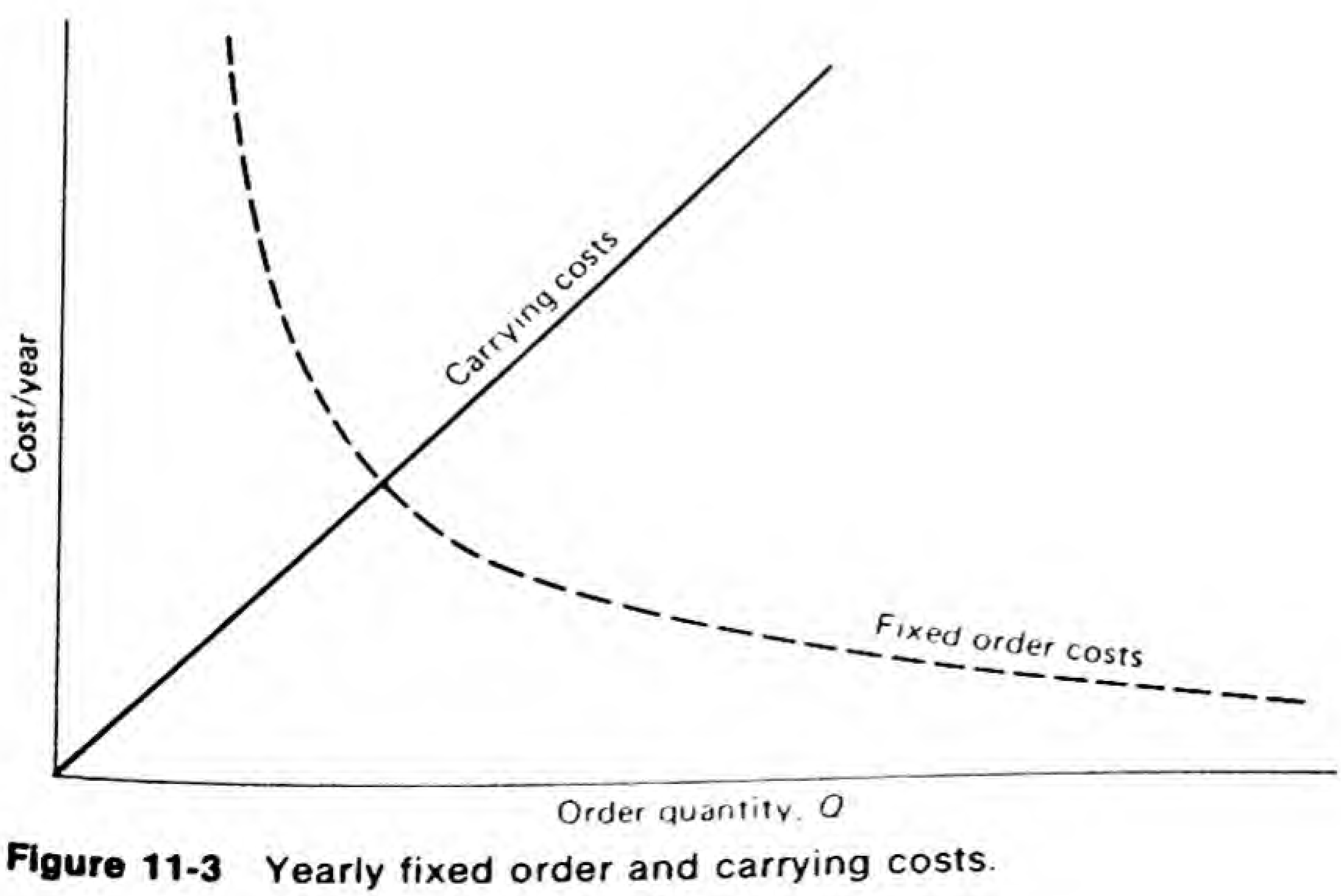


Figure 11-3 Yearly fixed order and carrying costs.

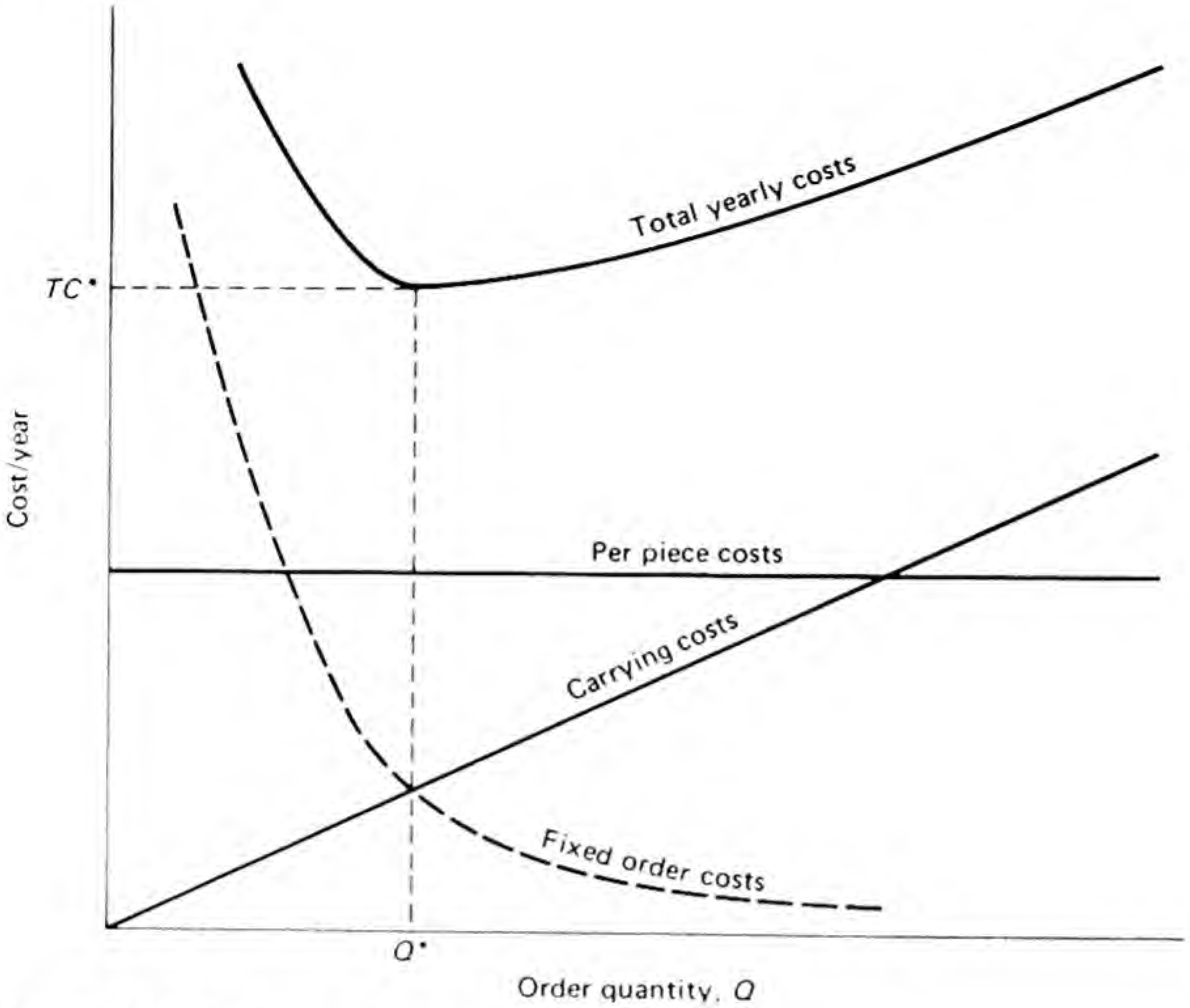


Figure 11-4 Total yearly inventory costs.

developed, and then that value of Q will be determined which minimizes these total costs.

Total Costs As we have seen in Figure 11-4, total yearly inventory cost can be broken down into three components: total per piece cost per year, total fixed order cost per year, and total inventory carrying cost per year.

$$\begin{pmatrix}\text{Total yearly}\\\text{inventory cost}\end{pmatrix} = \begin{pmatrix}\text{total per}\\\text{piece cost}\\\text{per year}\end{pmatrix} + \begin{pmatrix}\text{total fixed}\\\text{order cost}\\\text{per year}\end{pmatrix} + \begin{pmatrix}\text{total inventory}\\\text{carrying cost}\\\text{per year}\end{pmatrix}$$

If we let

Q = order quantity
C = per piece cost
A = fixed order cost per order
D = yearly demand estimate
I = cost of carrying inventory in percent per year

then we can develop a mathematical expression for total costs.

First, the total per piece cost is equal to the yearly demand multiplied by the per piece cost.

$$\begin{pmatrix}\text{Total per}\\ \text{piece cost}\\ \text{per year}\end{pmatrix} = CD$$

Second, the total fixed order cost per year is equal to the number of orders placed per year multiplied by the fixed order cost per order.

$$\begin{pmatrix}\text{Total fixed}\\ \text{order cost}\\ \text{per year}\end{pmatrix} = \begin{pmatrix}\text{number of}\\ \text{orders placed}\\ \text{per year}\end{pmatrix}\begin{pmatrix}\text{fixed}\\ \text{order cost}\\ \text{per order}\end{pmatrix}$$

To determine the number of orders placed per year, the yearly demand is divided by the order quantity associated with each order. Then to get the total fixed order cost per year, this result is multiplied by the fixed order cost per order.

$$\begin{pmatrix}\text{Total fixed}\\ \text{order cost}\\ \text{per year}\end{pmatrix} = \frac{D}{Q}A$$

Third, the inventory carrying cost per year is equal to the average inventory in units multiplied by the per unit procurement cost and then by the inventory carrying cost in percent per year.

$$\begin{pmatrix}\text{Total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix} = \begin{pmatrix}\text{average}\\ \text{inventory}\\ \text{in units}\end{pmatrix}\begin{pmatrix}\text{per unit}\\ \text{procurement}\\ \text{cost}\end{pmatrix}\begin{pmatrix}\text{inventory}\\ \text{carrying cost}\\ \text{in percent}\\ \text{per year}\end{pmatrix}$$

The computation of average inventory depends upon the assumption made about the time pattern of inventory depletion. We assumed that demand is such that inventory is depleted in a steady and predictable manner. For example, if demand is 2500 units per year and if there are 250 business days per year, demand and inventory depletion is 10 units each and every day. Continuing with the example, suppose that an order for $Q = 50$ units is placed every Monday and that the order arrives exactly 1 week later, on the following Monday. Every Monday morning the inventory level will therefore be 50 units, and since 10 units are demanded, the average inventory for Monday will be 45 units. The average on Tuesday will be 35 units, the average on Wednesday will be 25 units, the average on Thursday will be 15 units, and

the average on Friday will be 5 units. The weekly average can be computed in the following way:

$$\frac{45+35+25+15+5}{5} = \frac{125}{5} = 25$$

The average inventory under these assumptions can be expressed in the following way.

$$\begin{pmatrix}\text{Average}\\ \text{inventory}\\ \text{in units}\end{pmatrix} = \frac{Q}{2}$$

Returning to our example:

$$\frac{Q}{2} = \frac{50}{2} = 25$$

and we can conclude that the average inventory based on our assumptions about demand and inventory depletion is 25 units.

Using the expression for average inventory in units found in the last paragraph, the total inventory carrying cost per year can be summarized algebraically:

$$\begin{pmatrix}\text{Total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix} = \frac{Q}{2}CI$$

Returning to our original expression:

$$\begin{pmatrix}\text{Total yearly}\\ \text{inventory cost}\end{pmatrix} = \begin{pmatrix}\text{total per}\\ \text{piece cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total fixed}\\ \text{order cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix}$$

It can now be expressed algebraically in the following way.

$$\boxed{\text{Total yearly inventory cost} = CD + \frac{D}{Q}A + \frac{Q}{2}CI}$$

This is the total cost expression for any order quantity Q. Since the decision

maker has control only over this quantity, the relevant question is what order quantity Q will minimize total yearly inventory cost.

This mathematical expression has already been illustrated graphically in Figure 11-4, the the optimal order quantity was identified as Q^*. In the next section this optimal ordering strategy will be expressed mathematically.

Minimum Cost In order to find the value of Q which will minimize the total cost expression, it is necessary to employ the techniques of differential calculus. This is done in Appendix D, and the results are presented below:

$$Q^* = \sqrt{\frac{2DA}{IC}}$$

Therefore the optimal order quantity Q^* can be computed by performing the arithmetic operations shown in the formula given above. This quantity is frequently called the *economic order quantity*, or EOQ.

The total cost of this optimal strategy can be computed in the following way:

$$\begin{pmatrix}\text{Total yearly}\\ \text{inventory cost}\end{pmatrix}^* = CD + \frac{D}{Q^*}A + \frac{Q^*}{2}CI$$
$$= CD + \frac{D}{\sqrt{2DA/IC}}A + \frac{\sqrt{2DA/IC}}{2}CI^{1}$$

Now we will turn to examples of how these formulas can be used.

EXAMPLE

The yearly demand for a particular item has been forecast as 10,000 units. The per piece procurement cost is \$1, the fixed order cost is \$10 per order, and the inventory carrying cost has been estimated to be 20 percent per unit per year. No price discounts or stockouts are allowed.

The EOQ can be computed in the following way:

D = 10,000 units per year

A = \$10 per order

C = \$1 per unit

I = 20 percent per unit per year

[1] This can be simplified to:

$$\begin{pmatrix}\text{Total yearly}\\ \text{inventory}\\ \text{cost}\end{pmatrix}^* = \sqrt{2DAIC}$$

$$Q^* = \sqrt{\frac{2DA}{IC}}$$
$$= \sqrt{\frac{2(10{,}000)(10)}{.20(1)}}$$
$$= 1000$$

Therefore 1000 units should be ordered every time an order is placed. The total yearly cost of this strategy (TC^*) can be computed in the following way:

$$TC^* = CD + \frac{D}{Q^*}A + \frac{Q^*}{2}CI$$
$$= 1(10{,}000) + \frac{10{,}000}{1000}(10) + \frac{1000}{2}(1)(.20)$$
$$= 10{,}000 + 100 + 100$$
$$= 10{,}200$$

The total cost of this strategy including the per piece cost is $10,200 per year.

SENSITIVITY ANALYSIS

Seldom are the data required by the EOQ formula known precisely. It would therefore be useful to know the magnitude of error in the EOQ for errors in these data estimates. What, for example, would be the consequence on EOQ if demand were 1000 units higher than originally estimated? What would be the consequence of a fixed order cost of $10 rather than the original estimate of $20?

The consequence of these changes on the EOQ depends upon the sensitivity of the model to these changes. To explore the extent of this sensitivity, we will now turn to an example.

Suppose that the fixed order cost in the previous example actually turned out to be $20 rather than the original estimate of $10. The EOQ that should have been ordered can be computed in the following way:

$$Q^* = \sqrt{\frac{2(10{,}000)(20)}{.20(1)}}$$
$$= 1414$$

and the total cost of this strategy would have been

$$TC = 1(10{,}000) + \frac{10{,}000}{1414}(20) + \frac{1414}{2}(1)(.20)$$
$$= 10{,}282.80$$

Now the total cost of the EOQ strategy that was adopted on the basis of a \$10 fixed order cost can be calculated.

$$TC = 1\,(10{,}000) + \frac{10{,}000}{\underbrace{1000}_{\text{units actually ordered}}}\underbrace{(20)}_{\text{fixed order cost actually incurred}} + \frac{1000}{2}\,(1)(.20)$$

$$= \$10{,}300$$

Therefore with a 100 percent error in fixed order cost, the total inventory cost was higher by \$10,300 − \$10,282.80, or \$17.20, per year.[1] We can conclude that the computation of EOQ in this example was not very sensitive to errors in fixed order cost estimates. In fact EOQ calculations in general are not too sensitive to errors in any of the estimates.

INVENTORY SYSTEMS

Inventory systems can be classified as either outside or inside procurement systems. First we turn to a description of outside procurement systems, where orders are placed with outside suppliers.

Inventory Systems for Outside Procurement

An integrated inventory control system for outside procurement decisions combines three subsystems. The first is the order quantity and order point subsystem just developed in the previous sections. Two other subsystems are needed to support the first system. These include the forecasting and control subsystems.

Forecasting Subsystem Inventory is held for future demand. Therefore any well-designed inventory system requires that a forecast of future demand be available when inventory decisions are made.

In the EOQ model the role of the demand forecast is quite explicit. An estimate of D must be made before the value of Q^* can be computed.

Several forecasting methods are currently in use. These include judgmental, time series, and cause-effect forecasting.

In judgmental forecasting the forecaster makes an estimate based solely on good judgment and intuition. Unlike the next two, this method is not based upon *scientific* analysis of available data.

In time series analysis the forecast represents a historical extrapolation of available demand data. The simplest time series model uses the average of

[1] For firms with thousands of items in stock this "extra" cost could be substantial.

the historical data as a forecast for the future. Other time series models include moving averages, weighted averages, and exponential smoothing.

In cause-effect forecasting the underlying causes behind demand fluctuations are studied, and then these relationships are used to develop a forecast. For example, it might be determined that demand for refrigerators is related to housing starts. To forecast the demand for refrigerator sales, it would then be useful to see government projections of housing starts. The statistical methodology for drawing these relationships is called regression-correlation analysis.

All three of these forecasting methods are used, and it is difficult if not impossible to say which is the best. The choice of one rather than another would depend upon the available data, the time that can be devoted to the development of the forecast, and the accuracy required.

Order Quantity and Order Point Subsystem After the forecast has been made, the focus shifts to the problems of order quantity and timing. The EOQ system developed in this chapter is often used to make these decisions.

Doubts can be raised about the use of the EOQ model for *all* ordering decisions. The reason behind the doubts is that the use of the model requires that several assumptions be made about the real-world inventory problem. These assumptions include demand known with certainty, no price discounts, and no stockouts allowed. The model should therefore *not* be used for all inventory problems but only for those where these assumptions correspond to a reasonable degree with reality. As we have seen, the model is fairly insensitive to changes in some of the estimates. Therefore we might conclude that the model will provide reasonable results for minor departures from these assumptions. Major departures require that new models be developed. Several new models are developed in the appendixes to this chapter.

Control Subsystem Inventory control requires that inventory levels be monitored, and orders executed at the right time and for the right amount. Two methods for accomplishing this, currently in use, are the two-bin and perpetual-record systems.

In the two-bin system a lead time's worth of stock r is placed in a second bin and sealed. The remainder of the order is placed in the first bin. When stock is required, it is withdrawn from the first bin until it is depleted. Immediately upon breaking into bin 2, a reorder is placed. This system will therefore ensure that a reorder occurs when a lead time's worth of stock r is reached.

The second system requires that either manual or computer records of stock levels be maintained. Whenever items are withdrawn, the proper entry must be made in the records. When the records show that stock has been depleted to a lead time's worth of demand r, a reorder is placed.

Inventory Systems for Inside Procurement

When an order is to be produced within the firm, a new dimension is added to the problem, for the order will affect production scheduling and control decisions. Consequently the production subsystem must also be considered.

Many manufacturing firms use a system called material requirements planning (MRP). In this system the first step is the development of a product forecast. Then each product is broken down into its assemblies, subassemblies, and component parts. Identical parts are grouped together and economic order quantities are determined. Finally a production schedule is developed. The schedule must ensure that available machine and labor capacities are not violated and that the finished products will be ready when needed. If these requirements cannot be met, order quantities may have to be modified.

It is also necessary to ensure that production levels between consecutive periods is reasonably uniform. For example, it might make little sense to schedule overtime and extra hiring in one period if in the next workers would be scheduled for less than a full 40 hours and layoffs would be necessary besides. A better balance would be to smooth production over the two periods to whatever extent it is possible. Again, order quantities may have to be modified to meet this objective as well.

SUMMARY

The inventory control model developed in this chapter assumes that demand and lead time are known with certainty and that procurement is from outside sources. Although these assumptions may not apply to all inventory systems, the EOQ model may still be used if these are reasonable approximations. In fact the EOQ model has enjoyed widespread use in practice.

In Chapter 13 a model is developed for use when demand is not known with certainty but can be estimated by a probability distribution.

QUESTIONS

1 How sensitive is the optimal total cost TC^* to changes in the optimal order quantity Q^*? Illustrate your answer by using Figure 11-4.

2 What effect does inside procurement have on the inventory decision?

3 If demand for a product doubles, what effect should this have on the order quantity?

4 Suppose the cost of a stockout was very high. Can the simple EOQ model still be used?

5 An urban bus company must decide upon the number of buses that should be included in its fleet. Is this an inventory problem? If so, what characteristics does this problem share with other inventory problems?

PROBLEMS

11-1 An order is placed every 2 weeks (5 business days per week) for 200 units. Demand is 20 units per day. Prove that average inventory is $Q/2$ or 100 units.

11-2 An order is placed every week for 500 units. Daily demand is given below:

Day	Demand
1	50
2	50
3	100
4	100
5	200

Is the average inventory $Q/2$? If not, why? Compute the average inventory. Can the EOQ formula be used to determine the optimal order quantity?

11-3 The demand for a particular unit is 4000 per year, fixed order cost is \$100 per order, cost of carrying inventory is 20 percent per year, and the per piece procurement cost is \$1 per unit. What quantity should be ordered? If the lead time is 2 weeks, when should a reorder be placed?

11-4 The demand for a particular unit is 5000 pieces per year, fixed order cost is \$100 per order, cost of carrying inventory is 25 percent per year, and per piece procurement cost is \$4 per unit. What quantity should be ordered? If the lead time is 4 weeks, when should a reorder be placed?

11-5 The demand for a particular unit is 3600 pieces per year, fixed order cost is \$100 per order, cost of carrying inventory is 25 percent per year, and per piece procurement cost is \$8 per unit. What quantity should be ordered, and what is the total cost of this strategy? If lead time is 3 weeks, when should the order be placed?

11-6 Town and Country is a large retail store located in a suburban shopping center. It specializes in gifts and gourmet ware.

It stocks over 3000 retail items and purchases them mostly through sales representatives who have regular monthly appointments to see the buyer.

One of the most popular items in the store is dried flowers. Approximately 600 dozen per year are sold. Fixed order cost is \$10 per order. Inventory carrying cost is 20 percent, and the flowers cost \$6 per dozen. Lead time is 3 weeks.

a What ordering strategy would you recommend? When should this order be placed?

b Compare the cost of the strategy suggested in part a with the costs currently incurred.

c Since demand is not known with certainty, would it make sense to raise the reorder point considered in part a above?

d On the basis of the analysis you have undertaken, do you think that the store's inventory strategy is adequate? What would you do?

11-7 The production scheduling department of the Sprague Company must determine how many pieces to run on its new automatic machining center. The part which must be machined is used in several Sprague products and has an

expected yearly demand of 10,000. In the past the part has been machined on several different lathes and drill presses. Manufacturing costs have been close to \$1.40 per unit.

Now all the operations can be performed on the new computer-controlled machining center. Per unit production cost is expected to be low, about 80 cents per unit. The only problem, however, is that this new machine requires a 4-hour setup for the job. The cost of setup time is estimated at \$50 per hour. This includes not only labor cost but the opportunity cost of lost production while the machine is idle.

The cost of paperwork for placing an order is \$15, and the company's cost of carrying inventory is 20 percent.

The production scheduling department is about to recommend that orders be placed monthly rather than every 2 weeks as had been done in the past.

What would you recommend?

11-8 Last year the Somersworth Company ordered 1000 units of a particular item each time it placed an order. This year, however, demand has doubled, and the company has just decided to double its order quantities from 1000 units to 2000 units.

Last year's demand was 5000, and this year's is expected to be 10,000. Fixed order cost is \$100 per order, per piece procurement cost is \$4, and inventory carrying cost is 25 percent per unit per year.

Do you think Somersworth is about to make a wise decision?

11-9 The demand for a particular unit is 1000 per month. Fixed order cost is \$100 per order, cost of carrying inventory is 12 percent per year, and per piece procurement cost is \$2. What quantity should be ordered?

11-10 Mr. Renton, manager of inventory control, has just received a letter from the insurance firm that covers his inventory informing him that the cost of insurance will be increased next year. The present insurance cost is 2 percent of the per unit cost of the item per year. Next year it will be 3 percent.

Item number 1507-D is a typical item which is carried in Mr. Renton's warehouse. The per unit procurement cost is \$8, inventory carrying cost has been 24 percent, fixed order cost is \$100 per order, and yearly demand is 3600 pieces. Using this item as an example, do you think this increase in insurance cost should affect Mr. Renton's inventory strategy at all?

11-11 One year ago the demand for a particular unit was estimated to be 4000 units. The actual demand, however, turned out to be 6000 units. Given the following additional information, determine the error in total cost that was incurred.

$A = \$100$ per order

$I = 20$ percent per unit per year

$C = \$1$ per unit

During the past year each time an order was placed, the order quantity was 2000 units.

CASE STUDY: Springfield Alarm Company

The Springfield Alarm Company, located in Chicago, Illinois, produces 40 different burglar alarm systems. Since it was founded 10 years ago, it has experienced rapid growth.

This year the company reported $20 million in sales and a net profit after taxes of $894,000. Next year it expects an increase in sales to $25 million.

All its alarm systems are assembled in the company's Chicago plant, and in addition it manufactures approximately 70 percent of the parts used in these systems. The remainder are purchased from outside sources.

In the past the pressure has been on the marketing department to increase sales. It has been so successful that in the last six months the demands made upon the production department have increased dramatically. Orders have frequently been shipped late, backlogs have been high, and per unit production cost has increased. The pressure has now shifted to production.

George Butler, the manager of production, recently called a meeting which was attended by the assembly foreman, machine shop foreman, and inventory control foreman. The purpose of the meeting was to identify problem areas and uncover strategies for dealing with the problems.

At that meeting the first to make a presentation was the assembly foreman, Ralph Surette. "As far as I am concerned," said Ralph, "I could do a much better job if I had the parts. Lately it seems that every time I pull the stock from inventory to begin a certain job, at least one-third of the parts are out of stock. What this means is that I have to hold up the job and start something else or use substitute parts which may not be perfect but are often close. In fact, the out of stock problem is so severe that on several occasions we have used a hammer and file to alter substitute parts so that an order could be delivered to the customer. I think the fault lies with a lousy inventory control system."

The inventory control foreman, Bill Melchionda, trying to keep his temper under control, was slow to respond. "Ralph," he said, "don't forget that your people have violated every rule of our inventory system. When we instituted our system 3 years ago, it was clearly specified that only authorized stock is to be taken from the inventory bins. As you recall, all our inventory records are based on standard withdrawals from stock. When an assembly order is issued, the quantity of each item required for that assembly is automatically deducted from the item's balance on our inventory record. Once a month, when we examine these inventory records, we reorder those items which have low stock levels. Our problems with this system began when the assembly people started making substitute selections from stock. Since we have no way of knowing when a substitution is made, our records often show a higher balance than that which is actually in stock. I suggest that the first step in solving our problem is to prohibit the use of substitutes on assembly orders: if the part is not in stock, then the job must wait."

The next one to speak was James Malgeri, machine shop foreman. "My efficiency," he said, "has gone to hell since the inventory and assembly problems started. Let me give you an example of what you guys did to me yesterday. I had my numerically controlled drilling center running on job 1407-5. It had taken 4 hours to set up the job and the run time was to be about 6 hours. But at 11 A.M. Bill came running in to tell me that job 5703-2 was desperately needed. He said that a customer had called the marketing department and said that unless his order was shipped by the end of the week it would be canceled. But this system could not be assembled until the parts included in job 5703-2 were available. So, just as I have done countless times in the past, I broke down the machine, set it up for job 5703-2, ran the job, and then set up again for job 1407-5. No wonder our per unit production costs have gone way up."

Mr. Malgeri went on to say, "Another reason why our costs are so high is that our order quantities are so low. My machines are becoming more and more automated. A consequence of this is that setup times are longer but per piece production times are lower. My feeling is that once these machines are set up for a particular job, we ought to produce a large batch. I think our old policy of producing enough for 1 month's demand is absurd. The reason for this policy has been that the engineering department continually revises machining drawings and we are afraid that if we order too large a quantity, they will become obsolete when a new drawing is issued."

"Jim, why don't you pass out the copies of the report you prepared for me?" said George Butler. Jim then passed out copies to everyone present. The report is shown in Exhibit A.

Exhibit A Random Sample of Four Parts Manufactured in the Machine Shop

Part	Machine setup time hours	Fixed order paperwork cost	Yearly demand	Actual order quantities	Per piece production cost
2701-3	4	$10	10,000	1000	1.00
1406-1	2	$10	40,000	4000	.50
2704-8	2	$10	25,000	2000	.20
2694-5	1	$10	10,000	1000	.40

"What I did," said Jim, "was to randomly select four component parts which we have manufactured over the last year. On the basis of cost, this sample represents about 4 percent of the manufacturing costs incurred over this period."

Jim asked the group to look at the first item on the list. "This part was made on the new automatic drilling and machining center. It requires a 4-hour setup at $15 per hour, and every time an order is placed a fixed cost of $10 is incurred for paperwork. Yearly demand for the part is 10,000, and the actual

order quantities placed averaged 1000 units. I feel that since inventory carrying costs are a reasonable 20 percent, order quantities on this and the next three parts should have been substantially higher. I can assure you that if we produce larger lots, our production and inventory costs will drop."

At this point Mr. Butler decided to adjourn the meeting until the beginning of next week.

QUESTIONS

1 Is it justifiable to have many parts which are close substitutes for one another? Should this philosophy be encouraged?
2 What are the benefits and disadvantages of limiting the access to inventory? Should Springfield limit this access?
3 What inventory record-keeping system would you recommend? Should the records be based on standard or actual withdrawals?
4 Analyze the effectiveness of an EOQ system for Springfield by computing EOQ quantities and costs for the items given in Exhibit A.
5 Compare the actual costs incurred with those computed in question 4. Estimate the yearly saving if the EOQ system is used.
6 What problems may occur if the EOQ system is used to order those parts manufactured in Springfield's Chicago plant?
7 Can an EOQ system be used to order those parts purchased from outside suppliers?

CASE STUDY: Matrix Company

The Matrix Company, located in Los Angeles, California, manufactures component stereo receivers and speaker systems. Its product line includes twenty receivers in the moderate-price range ($200 to $400), five receivers in the high-price range ($400 to $800), ten speaker systems in the moderate-price range ($80 to $200), and one speaker system in the high-price range ($350).

The company was founded 14 years ago by two engineers who had previously worked together for a major hi-fi manufacturer. In their old job, they had developed a new speaker system which included several state-of-the-art innovations, but owing to the conservative nature of the company, their design was never used. They subsequently left the company, and 1 year later Matrix Company was founded.

Matrix has prospered since the beginning. Each year new models have been introduced. Most have been successful. At present the company is considering the expansion of its line to include turntables and compact portable stereo systems. The portable units will include an integrated amplifier, turntable, and speaker system.

In the past top management has stressed the engineering and marketing functions. That is, the emphasis has been on the development and marketing

of new sound systems. Production and inventory control have not been neglected, nor have they received any emphasis.

With the company about to embark on a major expansion program, Hank Lobeski, president, has decided to solve some inventory control problems which he feels may seriously limit expansion plans if they continue to be ignored.

Inventory Department

Most inventory items are purchased from outside sources and held in inventory until requested by the assembly department. The inventory department has no formal method for establishing when an order should be placed or how much to order. When the company was smaller, this informal system worked satisfactorily, but now, with some 10,000 items in stock, the old system has all but collapsed. Some items are frequently out of stock while others are overstocked. Mr. Lobeski has felt for some time that the lack of any formal system has cost the company a considerable amount of money.

Toward a Solution

Four months ago, Bucky Grader, a recent business school graduate, was hired by Mr. Lobeski for the purpose of developing a new inventory system. To become familiar with the system, Bucky selected five stockkeeping units at random and collected some inventory data on them. These data are shown in Exhibits A, B, and C. Bucky's intention was to carefully analyze these data, determine if a new system would be economical, and establish the methods and procedures that could be used. If this limited study proved successful, plans would be made to change the entire inventory system.

After examining these data Bucky could see that a forecasting system was needed. From Exhibit B he saw that demand for some items was increasing while for others it was decreasing. A good demand estimate would be

Exhibit A Random Sample of Five Items

Inventory item	Identification number	Actual demand, 1976	No. of orders actually placed that year	No. of times during year a stockout occurred	Per unit cost
Resistor	12715	1,000	10	4	$ 0.30
Transistor	14652	3,000	15	5	2.00
Cabinet	36565	300	4	3	15.00
Speaker	27891	250	10	1	10.00
Machine screw	17496	10,000	1	1	.02

Note 1. Cost of carrying inventory for Matrix Company is 20 percent.
Note 2. Fixed ordering cost is $30 per order.
Note 3. A stockout is recorded whenever the assembly department calls for a part and it is not found in the stockroom.

Exhibit B Yearly Demand for Five Selected Items

	Demand for item				
Year	12715	14652	36565	27891	17496
1964	550	—	—	—	6,600
1965	550	—	—	—	6,700
1966	600	—	—	—	6,700
1967	600	—	—	—	6,800
1968	600	—	—	200	6,800
1969	800	—	—	205	7,100
1970	1200	1400	—	205	7,100
1971	1400	1500	—	210	7,200
1972	1600	2000	150	210	7,400
1973	1400	2200	200	210	7,400
1974	1200	2500	240	240	7,500
1975	1020	2600	260	240	8,000
1976	1000	3000	300	250	10,000

Exhibit C Lead-Time Demand for Five Selected Items

The figures show the actual amount demanded between the times an order was placed and received in stock for six separate lead-time periods

Identification number	Demand during lead time for inventory cycle					
	1	2	3	4	5	6
12715	31	32	35	32	35	33
14652	15	17	16	19	17	18
36565	15	12	14	15	14	14
27891	9	12	8	10	10	11
17496	450	500	460	490	450	460

sensitive to these changes. Bucky, however, was not sure what kind of forecasting system should be used.

He also considered the use of an EOQ model for determining order quantity. His only reservation was that demand was not known with certainty. After analyzing the demand during several lead-time periods—shown in Exhibit C— he concluded that there was a small degree of variability in demand. He wondered what effect this would have on an EOQ system: How would this element of risk affect the reorder point r? The record-keeping system also needed consideration. Should a two-bin or perpetual inventory system be used? Three days ago a sales representative had dropped by to

explain an inventory control system that had a card file for keeping the records. On each card were recorded the outstanding orders, withdrawals, additions, and balance for each unit kept in stock. Bucky's reservation about this system was that two or three new clerks would have to be hired to maintain the records. Another alternative was to keep inventory records on a computer. The company was currently negotiating with a large computer manufacturer for the rental of a machine which could easily keep the necessary information.

Inventory access was a major issue in the company. Bucky felt that no one but stock-room personnel should have access to inventory. Only in this way could effective stock control be assured. At present anyone could withdraw parts, and for any reason.

The current inventory manager, Frank Greyson, had been on the job ever since the company started. He was an honest and sincere man but sometimes stubborn and defensive. He was firmly convinced that he could straighten out the inventory problems if he were given two additional clerks. To quote Frank, "Those confounded computers foul up every job they tackle. Just give me more people and you won't have problems."

Bucky could see that it was going to be difficult to get along with Frank. He was sure, however, that after his new system was developed, Frank would see it his way.

Bucky's last consideration would be the implementation of this system. He wondered whether the old system should be abruptly terminated and the new one begun or whether they should be run alongside each other for a while in order to gain the confidence of the users. If the two systems were run side by side, he would also have the opportunity to rectify any inconsistencies or errors before the system was in full operation.

QUESTIONS

1 Is the present system workable, and should Greyson be given the authorization to hire two new clerks?
2 How important is a forecasting system for Matrix Company? Design a simple forecasting system. Use it to forecast demand for 1977.
3 How effective would an EOQ system be for the Matrix Company? Compute the EOQ quantities for the five sample items. Compare the results of an EOQ system with those of the current system.
4 Determine reorder points for the five items.
5 What record-keeping system would be preferable?
6 Should access to inventory be limited? Why?
7 What behavioral problems could interfere with the success of the study? What could be done to minimize this conflict?
8 If the results of the study are positive, what suggestions could you make to ensure successful implementation of the system?

APPENDIX A: Quantity Discounts

The EOQ model assumes that one price prevails for all quantities purchased. Frequently, however, price breaks are offered for larger orders. Consider the following price schedule:

If $Q < q_0$, the price is C_1
If $Q \geq q_0$, the price is C_2

where Q = quantity ordered
q_0 = price-break quantity
$C_1 > C_2$

The decision process in the presence of this price break is quite complex, and it is helpful to develop the analysis as a sequence of decisions.

The first step is to determine the economic order quantity Q^* based on the higher price C_1. This is recorded as step 1 in Figure 11A-1. If Q^* is greater than the price-break q_0, the optimal strategy is to order the economic order quantity Q^{**} based on the lower price C_2.

$$Q^{**} = \sqrt{\frac{2DA}{IC_2}}$$

The reason behind this strategy can be seen from Figure 11A-2*a*. Whenever Q^* is greater than q_0, the total-cost curve associated with the lower price lies below the total-cost curve for the higher price in the region beyond q_0. Therefore, in the interest of cost minimization, Q^{**} is ordered.

If Q^* is not greater than the price-break q_0, the economic order quantity at the lower price Q^{**} must be computed and compared with q_0. If Q^{**} is greater than q_0, the cost functions must look like Figure 11A-2*b*. If you recall, we have already determined that Q^* lies below q_0, and now we have established that Q^{**} lies above q_0. Given this outcome, the optimal strategy is to order Q^{**}.

If Q^{**} is not greater than q_0, the situation depicted in either part *c* or *d* of Figure 11A-2 prevails. Both Q^* and Q^{**} must be less than q_0, and the only reasonable comparison is between the total cost associated with Q^* and the total cost incurred if exactly q_0 is ordered. Said another way, the decision is either to order the economic quantity based on the higher price or to order just enough to get the price break. Therefore the total cost for each alternative must be computed. First, the total cost incurred at Q^* is computed:

$$TC^* = C_1D + \frac{D}{Q^*}A + \frac{Q^*}{2}IC_1$$

Then the total cost of purchasing just the price-break quantity is computed.

$$TC_0 = C_2D + \frac{D}{q_0}A + \frac{q_0}{2}IC_2$$

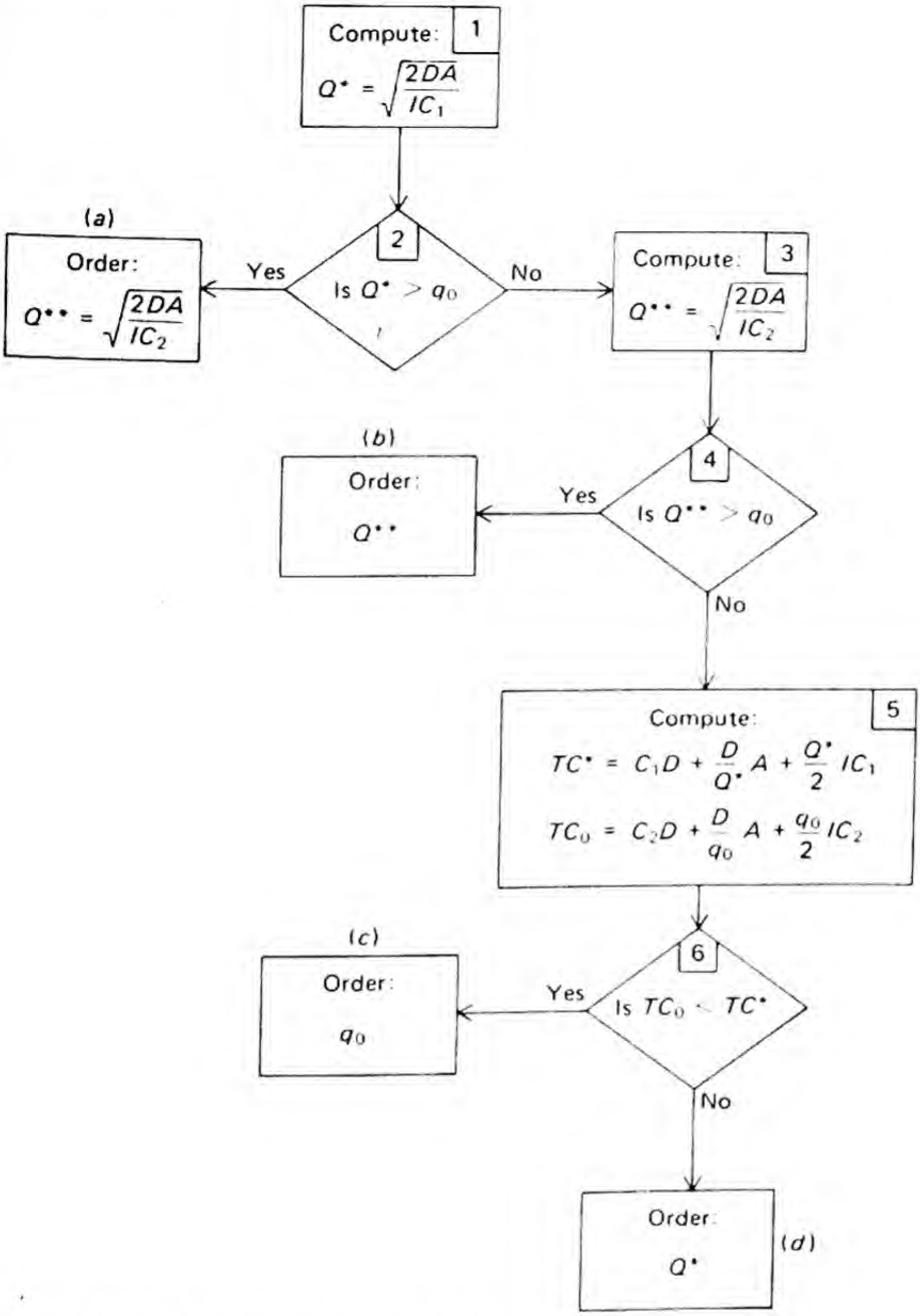


Figure 11A-1 Quantity discount decision model.

If TC^* is less than TC_0, Q^* is ordered and the behavior of the cost functions is as shown in Figure 11A-2*d*. If TC_0 is less than TC^*, the relevant cost functions are shown in Figure 11A-2*c*, and q_0 should be ordered.

EXAMPLE

Consider the following:

$D = 10{,}000$ units
$A = \$10$ per order

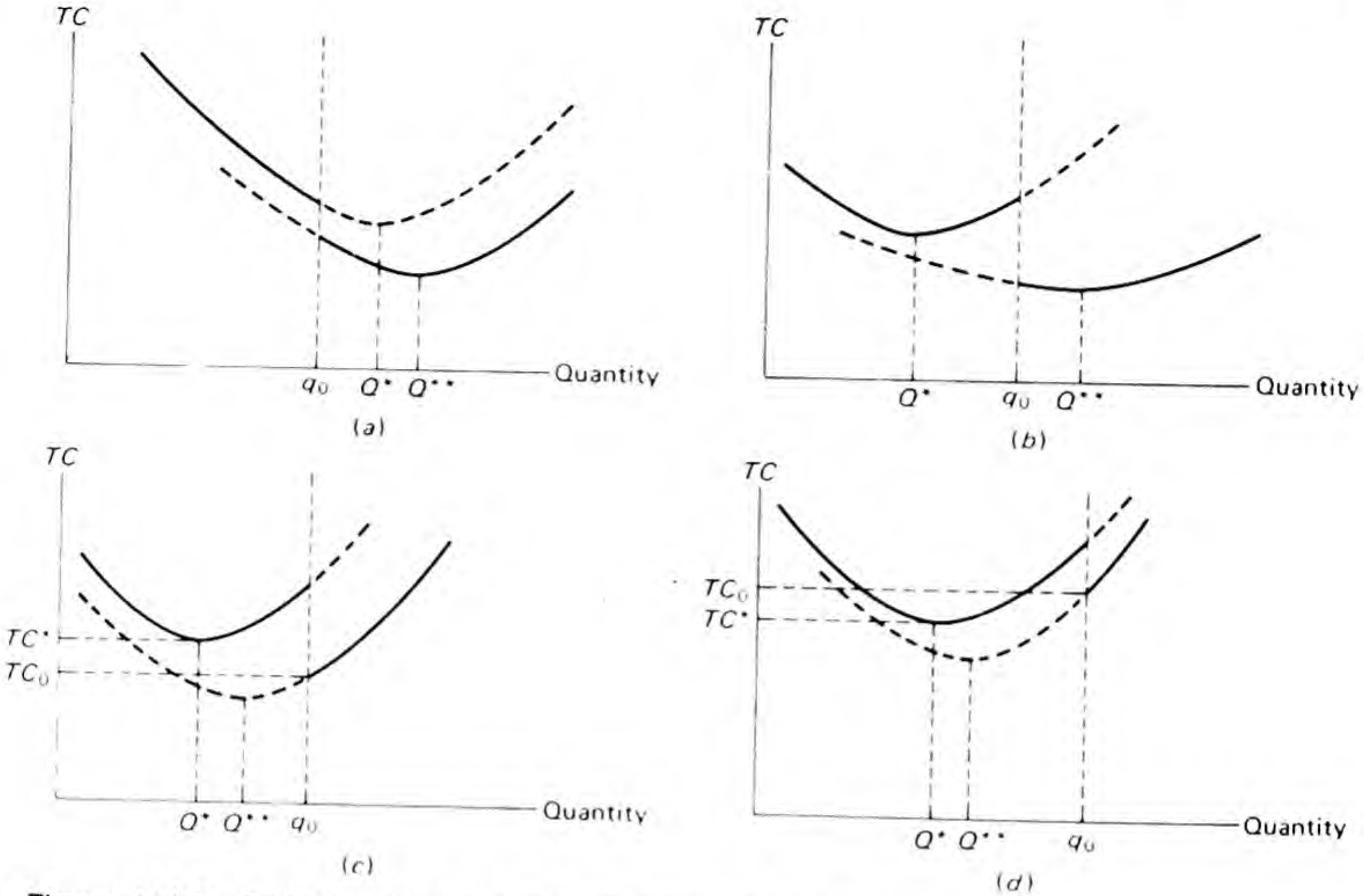


Figure 11A-2 Cost functions for various discount situations.

$I = 20$ percent

$q_0 = 1700$ units

If $Q < q_0$, $C_1 = \$1$

If $Q \geq q_0$, $C_2 = \$0.80$

Step 1: $Q^* = \sqrt{\dfrac{2(10{,}000)(10)}{.20(1)}} = 1000$

Step 2: $Q^* \not> 1700$

Step 3: $Q^{**} = \sqrt{\dfrac{2(10{,}000)(10)}{.20(.80)}} = 1580$

Step 4: $Q^{**} \not> q_0$

Step 5: $TC^* = 1(10{,}000) + \dfrac{10{,}000}{1000}10 + \dfrac{1000}{2}(.20)(1)$

$TC^* = 10{,}000 + 100 + 100$

$TC^* = 10{,}200$

$TC_0 = .80(10{,}000) + \dfrac{10{,}000}{1700}10 + \dfrac{1700}{2}(.20)(.80)$

$TC_0 = 8000 + 59 + 136$

$TC_0 = 8195$

Step 6: $TC_0 < TC^*$

Therefore,

$$\boxed{\text{Order } q_0 = 1700}$$

PROBLEMS

11A-1 A firm's demand for a particular product is 100,000 per year. Fixed order cost is $100 per order, and inventory carrying cost is 20 percent per unit per year. The per unit price paid depends upon the order quantity. If 5000 or fewer units are ordered, the price is $1 per unit. If more than 5000 units are ordered, the price drops to 80 cents per unit. How many units should be ordered?

11A-2 A firm's demand for a particular product is 1000 units per year. Fixed order cost is $100 per order, and inventory carrying cost is 20 percent per unit per year. The per unit procurement price depends upon the order quantity. If fewer than 1100 units are ordered, the price is $1 per unit. For quantities over 1100 units the price drops to 80 cents per unit. What quantity should be ordered?

11A-3 A firm's demand for a particular product is 7200 units per year. Fixed order cost is $10 per order, and inventory carrying cost is 20 percent. The per unit procurement cost depends upon the order quantity. If fewer than 650 units are ordered, the price is $2 per unit; if more than 650 units are ordered, the price drops to $1.50 per unit. How many units should be ordered?

11A-4 Consider the following pricing schedule:

If $Q < 850$, $C_1 = \$10$ per unit
If $Q \geq 850$, $C_2 = \$8$ per unit

If forecast demand is 8000 units, fixed order cost is $100, and inventory carrying cost is 25 percent, what quantity should be ordered?

11A-5 Consider the following pricing schedule:

If $Q < 2000$, $C_1 = \$10$ per unit
If $Q \geq 2000$, $C_2 = \$9.50$ per unit

If forecast demand is 8000 units, fixed order cost is $100, and inventory carrying cost is 25 percent, what quantity should be ordered?

11A-6 Given the following information, determine the appropriate order quantity:

$A = \$10$ per order
$I = 10$ percent per unit per year
$D = 500$ units per year
If $Q < 250$, $C_1 = \$10$ per unit
If $Q \geq 250$, $C_2 = \$2.50$ per unit

11A-7 The forecast demand for a particular unit is 40,000. The fixed order cost is $25,

and inventory carrying cost is 25 percent per unit per year. The manufacturer offers price breaks at the following two points:

If $0 \leq Q < 100$, $C_1 = \$8$ per unit
If $100 \leq Q < 800$, $C_2 = \$7$
If $Q \geq 800$, $C_3 = \$6$

What quantity should be ordered?

APPENDIX B: Noninstantaneous Replenishment

PRODUCTION RATE AND AVERAGE INVENTORY

The EOQ model assumes that stock is replenished instantaneously by an amount equal to the order quantity Q Rather than having all Q units arrive at once, it is possible for units to be replenished in stock one at a time until the order is complete. This is especially true of companies that manufacture their own inventory.

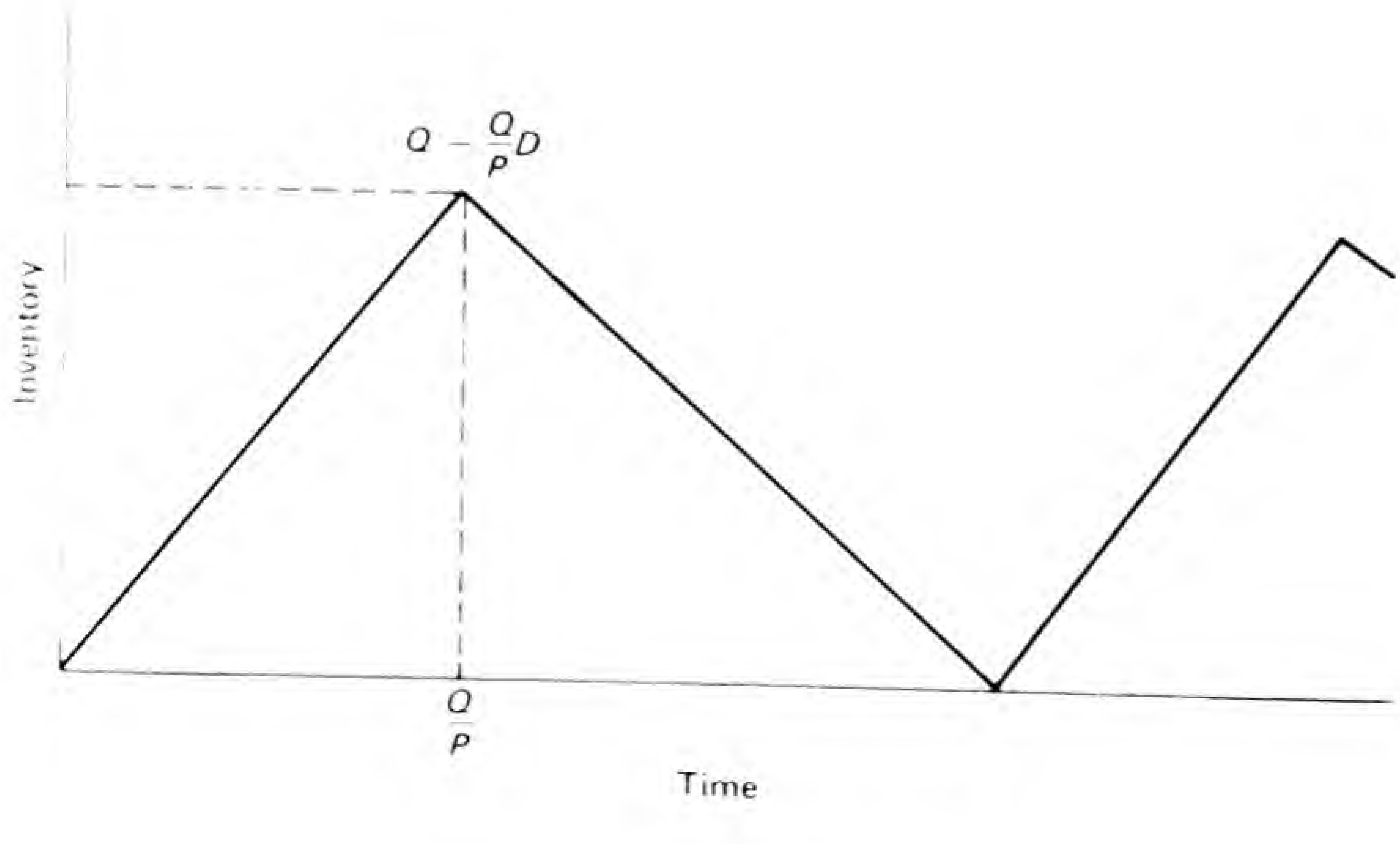
We will assume the units are demanded at the rate D and replenished at the rate P until all Q units have been produced. This pattern is illustrated in Figure 11B-1.

The production rate P and demand D must be given in the same time dimensions. We will assume it is yearly demand and yearly production rate.

Reference to Figure 11B-1 shows that the time it takes to produce the order is Q/P. At this time $(Q/P)D$ has been demanded and used. Therefore the inventory at the highest point is

$$Q-(Q/P)D$$

or



$$\frac{Q}{P}(P - D)$$

Average inventory is half this amount.

$$\text{Average inventory} = \frac{Q}{2P}(P - D)$$

TOTAL COSTS

Returning to the statement of total yearly inventory cost made in the chapter, we have the following:

$$\begin{pmatrix}\text{Total yearly}\\\text{inventory cost}\end{pmatrix} = \begin{pmatrix}\text{total per}\\\text{piece cost}\\\text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\\text{fixed order}\\\text{cost per}\\\text{year}\end{pmatrix} + \begin{pmatrix}\text{total}\\\text{inventory}\\\text{carrying cost}\\\text{per year}\end{pmatrix}$$

The only change that needs to be made is the total inventory carrying cost.

$$\begin{pmatrix}\text{Total}\\\text{inventory}\\\text{carrying cost}\\\text{per year}\end{pmatrix} = \begin{pmatrix}\text{average}\\\text{inventory}\\\text{in units}\end{pmatrix}\begin{pmatrix}\text{per unit}\\\text{procurement}\\\text{cost}\end{pmatrix}\begin{pmatrix}\text{inventory}\\\text{carrying cost}\\\text{in percent}\\\text{per year}\end{pmatrix}$$

$$= \frac{Q}{2P}(P - D)CI$$

We can now write the expression for total yearly inventory cost:

$$\text{Total yearly inventory cost} = CD + \frac{D}{Q}A + \frac{Q}{2P}(P - D)CI$$

MINIMUM COSTS

The value of Q which will minimize this expression is found by employing the techniques of calculus. This is done in Appendix D. The results are presented below.

$$Q^* = \sqrt{\frac{2DA}{IC}\left(\frac{P}{P - D}\right)}$$

Consider the following example:

D = 10,000 per year
I = 25 percent per unit per year
C = \$4 per unit

$A = \$2$ per order

$P = 50{,}000$ per year

$$Q^* = \sqrt{\frac{2(10{,}000)2}{.25(4)} \; \frac{50{,}000}{40{,}000}}$$

$Q^* = 224$

PROBLEMS

11B-1 The yearly demand for a particular item is 10,000 units. It is produced within the plant at the rate of 40,000 units per year. Ordering and setup costs are \$100 per order, inventory carrying cost is 25 percent, and production cost is \$1 per unit.

- **a** What quantity should be ordered?
- **b** What will be the maximum amount in inventory?
- **c** If it takes 1 month to receive the first units from production, when should a reorder be placed?
- **d** What is the cost of this strategy?

11B-2 The yearly demand for a particular unit is 1000. It is produced within the plant at the rate of 100,000 units per year. Ordering and setup costs are \$100 per order, inventory carrying cost is 25 percent, and production cost is \$1 per unit.

- **a** What quantity should be ordered?
- **b** What can be said about the order quantity in the noninstantaneous replenishment case when the rate of production is substantially higher than the rate of demand?

11B-3 The yearly demand for a particular item is 10,000 units. It is produced within the plant at the rate of 20,000 units per year. Ordering and setup costs are \$100 per order, inventory carrying cost is 25 percent, and production cost is \$2 per unit.

- **a** What quantity should be ordered?
- **b** What will be the maximum amount in inventory?
- **c** If it takes 1 month to receive the first units from production, when should a reorder be placed?

11B-4 The yearly demand for a particular item is 5000. It is produced within the plant at a rate of 1000 units per week. Ordering and setup costs are \$100 per order, inventory carrying cost is ½ percent per week, and production cost is \$2 per unit. How many units should be ordered?

APPENDIX C: EOQ Model with Stockouts Allowed

Again we will return to the simple EOQ model, but this time we will relax the assumption that no stockouts are allowed. The first step is to specify the new total-cost expression.

TOTAL COSTS

Writing the total-cost expression for those costs which are affected by the order quantity Q, we have

$$TC = \begin{pmatrix}\text{total per}\\ \text{piece cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{fixed order}\\ \text{cost per}\\ \text{year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{back-order}\\ \text{cost per}\\ \text{year}\end{pmatrix}$$

The total per piece and fixed order costs are determined just as before. The other costs, however, require that we examine the time pattern of inventory levels. This pattern is shown in Figure 11C-1.

Turning to the total inventory carrying cost per year, we have the following:

$$\begin{pmatrix}\text{Total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix} = \begin{pmatrix}\text{average}\\ \text{inventory}\end{pmatrix}\begin{pmatrix}\text{per piece}\\ \text{procurement}\\ \text{cost}\end{pmatrix}\begin{pmatrix}\text{inventory}\\ \text{carrying cost}\\ \text{in percent}\\ \text{per year}\end{pmatrix}$$

$$\begin{pmatrix}\text{Total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix} = \begin{pmatrix}\text{average}\\ \text{inventory}\end{pmatrix} CI$$

Average inventory, however, is quite different from what it was in the simple case. Maximum inventory is defined as M and the minimum is zero. The average inventory of $(M+0)/2 = M/2$, however, is carried for only t_1/T of the cycle. The average inventory for the remainder of the cycle is zero. We can therefore write the

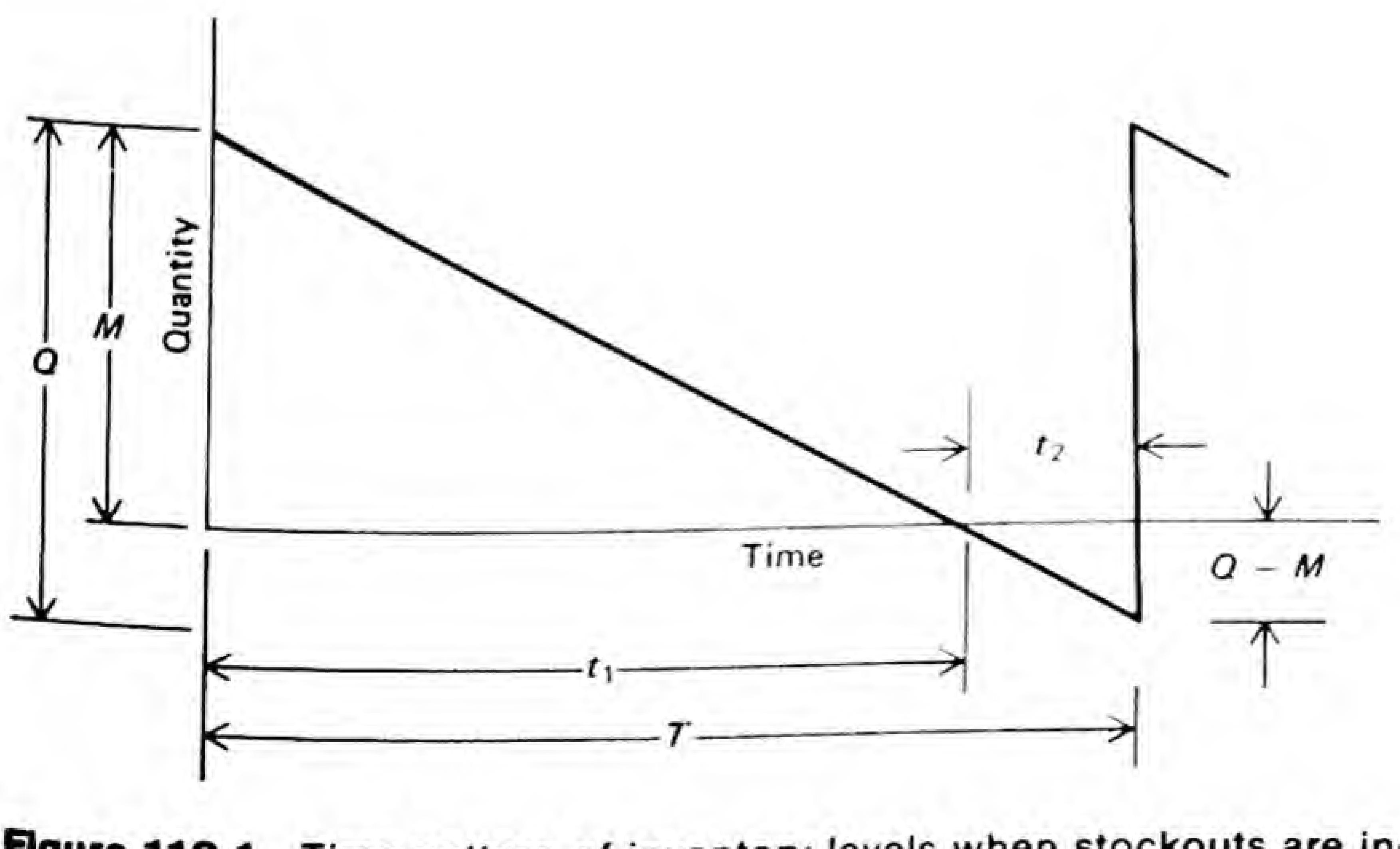


Figure 11C-1 Time pattern of inventory levels when stockouts are incurred.

average inventory as $(M/2)(t_1/T)$. Therefore,

$$\begin{pmatrix} \text{Total} \\ \text{inventory} \\ \text{carrying cost} \\ \text{per year} \end{pmatrix} = \frac{M}{2}\left(\frac{t_1}{T}\right)CI$$

Next we turn to the back-order cost per year. This cost can be determined in the following way:

$$\begin{pmatrix} \text{Total} \\ \text{back-order} \\ \text{cost per} \\ \text{year} \end{pmatrix} = \begin{pmatrix} \text{average number} \\ \text{of back-orders} \\ \text{per year} \end{pmatrix} \begin{pmatrix} \text{back-order} \\ \text{cost} \end{pmatrix}$$

The average number of back-orders while back-orders are taking place is equal to the maximum number of back-orders divided by 2, or $(Q - M)/2$. Since back-orders occur only during t_2/T of the cycle, the average number of back-orders per cycle is $[(Q - M)/2](t_2/T)$. This will also be the average number of back-orders per year. We therefore have

$$\begin{pmatrix} \text{Total} \\ \text{back-order} \\ \text{cost per} \\ \text{year} \end{pmatrix} = \frac{Q - M}{2}\frac{t_2}{T}B$$

where B is the cost incurred for back-ordering one unit of demand for one year.

Returning to the expression for total cost, we now have the following:

$$TC = CD + \frac{D}{Q}A + \frac{M}{2}\frac{t_1}{T}CI + \frac{Q - M}{2}\frac{t_2}{T}B$$

It is possible to remove t_1, t_2, and T from this expression in the following way:

$$t_1 = \frac{M}{D}$$

$$T = \frac{Q}{D}$$

$$t_2 = T - t_1 = \frac{Q}{D} - \frac{M}{D} = \frac{Q - M}{D}$$

Therefore,

$$\frac{t_1}{T} = \frac{M}{Q}$$

$$\frac{t_2}{T} = \frac{Q - M}{Q}$$

Returning to the total-cost expression, we now have

$$TC = CD + \frac{D}{Q}A + \frac{M}{2}\left(\frac{M}{Q}\right)CI + \frac{Q-M}{2}\left(\frac{Q-M}{Q}\right)B$$

$$TC = CD + \frac{D}{Q}A + \frac{M^2}{2Q}CI + \frac{(Q-M)^2}{2Q}B$$

The next step is to find that order strategy Q which minimizes these costs.

MINIMUM COST

Since we now have a function in two decision variables Q and M, it becomes a little more difficult to solve for the lowest total cost TC^*. In Appendix D the following solution is found:

$$Q^* = \sqrt{\frac{2DA}{IC}}\sqrt{\frac{CI+B}{B}}$$

and

$$M = \frac{QB}{CI+B}$$

Notice that if back-order cost B is very large, the value of the second radical approaches 1 and economic order quantity derived for the simple case holds.

As an example, consider the following:

$D = 10{,}000$ units per year

$A = \$10$ per order

$I = 20$ percent per unit per year

$C = \$1$ per unit

$B = \$5$ per each back-order incurred

$$Q^* = \sqrt{\frac{2(10{,}000)(10)}{.20(1)}}\sqrt{\frac{1(.20)+5}{5}}$$

$$Q^* = 1000(1.02)$$

$$Q^* = 1020$$

$$M = \frac{1020(5)}{1(.20)+5} = 980 \text{ units}$$

Therefore the maximum amount back-ordered is

$$Q - M = 1020 - 980 = 40 \text{ units}$$

PROBLEMS

11C-1 The demand for a particular unit is 4000 per year, fixed order cost is $100 per order, cost of carrying inventory is 20 percent, and the per piece procurement cost is $1 per unit. Back-order cost is considered to be very high: the latest estimate is $1000 for every unit back-ordered. What quantity should be ordered? What conclusions can be reached when back-order cost is high?

11C-2 The demand for an item is 4000 per year, fixed order cost is $100 per order, cost of carrying inventory is 20 percent per year, the per piece procurement cost is $100 per unit, and the back-order cost is $5 per unit back-ordered. What quantity should be ordered? What is the maximum amount back-ordered?

11C-3 The demand for an item is 3600 units per year, fixed order cost is $100 per order, cost of carrying inventory is 25 percent per year, the per piece cost is $8 per unit, and the back-order cost is $10 per unit back-ordered. What quantity should be back-ordered? What is the maximum amount back-ordered? What is the cost of this strategy?

11C-4 The Dawson Company sells a patented electric rotor to several large firms who manufacture printing presses. The manufacturing cost of the rotor is $50. Demand over the past few years has been growing. Next year's demand estimate is 1000 units. Fixed order cost, including paperwork and machine setup costs, is $200 per order. Inventory carrying cost is 20 percent per unit per year.

Since the rotor is a highly specialized and patented part, it is felt that the firms who purchase the unit will wait a reasonable time if the unit is out of stock. Back-order cost is therefore relatively low and has been estimated to be $2 per unit.

a How many parts should be ordered? What is the maximum number of parts back-ordered?

b What conclusions can be reached when back-order cost is small in relation to inventory carrying cost?

c Is this an example of inside or outside procurement? What precautions should be taken if EOQ is to be used in this situation?

d Do you think that back-order cost is too low? What factors should be included in the cost of a back-order?

11C-5 Determine the reorder point for problem 11C-4. The length of time to complete a production run is 4 weeks.

APPENDIX D: Derivations

DERIVATION OF EOQ

$$TC = CD + \frac{D}{Q}A + \frac{Q}{2}CI$$

Taking the first derivative with respect to Q, we have:

$$\frac{d(TC)}{dQ} = -DAQ^{-2} + \frac{CI}{2}$$

Setting this expression equal to zero and solving for Q, we have the following:

$$Q^* = \sqrt{\frac{2DA}{IC}}$$

DERIVATION OF EOQ FOR NONINSTANTANEOUS REPLENISHMENT

$$TC = CD + \frac{D}{Q}A + \frac{Q}{2P}(P - D)CI$$

$$\frac{d(TC)}{dQ} = -DAQ^{-2} + \frac{(P - D)CI}{2P}$$

Setting this expression equal to zero and solving for Q, we have the following:

$$Q^* = \sqrt{\frac{2DA}{IC}\left(\frac{P}{P - D}\right)}$$

DERIVATION OF ECONOMIC ORDER QUANTITY WITH BACK ORDERS ALLOWED

The process by which the optimal value of Q is found involves taking the partial derivative of the total-cost expression first with respect to the variable Q and then with respect to the variable M. Then the partial derivatives are set equal to zero and solved for Q.

The total-cost expression is repeated for convenience:

$$TC = CD + \frac{D}{Q}A + \frac{M^2}{2Q}CI + \frac{(Q - M)^2}{2Q}B$$

Its partial derivative with respect to M is

$$\frac{\partial(TC)}{\partial M} = \frac{M}{Q}CI - \frac{(Q - M)}{Q}B$$

Then we set it equal to zero,

$$\frac{M}{Q}CI - \frac{(Q - M)}{Q}B = 0$$

and solve for M:

$$M = \frac{QB}{CI + B}$$

Next we take the partial derivative of the total-cost expression with respect to Q.

$$\frac{\partial(TC)}{\partial Q} = -\frac{DA}{Q^2} - \frac{M^2}{2Q^2}CI + \frac{B}{2} - \frac{M^2B}{2Q^2}$$

Setting it equal to zero, we have the following:

$$-\frac{DA}{Q^2} - \frac{M^2}{2Q^2}CI + \frac{B}{2} - \frac{M^2B}{2Q^2} = 0$$

Finally, by substituting M in this equation and simplifying, we get the expression for Q^*:

$$Q^* = \sqrt{\frac{2DA}{IC}}\sqrt{\frac{CI + B}{B}}$$

Chapter 12

Inventory Models: Risk

INTRODUCTION

The EOQ models developed in the last chapter are used when it is reasonable to assume that demand is known with certainty. Often, however, this assumption cannot be made, and it is more appropriate to describe demand by a probability distribution. The purpose of this chapter is to develop a model for these occasions.

Reorder Points

When demand is not known with certainty, the inventory system may suffer stockouts during the lead-time period. Some protection against these stockouts is required.

Consider the case of the Eastern Auto Parts store. Its retail stock is ordered from a wholesaler who always fills an order in 2 weeks. Lead time is therefore known with certainty. Demand for their products, however, is not known with certainty. Wide fluctuations above and below the average level of demand frequently occur.

Eastern Auto Parts uses the two-bin inventory system. When the items in bin 1 are depleted, an order is placed. It is while items are being withdrawn from bin 2 that stockouts may occur.

Eastern Auto's present strategy is to fill the second bin with a quantity of stock equal to the average demand expected over the lead time of that item.

The average demand during lead time can be computed from historical records. For example, the record for a particular spark plug is given in Table 12-1. It was compiled during 100 lead-time periods. To compute the average demand $\bar{X}$, first the product of each lead-time demand X and its probability of occurrence $P(X)$ is taken.

$$XP(X)$$

Then they are summed.

$$\bar{X} = \Sigma XP(X)$$

The result is an average lead-time demand of 50 cases. Eastern Auto has therefore filled the second bin with 50 cases of spark plugs. This strategy, however, has *not* worked out very well for them. Stockouts have occurred during more than half the lead-time periods.

A closer analysis of the Eastern Auto Parts system reveals why frequent stockouts should indeed be expected. If a reorder is placed when an *average* lead time's worth of stock is reached, then during 54 percent of the lead-time periods demand will be greater than available supply.[1] If greater protection against stockouts is desired, the reorder point will have to be raised. This can be accomplished by including a larger number of items in the second bin.

[1]From Table 12-1 it can be determined that there is a 54 percent chance that lead-time demand will exceed 50 units.

Table 12-1 Demand during Lead Time for Spark Plug AC-P210 (in Cases) during 100 Lead-Time Periods

Lead-time demand, X	Frequency of occurrence	Probability, $P(X)$	Lead-time demand, X	Frequency of occurrence	Probability, $P(X)$
40*	2	.02	50	7	.07
41	2	.02	51	11	.11
42	3	.03	52	10	.10
43	3	.03	53	9	.09
44	4	.04	54	8	.08
45	4	.04	55	7	.07
46	4	.04	56	5	.05
47	5	.05	57	3	.03
48	6	.06	58	1	.01
49	6	.06			

*Number of cases

The question to be answered in this chapter is: To what level should the reorder point be raised in order to provide adequate protection against a stockout?

Inventory Patterns

When demand is known with certainty, it is possible to place a reorder when inventory is exactly equal to a lead time's worth of stock and thereby avoid the possibility of a stockout. With reference to Figure 12-1, a reorder would be placed at r and would arrive a lead time later. When it arrives, stock is replenished to Q units and no stockouts have occurred.

But when demand is not known with certainty, inventory depletion will not behave in a predictable way. An example of this is shown in Figure 12-2. The reorder point r has been set at an average lead time's worth of demand (just as in the Eastern Auto Parts example). In the first cycle demand during lead time is close to average and the order arrives just when stock is depleted to zero: no stockouts have been incurred. In the second cycle demand during lead time is higher than average and the stock is depleted before the order arrives: stockouts have therefore been incurred. Finally, in the third cycle we see an example of below-average demand during lead time. When the order arrives, there is still some inventory on hand (bin 2 has not been completely depleted) and consequently there are no stockouts.

Safety Stocks

To ensure stockout protection, extra stock must be included in bin 2. Or to say it another way, the reorder point r must be raised. This extra stock must be above and beyond an *average* lead time's worth of stock, and the protection against stockouts will increase as more and more extra stock is used. Technically, this extra stock is called either safety stock or buffer stock.

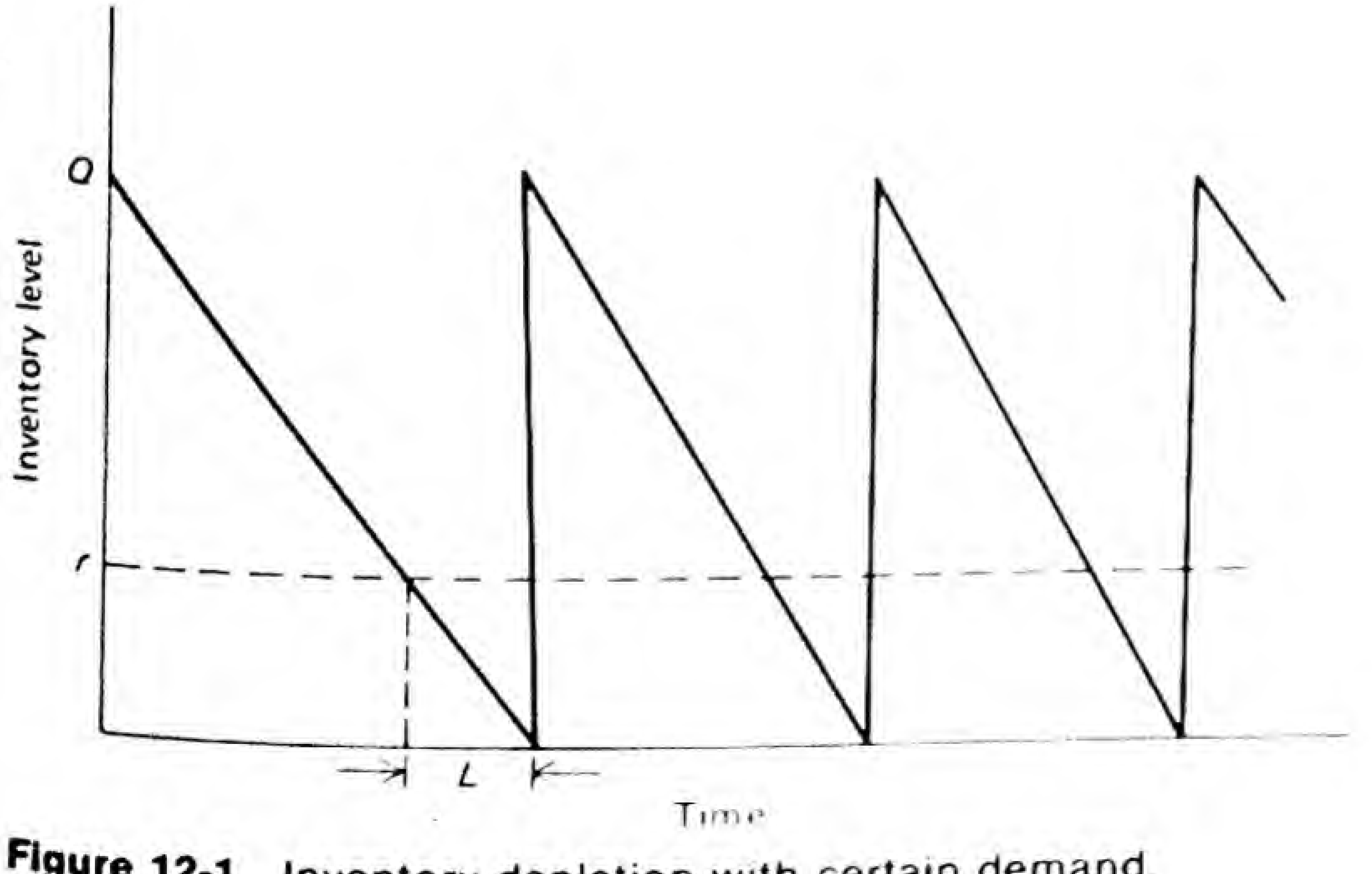


Figure 12-1 Inventory depletion with certain demand.

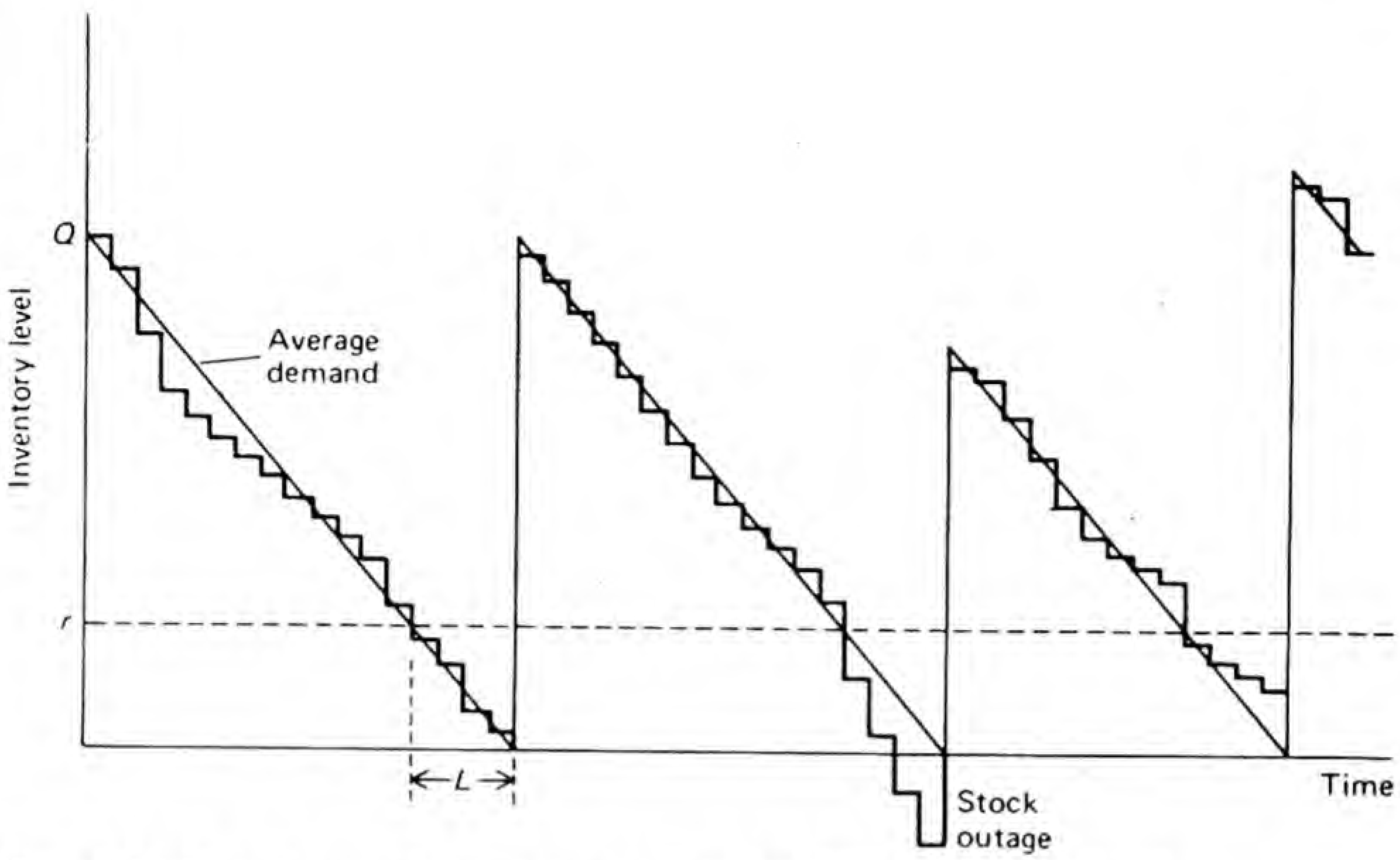


Figure 12-2 Inventory depletion: demand not known with certainty.

If we let S represent safety stock, then

$$S = r - \bar{X}$$

or safety stock is equal to the difference between the reorder point and the average lead-time demand.

For example, if average lead-time demand is 50 units and the reorder point is 80 units, 30 units are maintained as safety stock to protect against stockouts.

TOTAL COST WHEN STOCKOUT COST IS GIVEN

The total cost of operating an inventory system under certainty was developed in Chapter 11 and is repeated below:

$$\begin{pmatrix}\text{Total}\\ \text{yearly}\\ \text{inventory}\\ \text{cost}\end{pmatrix} = \begin{pmatrix}\text{total}\\ \text{per piece}\\ \text{cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{fixed}\\ \text{order cost}\\ \text{per year}\end{pmatrix} + \begin{pmatrix}\text{total}\\ \text{inventory}\\ \text{carrying cost}\\ \text{per year}\end{pmatrix}$$

$$\text{Total yearly inventory cost} = CD + \frac{D}{Q}A + \frac{Q}{2}CI$$

The assumptions of this model were that demand and lead time were known with certainty and that no stockouts were allowed.

When demand is no longer known with certainty, stockouts can occur, and safety stocks are used to provide some protection against this possibility.

Stockouts can be expensive. Each time one occurs, there is a cost to the firm. For some firms and for certain products these costs may be high; for others they may be low. The safety stocks which are used to prevent these stockouts can also be expensive because they represent additional stock which must be carried in inventory.

An economic analysis therefore requires that both costs—stockout and safety stock—be included in the total cost equation which was just presented. This is done in the following way:

$$\begin{pmatrix}\text{Total}\\\text{yearly}\\\text{inventory}\\\text{cost}\end{pmatrix}=\begin{pmatrix}\text{total}\\\text{per piece}\\\text{cost}\\\text{per year}\end{pmatrix}+\begin{pmatrix}\text{total}\\\text{fixed}\\\text{order cost}\\\text{per year}\end{pmatrix}+\begin{pmatrix}\text{total}\\\text{inventory}\\\text{carrying cost}\\\text{per year}\end{pmatrix}$$
$$+\begin{pmatrix}\text{total}\\\text{safety stock}\\\text{carrying cost}\\\text{per year}\end{pmatrix}+\begin{pmatrix}\text{total}\\\text{stockout}\\\text{cost}\\\text{per year}\end{pmatrix}$$

Solving the Problem by the Decomposition Method

Given the total-cost expression, the problem to be solved is this: *How many* (Q^*) items should be ordered each time, and when (r^*) should the order be placed so that costs are minimized? The method which we will use requires that the problem be broken into two separate parts. The first is the order quantity problem and requires that the first three terms in the expression be solved for a minimum; the second is the reorder point problem and requires that the last two terms in the expression be solved for a minimum.

Theoretically, this is not the proper way to solve the problem, but it reduces the amount of computation significantly. In addition it is very intuitive and provides answers which are close to optimal.

Next we turn to the first part of this two-part problem, the order quantity.

ORDER QUANTITY

Returning to the total-cost expression and isolating the first three terms, we have the following:

$$\begin{pmatrix}\text{Total}\\\text{per piece}\\\text{cost}\\\text{per year}\end{pmatrix}+\begin{pmatrix}\text{total}\\\text{fixed}\\\text{order cost}\\\text{per year}\end{pmatrix}+\begin{pmatrix}\text{total}\\\text{inventory}\\\text{carrying cost}\\\text{per year}\end{pmatrix}$$

This should be very familiar to you, as it is the total-cost expression which was developed in the last chapter. Substituting, we have:

$$DC + \frac{D}{Q}A + \frac{Q}{2}CI$$

and minimizing the expression, the optimal order quantity is found to be the following:

$$Q^* = \sqrt{\frac{2DA}{IC}}$$

It can therefore be concluded that when demand is *not* known with certainty, the EOQ expression can still be used to approximate the order quantity.

As an example, we will return to the Eastern Auto Parts case. The following additional information has been obtained.

D = 1000 cases per year

C = \$10 per unit

A = \$40 per order

I = 20 percent per unit per year

Computing the EOQ, we have:

$$Q^* = \sqrt{\frac{2(1000)40}{10(.20)}}$$

$$Q^* = 200$$

Therefore, every time an order is placed, 200 cases of spark plugs should be ordered.

CALCULATING THE REORDER LEVEL WHEN BACK-ORDER COST IS GIVEN

Now we turn to the last two terms in the total-cost expression. Before representing these costs with algebraic notation, let's examine their economic behavior over a few different strategies.

The Economics of Safety Stocks and Stockouts

Consider a strategy where safety stocks are kept very low. A consequence of this may be frequent stockouts. The cost associated with these stockouts may be large. However, the cost of carrying these extra units in stock will be low.

Consider another strategy in which safety stocks are high. A consequence of this will be infrequent stockouts. The cost associated with

incurring these infrequent stockouts may be low, but the cost of carrying these extra units in stock will be high.

Although either of these extreme strategies could be used, it is reasonable to expect that the strategy finally adopted would try to minimize the sum of both costs: inventory carrying cost for safety stock and stockout cost.

In the next sections a method will be developed for determining the reorder point—and therefore the safety stock level—for which the sum of these two costs is minimized.

Back Orders

The model that will be developed assumes that all stockouts are back-ordered and do not become lost sales. In addition, when a back order does occur, a fixed back-order charge is incurred only once. For example, if a stockout occurs for spark plugs, a fixed charge of \$5 per case will be incurred owing to the expenses associated with a back order. The letter B is used to represent this cost.

Quantifying Total Safety Stock and Back-Order Costs

When safety stocks are added to average lead-time demand, they must be carried for the entire year. The cost of carrying these stocks can be quantified in the following way:

$$\begin{pmatrix}\text{Total safety}\\ \text{stock carrying}\\ \text{cost per year}\end{pmatrix} = \begin{pmatrix}\text{safety}\\ \text{stock}\end{pmatrix}\begin{pmatrix}\text{per piece}\\ \text{cost}\end{pmatrix}\begin{pmatrix}\text{inventory}\\ \text{carrying cost}\end{pmatrix}$$

$$= SCI$$

$$= (r - \overline{X})CI$$

Back orders can occur during the lead time of each cycle. If $E(N)$ is the expected number of back orders per cycle, the total number of back orders per year is equal to the number of inventory-ordering cycles per year multiplied by the expected number of back orders per cycle.

$$\frac{D}{Q^*}[E(N)]$$

To get the cost of these back orders, this product is multiplied by the cost of a back order B.

$$\text{Total cost of back orders per year} = \frac{D}{Q^*}[E(N)]B$$

Summarizing these two components of the total-cost expression, we have

the following:

$$\begin{pmatrix} \text{Total} \\ \text{safety stock} \\ \text{carrying cost} \\ \text{per year} \end{pmatrix} + \begin{pmatrix} \text{total} \\ \text{cost of} \\ \text{back orders} \\ \text{per year} \end{pmatrix} = (r - \overline{X})CI + \frac{D}{Q*}[E(N)]B$$

In the next section the method for minimizing the sum of these two costs will be introduced.

Tabular Form for Cost Minimization

The sum of the back-order and safety stock costs depends upon the reorder level chosen. It will be necessary, then, to compare the cost consequences for several different reorder levels. This will be done using the tabular form shown in Table 12-2.

Across the top are entered the alternative reorder levels starting with the average lead-time demand. The first entry in the body of the table is the safety stock. For example, if the reorder level is 54, the safety stock is determined in the following way:

$$S = r - \bar{X}$$
$$S = 54 - 50$$
$$S = 4$$

The next entry in the table completes the computation of the safety stock carrying cost.

$$\text{Total safety stock carrying cost per year} = (r - \bar{X})CI$$

For example, if the reorder level is 54, safety stock is 4 and the carrying costs are:

$$4(10)(.20) = \$8$$

The second row, therefore, summarizes the carrying costs for each alternative reorder level strategy. The next two rows in the table summarize back-order costs for each of these strategies. The first of these rows requires that the expected number of back orders per cycle be computed. The next section is devoted to these computations.

Determining the Expected Number of Back Orders

The expected number of back orders per order cycle depends upon the reorder level. If the reorder level is high, few back orders will be expected. If, on the other hand, the reorder level is low, many back orders will be expected.

Table 12-2 Safety Stock and Back-Order Costs

	Reorder level r								
	50	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**
Safety stock, $r - \bar{X}$	0	1	2	3	4	5	6	7	8
Yearly carrying cost, $(r - \bar{X})CI$	\$0	\$2	\$4	\$6	\$8	\$10	\$12	\$14	\$16
Expected number of back orders, $E(N)$	1.84	1.30	.87	.54	.30	.14	.05	.01	.00
Back-order costs, $\frac{D}{Q^*}[E(N)]B$	\$46.00	\$32.50	\$21.75	\$13.50	\$7.50	\$3.50	\$1.25	\$0.25	\$0.00
Total, $(r - \bar{X})CI + \frac{D}{Q^*}[E(N)]B$	\$46.00	\$34.50	\$25.75	\$19.50	\$15.50	\$13.50	\$13.25	\$14.25	\$16.00
							$\uparrow$ $r^* = 56$		

Table 12-3 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 50

(1) If $r = 50$ and actual demand is X...	(2) ... the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders $(X - r)P(X)$
≤ 50	0	.46	.00
51	1	.11	.11
52	2	.10	.20
53	3	.09	.27
54	4	.08	.32
55	5	.07	.35
56	6	.05	.30
57	7	.03	.21
58	8	.01	.08
			$E(N) = 1.84$

Table 12-4 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 51

(1) If $r = 51$ and actual demand is X...	(2) ... the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 51	0	.57	.00
52	1	.10	.10
53	2	.09	.18
54	3	.08	.24
55	4	.07	.28
56	5	.05	.25
57	6	.03	.18
58	7	.01	.07
			$E(N) = 1.30$

Returning to the Eastern Auto Parts case, we will assume that the reorder level will be set somewhere between the average lead-time demand, $\bar{X} = 50$, and the highest lead-time demand recorded in our sample, $X = 58$ (see Table 12-1).

Next, we will compute the expected number of back orders per order cycle for each of these alternative reorder levels.

First, the reorder level will be set to 50. If the actual demand is less than or equal to 50, no back orders will occur. This fact is summarized in Table 12-3. If the actual demand is 51, the back order would be 1 unit. If the actual demand is 52, the back order would be 2 units. These and all other demand

and back-order possibilities are entered in the first two columns of Table 12-3.

From Table 12-1 it can be determined that there is a 46 percent chance that demand during lead time will be less than or equal to 50. This fact is entered in column 3 of Table 12-3. The probabilities for other demand levels also can be found in Table 12-1, in column 3.

From columns 2 and 3 in Table 12-3 it can be seen that 46 percent of the time no back order would occur, 11 percent of the time one back order would occur, 10 percent of the time two back orders would occur, and so on. If the product of these two columns is taken and these values are added, the result will be the expected value or the average number of back orders expected each cycle. This is done in the last column of the table, and it can therefore be concluded that if the reorder level is set to 50, 1.84 back orders can be expected every order cycle.

The same procedure must be followed for reorder levels of 51 to 58. This is done in Tables 12-4 through 12-11. It should be clear by looking at the results that as the reorder level is raised, the expected number of back orders per cycle drops.

Finally these values are entered in Table 12-2.

Computing Total Back-Order Cost

Total back-order cost for each reorder level can be computed by using the following formula:

$$\text{Total yearly back-order cost} = \frac{D}{Q^*}[E(N)]B$$

The value of Q^* has already been computed in the first part of the analysis; it is used in this formula to compute total back-order cost. For example, if the

Table 12-5 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 52

(1) If $r = 52$ and actual demand is X...	(2) ...the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 52	0	.67	.00
53	1	.09	.09
54	2	.08	.16
55	3	.07	.21
56	4	.05	.20
57	5	.03	.15
58	6	.01	.06
			$E(N) = .87$

Table 12-6 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 53

(1) If $r = 53$ and actual demand is X...	(2) ...the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 53	0	.76	.00
54	1	.08	.08
55	2	.07	.14
56	3	.05	.15
57	4	.03	.12
58	5	.01	.05
			$E(N) = .54$

Table 12-7 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 54

(1) If $r = 54$ and actual demand is X...	(2) ...the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 54	0	.84	.00
55	1	.07	.07
56	2	.05	.10
57	3	.03	.09
58	4	.01	.04
			$E(N) = .30$

Table 12-8 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 55

(1) If $r = 55$ and actual demand is X...	(2) ...the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 55	0	.91	.00
56	1	.05	.05
57	2	.03	.06
58	3	.01	.03
			$E(N) = .14$

Table 12-9 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 56

(1) If $r = 56$ and actual demand is X...	(2) ... the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 56	0	.96	.00
57	1	.03	.03
58	2	.01	.02
			$E(N)$ = .05

Table 12-10 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 57

(1) If $r = 57$ and actual demand is X...	(2) ... the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 57	0	.99	.00
58	1	.01	.01
			$E(N)$ = .01

Table 12-11 Computation of Expected Number of Back Orders per Cycle When the Reorder Level r Is 58

(1) If $r = 58$ and actual demand is X...	(2) ... the back order would be $X - r$	(3) The probability of this is $P(X)$	(2) × (3) Computation of the expected number of back orders per cycle $(X - r)P(X)$
≤ 58	0	1.00	.00
			$E(N)$ = .00

reorder level is 54 units, total yearly back-order cost is

$$\frac{1000}{200}(.30)(5) = \$7.50$$

where the cost of a back-order B has already been given as \$5. The results of the computations for the remaining reorder levels are given in Table 12-2.

Computing the Total of Back-Order and Safety Stock Costs

The safety stock and back-order costs have already been calculated for the various reorder levels. What remains is to add them together and compare. This is done at the bottom of Table 12-2. It can be seen that if the reorder level is set to $r^* = 56$ units, the sum of back-order and safety stock costs will be minimized at \$13.25.

The Solution

We can conclude that a reorder should be placed when the number of spark plug cases in the second bin is reduced to $r^* = 56$. Since average lead-time demand is 50, the safety stock will be six cases.

The Total Yearly Inventory Cost

The total yearly inventory carrying cost can be expressed in the following way:

$$TC = CD + \frac{D}{Q}A + \frac{Q}{2}CI + (r - \bar{X})CI + \frac{D}{Q}[E(N)]B$$

The total cost for the problem just completed can be computed by substituting the appropriate data.

$$\begin{aligned} TC &= 10(1000) + \frac{1000}{200}(40) + \frac{200}{2}(10)(.20) + 6(10)(.20) + \frac{1000}{200}(.05)(5) \\ &= 10{,}413.25 \end{aligned}$$

It can be concluded that the total cost of the strategy to order 200 cases when inventory falls to 56 cases is \$10,413.25.

Is This the Optimal Strategy?

We cannot say that the total cost of the strategy chosen in the last section ($Q^* = 200$, $r^* = 56$) is the best one. Recall that in the process of solving the total yearly inventory cost expression, we simplified the problem by decomposing the expression into two parts. First we solved for Q^* and then for r^*.

By using only the last two terms to determine r^*, we have implied that the expected number of stockouts per year is strictly a function of r, the reorder level. This is only part of the picture, for the choice of r will determine

the expected number of stockouts *per cycle*, but the expected number of stockouts *per year* depends on the product of the expected number of stockouts per cycle and the number of orders (or cycles) per year. Therefore, the number of stockouts that occurs per year depends on *both* Q and r, and in our simplified approach we have chosen Q^* independently of r^*. What is the magnitude of the error? You may recall that in the last chapter the total-cost expression was found to be rather insensitive in the vicinity of its minimum. It is therefore reasonably safe to conclude that our simplified method presented in this chapter will result in values of Q^* and r^* which are close to the optimal ones.

SUMMARY

In this chapter a model has been developed for solving inventory problems when demand is not known with certainty but can be expressed as a probability distribution. The problem of how much to order was solved in exactly the same way as it was in Chapter 12, but the problem of when to reorder had to take into consideration the size of the safety stock necessary to protect against stockouts.

QUESTIONS

1 Can a stockout ever occur before the lead-time period?

2 In the Eastern Auto Parts case, under what condition would it be necessary to set the reorder level to 58 units?

3 Would it ever be reasonable to set the reorder level below average lead-time demand?

4 When a company carries large safety stocks, what assumptions has it made?

5 How is the cost of a back order determined?

PROBLEMS

12-1 If the average lead-time demand for an item is 250 units and the reorder point is 275, what is the safety stock level?

12-2 The demand for a particular item is 10,000 units per year. The EOQ is 200 units, the back-order costs are \$3 per unit, and the expected number of back orders per cycle is five. What is the total cost of back orders per year.

12-3 The reorder level for a particular item has been set to $r = 25$. The probability that demand will be less than or equal to this reorder level is 75 percent. At most demand during this lead-time period can be 26 units. What is the expected number of back orders per cycle?

12-4 The reorder level for a particular item has been set to $r = 300$. The probability that demand will be less than or equal to this reorder level is 80 percent. At most demand during this lead-time period can be 301 units. What is the expected number of back orders per cycle?

12-5 Verify that the average demand in Table 12-1 is 50.

12-6 Show that if the safety stock carried for a particular stockkeeping item is 30 units, the average extra stock carried in inventory is 30 units, not $^{30}/_{2} = 15$ units.

12-7 The Mayfield Company sells precision tools to machinists. The demand for one such tool, a micrometer, is expected to be 1000 units. Its cost is \$10 each.

The micrometer is ordered from the Micro Tool Company, and predictably arrives 1 week after the order is placed. The fixed cost associated with placing this order is \$40, and inventory carrying cost is 20 percent per unit per year.

Stockouts are expensive for the Mayfield Company, but in most cases the customer will wait for the item to arrive. The cost of this back order has been estimated to be \$5 per unit.

For several years management has been collecting data on demand during lead time. The results are given below:

Demand	Number of occasions on which this demand was observed	Demand	Number of occasions on which this demand was observed
20	0	25	20
21	5	26	10
22	10	27	5
23	20	28	0
24	30		

Find the reorder quantity, reorder level, and total cost of this strategy.

12-8 Determine the reorder level, the order quantity, and the total cost of this strategy for the following inventory problem:

$D = 1800$ units per year

$A = \$30$ per order

$I = 15$ percent per unit per year

$B = \$1$ per unit back-ordered

$C = \$2$ per unit

Lead-time demand	Probability	Lead-time demand	Probability
48	.02	54	.20
49	.03	55	.07
50	.06	56	.06
51	.07	57	.03
52	.20	58	.02
53	.24		

12-9 The probability distribution for lead-time demand is estimated to be the following:

Lead-time demand	Probability	Lead-time demand	Probability
30	.07	34	.12
31	.15	35	.08
32	.26	36	.07
33	.18	37	.07

Demand is 2000 units per year; fixed order cost is $25 per order; per piece cost is $2 per unit; inventory carrying cost is 25 percent per unit per year; and back-order cost is $1 per unit back-ordered.

When should an order be placed, and how many should be ordered?

CASE STUDY: Precision Machine Tool, Inc.

Precision Machine Tool, Inc., manufactures a broad line of machine tools, including grinders, lathes, and jig borers. A recent management audit conducted by the management services division of a nationally recognized accounting firm uncovered several weaknesses in the inventory control function. In comparison with other machine tool manufacturers, the auditors found that Precision Machine had unusually high inventory costs and recommended that its inventory ordering system be redesigned.

The president, Kenneth King, was discussing the problem with the inventory control manager, Peter Barbera.

"Pete, I think we ought to select the electric motors that go into our grinders and reevaluate the inventory strategy that we use on them. If we can come up with a better strategy, we can apply it to the other units in stock."

Peter replied, "I'll pull the records this morning, but I'm not sure we can do very much. As you know, for 10 years we have been taking inventory on major items every Monday morning. Each of these items has a desired level. The actual level is compared against this desired level, and we place an order for the difference. For example, our desired level for the grinder motor is 8 units. If Monday's inventory reveals 4 units in stock, an order is issued for 4 units more. Now here is the problem, Ken. If the desired level of inventory is lowered, more stockouts will be incurred. As we all know, more stockouts mean production delays, schedule changes, late shipments to customers, and higher production costs. I'll be glad to pull the records, Ken, but I don't think there is much we can do. Given the kind of business we're in, I think we're doing a great job. Why should we listen to management consultants? Have they ever made a grinder?"

"You've made several good points," said Ken, "but I think we should still go ahead with our reevaluation. I have a suspicion that orders are placed too frequently. Our ordering cost for those motors is about $20, which includes writing the purchase order, follow-up on the order, handling cost, and incoming inspection cost. Since we order every week, with the exception of the 2-week vacation period, this means that we spend $1000 per year on ordering alone! I'll bet we could cut this cost in half!"

"But wait a minute," Peter interrupted, "you're forgetting about our carrying cost. At 15 percent per unit per year these costs make it unprofitable to order larger quantities on a less frequent basis. Since the motors cost $500 each, a larger average inventory would add substantially to your inventory cost."

"Yes," said Ken, "but this extra cost might be more than offset by the reduction in ordering cost. Why don't you look into using the EOQ method? It takes both of these costs into consideration."

Pete replied, "The EOQ system requires that demand and lead time are known with certainty. In the case of the grinder motor neither is known with certainty. Demand during lead time may be either 0, 1, 2, 3, or 4, and it may

take either 1 or 2 days to get the motor from our supplier once the order is placed. Consequently, stockouts can be incurred if insufficient safety stocks are carried. If you recall, we computed the cost of a stockout at $40."

Ken concluded, "Pete, why don't you pull the records (see Exhibit A) and come back to me in 2 days with some suggestions on reorder quantities and reorder levels? I would also like to see a comparison between the inventory cost for this motor using our present strategy and the cost using any other strategy which you might propose."

Exhibit A Inventory Record, Grinder Motor A375-1

Day	Beginning Inventory	Order received	Demand	Ending inventory
1*	4	0	0	4
2	4	4	1	7
3	7	0	1	6
4	6	0	1	5
5	5	0	2	3
6*	3	0	2	1
7	1	5	1	5
8	5	0	1	4
9	4	0	0	4
10	4	0	0	4
11*	4	0	2	2
12	2	4	0	6
13	6	0	0	6
14	6	0	1	5
15	5	0	0	5
16*	5	0	0	5
17	5	3	1	7
18	7	0	2	5
19	5	0	2	3
20	3	0	0	3
21*	3	0	0	3
22	3	5	0	8
23	8	0	1	7
24	7	0	0	7
25	7	0	1	6
26*	6	0	2	4
27	4	2	0	6
28	6	0	0	6
29	6	0	0	6
30	6	0	2	4
31*	4	0	0	4
32	4	4	1	7

Exhibit A (continued)

Day	Beginning inventory	Order received	Demand	Ending inventory
33	7	0	0	7
34	7	0	1	6
35	6	0	0	6
36*	6	0	2	4
37	4	0	0	4
38	4	2	1	5
39	5	0	2	3
40	3	0	2	1
41*	1	0	1	0
42	0	7	1	6
43	6	0	0	6
44	6	0	0	6
45	6	0	0	6
46*	6	0	0	6
47	6	2	0	8
48	8	0	0	8
49	8	0	1	7
50	7	0	0	7
51*	7	0	1	6
52	6	0	0	6
53	6	1	2	5
54	5	0	0	5
55	5	0	1	4
56*	4	0	0	4
57	4	4	0	8
58	8	0	0	8
59	8	0	1	7
60	7	0	0	7
61*	7	0	0	7
62	7	1	1	7
63	7	0	0	7
64	7	0	2	5
65	5	0	2	3
66*	3	0	0	3
67	3	0	1	2
68	2	5	2	5
69	5	0	1	4
70	4	0	0	4
71*	4	0	0	4
72	4	4	0	8

Exhibit A (continued)

Day	Beginning inventory	Order received	Demand	Ending inventory
73	8	0	2	6
74	6	0	0	6
75	6	0	1	5
76*	5	0	0	5
77	5	3	0	8
78	8	0	2	6
79	6	0	1	5
80	5	0	2	3
81*	3	0	0	3
82	3	5	1	7
83	7	0	0	7
84	7	0	0	7
85	7	0	2	5
86*	5	0	0	5
87	5	3	1	7
88	7	0	0	7
89	7	0	0	7
90	7	0	1	6
91*	6	0	1	5
92	5	0	0	5
93	5	2	0	7
94	7	0	2	5
95	5	0	0	5
96*	5	0	1	4
97	4	3	2	5
98	5	0	0	5
99	5	0	1	4
100	4	0	1	3

* Order placed every Monday.
Note: Assume 250 working days per year.

QUESTIONS

1 Determine the probability that demand during lead time will be 0, 1, 2, 3, or 4.
2 What is the average lead-time demand for the product?
3 Compute the economic order quantity.
4 When should reorders be placed?
5 Compare the cost of the fixed order quantity system developed in questions 3 and 4 with that of the fixed order interval system now used. Which would you recommend?

APPENDIX A: Setting the Reorder Level When a Service Criterion Is Given

DISCRETE DISTRIBUTIONS OF LEAD-TIME DEMAND

In many situations, it might be quite difficult to determine the actual cost of a stockout. An indirect way to assess this is to specify a service criterion. For example, the inventory manager of the product whose lead-time demand was depicted in Figure 12-1 might feel that the likelihood of a stockout should not exceed 4 percent during any one cycle.

To find the appropriate reorder level that meets this criterion, a "cumulative greater than" distribution must be compiled. This is accomplished by determining the probability that demand will exceed each possible lead-time demand level X. Starting from Table 12-1, a "cumulative greater than" distribution is presented in Table 12A-1.

From Table 12A-1 it can be seen that if the reorder level is $r^* = 56$, the likelihood of a stockout will not *exceed* 4 percent.

CONTINUOUS DISTRIBUTIONS OF LEAD-TIME DEMAND

Suppose that the distribution of lead-time demand could be approximated by a normal distribution as shown in Figure 12A-1. The average lead-time demand is $\bar{X}$ and its standard deviation is σ. If the reorder level is set at r, the safety stock is S, and the probability that demand will exceed the reorder level r is shown as the shaded area.

Consider an example where $\bar{X} = 400$ and $\sigma = 15$. The criterion is that the likelihood of a stockout during any one cycle should be less than 1 percent.

In the normal table at the back of the book, it is found that the area to the right of 2.33 standard deviations from the mean is approximately 1 percent. Therefore the reorder level should be set at the mean plus 2.33 standard deviations.

Table 12A-1 Cumulative Greater than Distribution for Lead-Time Demand

Lead-time demand, X	$P(X)$	Probability that demand will exceed X	Lead-time demand, X	$P(X)$	Probability that demand will exceed X
39	.00	1.00	49	.06	.61
40	.02	.98	50	.07	.54
41	.02	.96	51	.11	.43
42	.03	.93	52	.10	.33
43	.03	.90	53	.09	.24
44	.04	.86	54	.08	.16
45	.04	.82	55	.07	.09
46	.04	.78	56	.05	.04
47	.05	.73	57	.03	.01
48	.06	.67	58	.01	.00

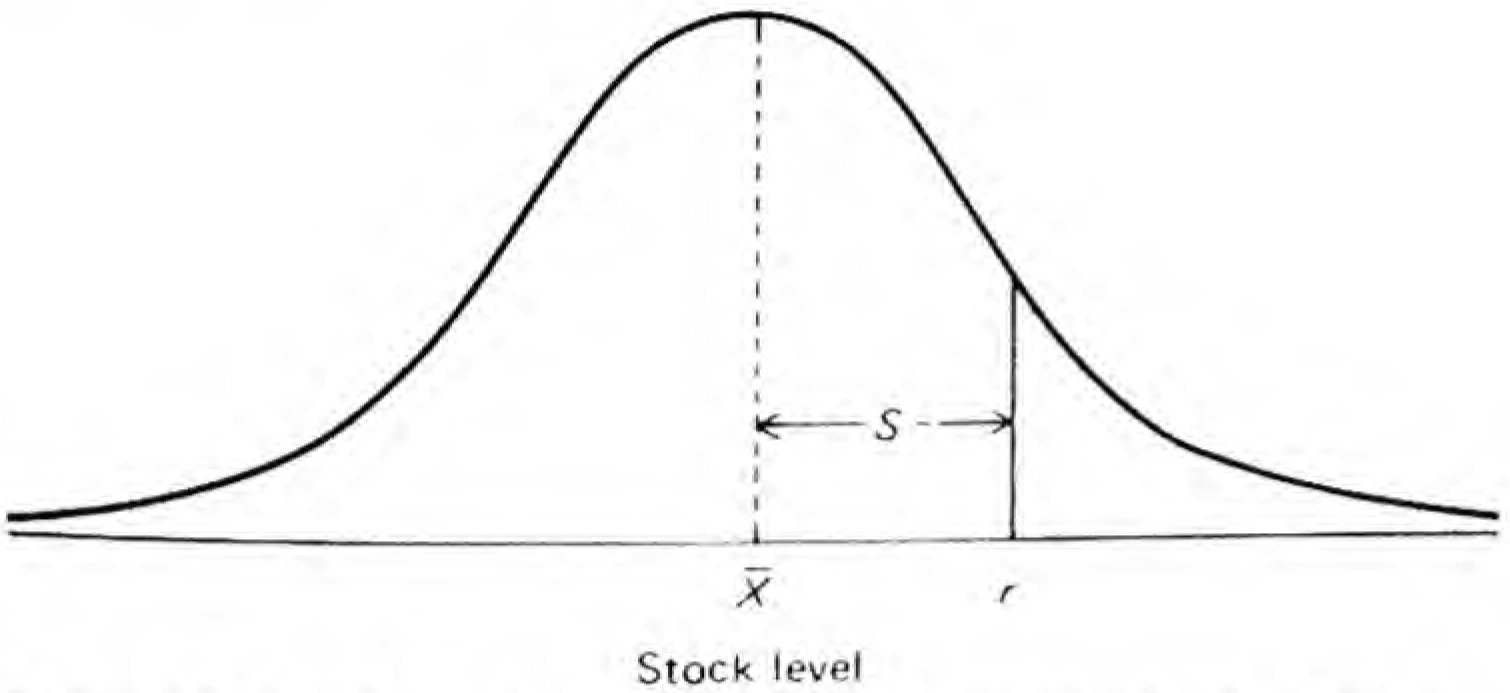


Figure 12A-1 Lead-time demand represented as a continuous distribution.

$$r = \bar{X} + 2.33(\sigma)$$
$$r = 400 + 2.33(15)$$
$$r = 435$$

If the reorder level is set to 435 units, the likelihood of a stockout during any one cycle is 1 percent or less.

PROBLEMS

12A-1 Suppose the criterion for the problem illustrated in Table 12A-1 was that the probability of a stockout was not to exceed 16 percent during any one cycle. Find r^*.

12A-2 Suppose that the criterion for the problem illustrated in Table 12A-1 was that the probability of a stockout was not to exceed 12 percent. Find r^*

12A-3 Demand during lead time for a particular unit can be described by a normal distribution with an average demand of 5000 units and a standard deviation of 1000 units. Suppose that management does not want more than a 3 percent chance of a stockout during any one ordering cycle; when should it reorder?

12A-4 Demand during lead time for a particular unit can be described by a normal distribution with an average demand of 1000 units and a standard deviation of 100 units. Suppose that management does not want more than a 10 percent chance of a stockout during any one ordering cycle; when should it reorder?

12A-5 Return to problem 12A-3. Suppose that management would like the probability of stockout not to exceed 12 percent during any one inventory cycle. When should a reorder be placed?

Chapter 13

Queuing Theory

INTRODUCTION

Waiting lines are common. They can be found in almost any organization. Lines form in airline terminals, bus stations, supermarkets, banks, manufacturing facilities, health clinics, restaurants, and at tollbooths.

In any of these examples it would be unreasonable to provide a level of service that could prevent the formation of a waiting line. Instead, adequate levels of service are provided and both the customer and the organization must accept the presence of these lines.

What is an adequate level of service? The answer depends upon the willingness of the customer to wait, or the cost of waiting, and upon the cost of providing service.

The purpose of this chapter is to explore several models that can be used by decision makers to compare the cost of waiting and service for several different service-level strategies.

The case study which follows will be used to introduce the concepts necessary in the development of these models.

CASE STUDY: Mission Hill Neighborhood Health Clinic

The Mission Hill Neighborhood Health Clinic is located in a predominantly black section of Boston, Massachusetts. It was established seven years ago under a limited grant from the federal government. At present the staff includes three physicians, one laboratory assistant, and one secretary.

The medical needs of the community far exceed the service that these physicians can provide. Frequently patients must wait 2 or 3 hours before they can be seen by the staff. In fact, the medical director, Dr. Gail Swanson, feels that this waiting period often discourages patients from coming to the clinic.

When patients come to the clinic they must register with the secretary and wait for the first available physician. They proceed into an examination room when the physician becomes available and may spend from 10 minutes to an hour with the doctor. Upon completion of the visit they leave the clinic.

Two months ago Dr. Swanson requested the addition of two physicians to her staff. She argued that this increase would not only reduce patient waiting time but would probably encourage more people to use the facilities of the clinic.

In response to this request the federal funding agency asked Dr. Swanson to estimate the impact of this staff increase on patient waiting time. In its letter the agency expressed concern over the magnitude of the benefits that could be attributed to this substantial increase in operating cost. The letter further suggested that with an increase in staff to five physicians these physicians might well experience periods during the day when there would not be enough patients to keep the staff busy. The agency concluded that this could result in a waste of federal funds.

The Mission Hill Clinic as a Queuing System

The operation of the Mission Hill Clinic can be described as a queuing system. The patients arrive, wait their turn, are serviced by a physician, and finally exit from the system. In the next section the basic characteristics of this and other queuing systems will be described.

THE BASICS

There are two basic aspects of any waiting line: arrivals and service. Figure 13-1 represents the very simplest waiting-line system. In fact it is identical to a system that could exist at a health clinic with only one physician. Patients arrive at the clinic, the service is supplied by the physician, and upon completion of the service the patient exits from the system. If a particular service takes much longer than average, or if arrivals occur sooner than average, a waiting line may form.

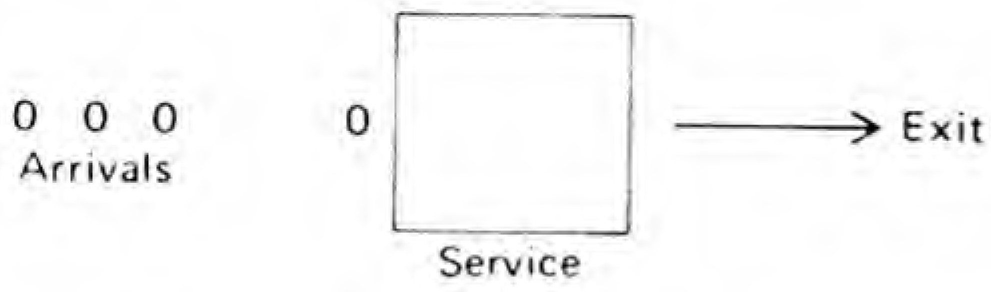


Figure 13-1 Arrivals and service in a simple queue.

How a Waiting Line Is Formed

Suppose that the average arrival rate at this simple clinic is six per hour (one every 10 minutes) and the average service time is 8 minutes. If a patient arrived precisely every 10 minutes, and if service took precisely 8 minutes, a waiting line should never form. The reason that it does indeed form is that there exists a randomness about these average values. Even though the long-run average arrival time is every 10 minutes, some patients may arrive 5 minutes after the previous one, some 14 minutes, and some 11 minutes, and so on. Similarly, service on any patient may take 4 minutes, 11 minutes, 9 minutes, and so on even though the long-run *average* of these service times is 8 minutes. When several arrivals occur sooner than the average time or when service takes longer than the average time, a line may form.

We can therefore conclude that owing to the randomness in the arrival and service patterns it is quite possible for waiting lines to build even when the *average* service time is less than the *average* time between arrivals. Any theory of waiting lines must therefore take this phenomenon of randomness into consideration.

CLASSIFICATION OF QUEUES

A waiting-line problem can be classified according to the following criteria.

1 Whether the source of the arrivals is finite or infinite
2 The number of lines and service stations
3 The priority of the arrivals

Finite and Infinite Sources

In most waiting-line problems the source of the arrivals is very large. In fact it is so large that we assume it is infinite. Consider the source of arrivals at the Mission Hill Clinic. The number of residents in the area is so large that we say the source of the arrivals is infinite.

In some waiting-line problems, however, the source of arrivals is small in number. In this case we say the source is finite. Consider a repairman who is responsible for eight electronic installations. When one breaks down, we say that it has "arrived" for service. Clearly the source of arrivals is *not* infinite.

The Number of Lines and the Number of Stations

The simplest of all waiting-line problems was presented in Figure 13-1. It represented *one line, one service station*. This is but one possible configuration. Another would be the situation depicted in Figure 13-2: *one line, multiple*

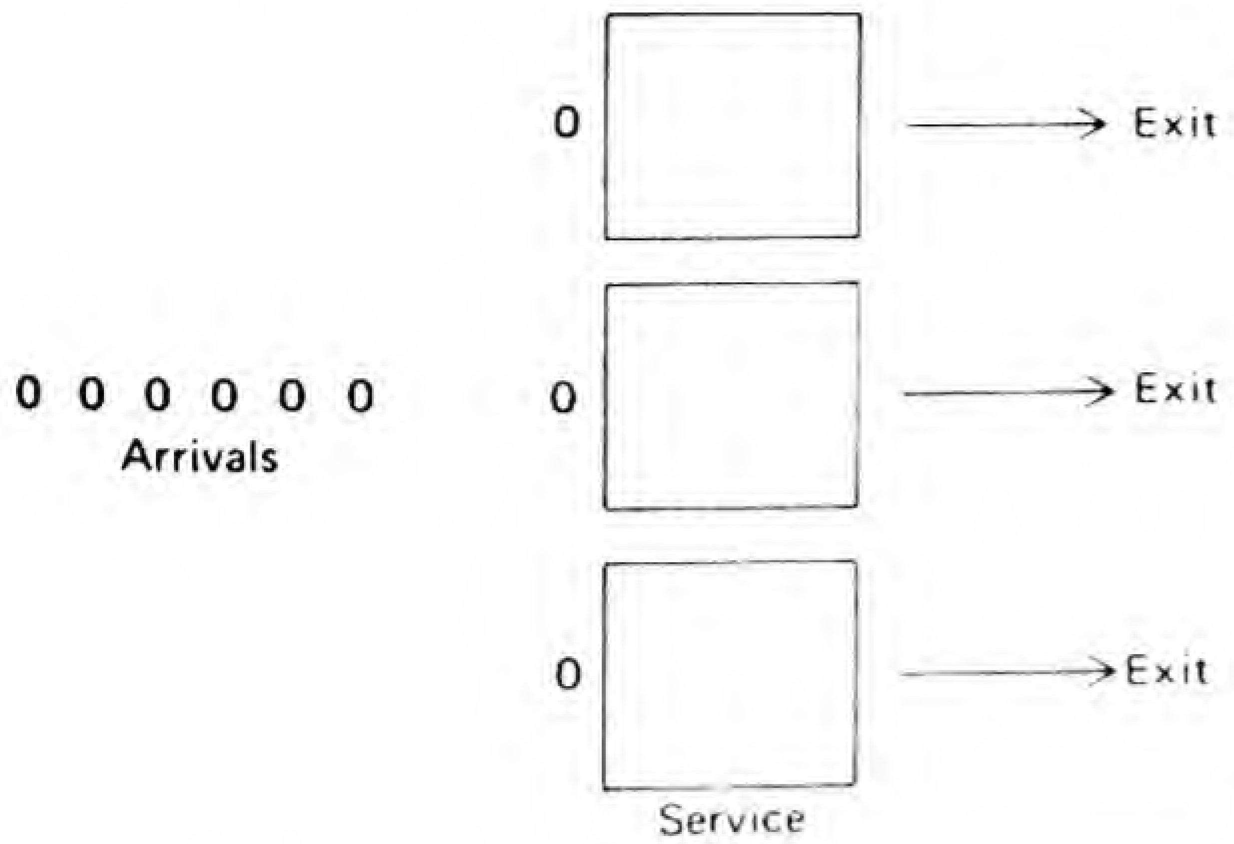


Figure 13-2 One line, multiple service.

service. In this situation all arrivals form one line, and when a service station is available, the person at the head of the line proceeds to that station. Many banks use this method. All arrivals stand in one single line rather than queuing up in front of each window. A service assistant at the head of the line directs customers to available windows. In the conventional system most people choose the line with the fewest *number* of people. To their dismay, however, there often are customers in front of them who take much longer than average in transacting their business. Although their line had the fewest number of people in the beginning, waiting time was the *longest*. A system with one line and multiple service would eliminate these unlucky choices and improve customer service. It is this method that is used at the Mission Hill Clinic.

The conventional bank system utilizes *multiple lines* and *multiple services*, as depicted in Figure 13-3. Other examples include autos lining up during rush hour to pay their toll on the New York Thruway, and students lining up by alphabet groups (A–M, N–Z) to register for courses.

Priority of Arrivals

The most common queue discipline is first come, first served. This means that the first in line is the first served, the second in line is the second served, and so on. Perhaps the reason why this type of discipline is so popular is that it is easy to administer.

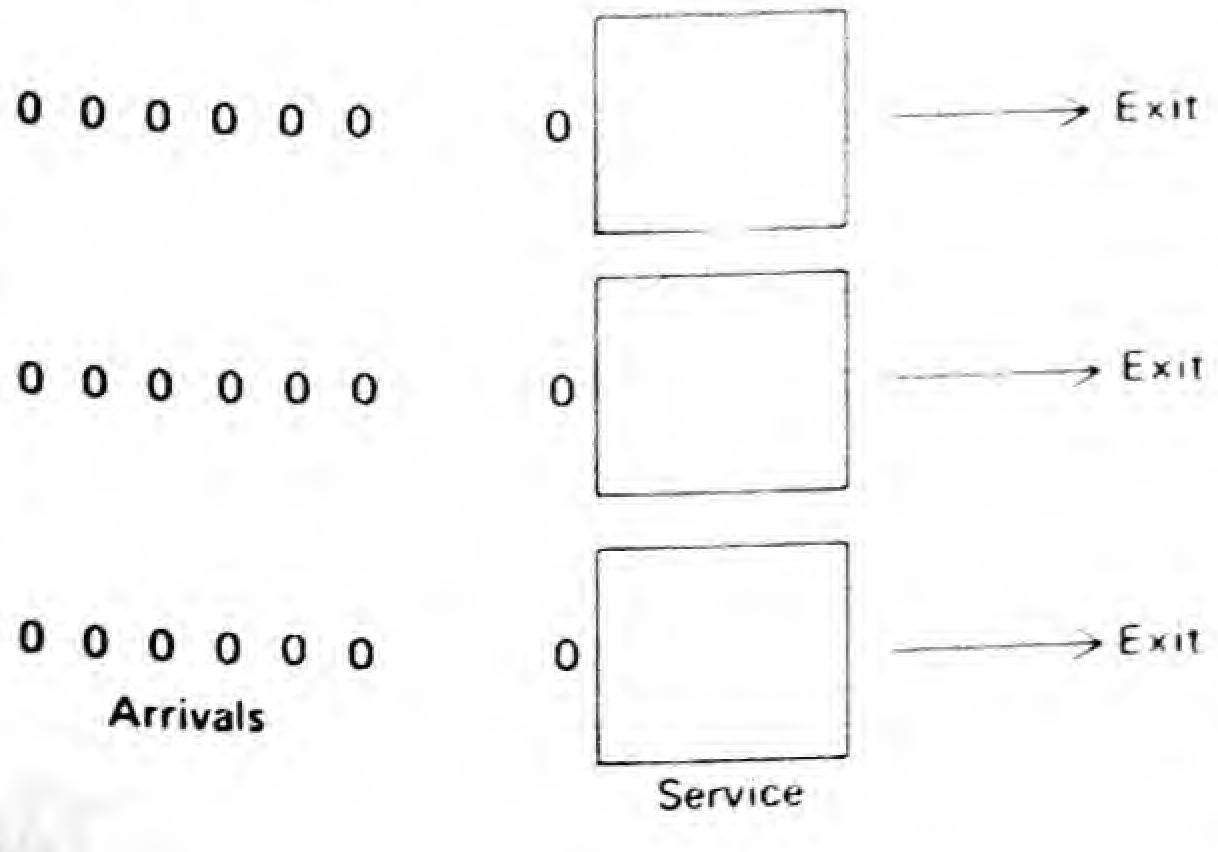


Figure 13-3 Multiple lines, multiple service.

Countless other queue disciplines have been used. In a factory, jobs may be scheduled by their due dates. Those jobs with the earliest due dates are scheduled first. In a hospital lab, patients may be taken on a first come, first served basis, but the queue may be interrupted for high-priority patients.

The models developed in this chapter all assume that the queue discipline is first come, first served. A more advanced treatment of the subject would consider the consequence of other queue disciplines.

ECONOMICS OF WAITING LINES

It would be totally unthinkable for the Mission Hill Clinic to be provided with enough physicians so that no patient would ever have to wait in line. Why? Such a strategy would be too costly. Although no patient would have to wait, many physicians would experience long idle periods. Salaries for this strategy would be prohibitive. Consequently it would appear that the existence of waiting lines must be accepted. But waiting lines are also costly! When waiting lines get too long, patients become impatient and disenchanted with the service being offered and may leave. To the patient, or customer, the cost of waiting may be too high. Clearly, long lines are a disadvantage.

The number of physicians finally chosen is to some extent an economic decision. To make this decision two costs must be balanced. The first is the cost of waiting, and the second is the cost of providing additional service. With too many physicians the cost of providing this service will be high while the cost of waiting might be low. With too few physicians the cost of providing this service will be low, while the cost of waiting might be high. In general any waiting-line problem faces these opposing costs. As the number of service stations increases and the cost of service increases, the waiting times decrease as does the cost of waiting. There is some intermediate number of service stations for which the sum of these two costs is at a minimum.

The waiting-line models that will be presented in this chapter will not be capable of identifying the precise number of service stations that will minimize the sum of these costs. They can, however, be used to determine the waiting times associated with several different strategies, each one for a different number of service stations. These waiting times can then be used to compute the economic consequence of these strategies. Finally the strategies can be compared and the choice made. In the next section a model will be developed which is capable of determining the waiting time for the very simplest of circumstances: one line, one service station.

ANALYSIS OF ONE LINE, ONE SERVICE STATION, AND INFINITE SOURCE

Assumptions

The simplest waiting line occurs when there is one line and one service station. We can model this simple process mathematically if we make some

assumptions. First, we must assume that the source of the arrivals is infinite. Second, we must assume that the queue discipline is such that the first one to arrive is the first one served, and so on. Third, we must assume that the arrivals and services are independent of one another. When arrivals are independent, the occurrence of one does not influence when the next one will occur. Similarly when service times are independent, the length of time it takes to service one customer has no influence on the length of time to service the next one.

Finally we must make some assumption about the pattern of arrivals and services. It is not enough to know the average time between arrivals and the average service time. Something must also be said about the likelihood that arrivals and services will fall on either side of these averages. It can be shown that the Poisson probability distribution best describes the randomness associated with these arrivals and services. The models which follow therefore assume that arrivals and services vary according to this distribution.

The Model

By utilizing the assumptions presented in the previous section, the following characteristics of the waiting-line system can be developed:[1] the average number in the system,[2] the average number in the waiting line, the average waiting time, and the probability of n units in the system.

The Average Number in the System The average number in the system (L), including the one being serviced, can be computed in the following way:

$$L = \frac{\lambda}{\mu - \lambda}$$

where λ = the average number of arrivals in an interval of time

μ = the average number of services which the server is capable of accommodating in an interval of time

For example, if the average number of arrivals at a supermarket check-out stand is 60 per hour and the average number of services in 1 hour is 100, the average number of people in the system, including the one being serviced, can be computed in the following way:

$$L = \frac{\lambda}{\mu - \lambda}$$

and, since $\lambda = 60$ and $\mu = 100$,

$$L = \frac{60}{100 - 60} = \frac{60}{40} = 1.5$$

[1] See Appendix A for derivation of these formulas.
[2] Long-run average.

We can therefore conclude that the average number of people in the system, including the one being serviced, is 1.5.

The Average Number in the Waiting Line The average number in the waiting line (L_q), *not* including the one being serviced, can be computed in the following way:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

In our example we have the following:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(60)^2}{100(100 - 60)} = .9$$

The average number in the waiting line is therefore .9.

The Average Waiting Time The average waiting time of an arrival excluding service time, can be computed in the following way:

$$W_q = \frac{L_q}{\lambda}$$

Returning to our example, we have the following:

$$W_q = \frac{L_q}{\lambda} = \frac{.9}{60} = .015$$

We can therefore conclude that the average waiting time for an individual is .015 hours, or $60 \times .015 = .9$ minute.

Probability of n Units in the System The probability of n units in the system at any one time can be computed from the following formula:

$$P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

In our example, the probability of $n = 2$ people at any one time can be computed in the following way.

$$\begin{aligned} P_n &= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\ &= \left(1 - \frac{60}{100}\right)\left(\frac{60}{100}\right)^2 = .144 \end{aligned}$$

The likelihood that two people will be in the system at any one time is .144, or 14.4 percent.

MULTIPLE-SERVICE FACILITIES

In the last section we considered systems with one line and one service station. In this section we will consider systems with one line and multiple-service stations (Figure 13-2).

An example of a one-line, multiple-service system can be found at the Greyhound Bus Terminal in Boston, Massachusetts. Guardrails are used to converge arrivals into one line. At the end of this line is a sign, "Wait here and then proceed to first vacant window." Beyond the sign are six ticket windows. The customer proceeds to one of these windows when it is free.

To most who have used this system it is a substantial improvement over the multiple-line, multiple-service system. It is better because it eliminates the possibility of customers waiting for a long time in a slow line.

Since people in multiple-line, multiple-service systems have a tendency to jump lines, the one-line, multiple-service models about to be developed have been used to approximate the multiple-line, multiple-service situation.

The Model

In the following sections several characteristics of the one-line, multiple-service system are developed.

The Probability of Zero Units in the System The likelihood of zero units or zero individuals in the system, P_0, can be calculated in the following way:

$$P_0 = \left\{ \frac{(\lambda/\mu)^S}{S![1 - \lambda/(S\mu)]} + 1 + \frac{(\lambda/\mu)^1}{1!} + \frac{(\lambda/\mu)^2}{2!} + \cdots + \frac{(\lambda/\mu)^{S-1}}{(S-1)!} \right\}^{-1}$$

where λ = the average number of arrivals in an interval of time
μ = the average number of services which a server is capable of accommodating in an interval of time
S = the number of service channels

For example, if we have a one-line arrival system with three servers, and if the average number of arrivals is 100 per hour and the average number of services per server is 40 per hour, the likelihood of zero people in the system can be calculated from the above equation:

$\lambda = 100$
$\mu = 40$
$S = 3$

$$P_0 = \left[\frac{(^{100}/_{40})^3}{3!(1 - {}^{100}/_{120})} + 1 + \frac{^{100}/_{40}}{1} + \frac{(^{100}/_{40})^2}{2}\right]^{-1}$$

$$P_0 = (15.625 + 1 + 2.5 + 3.125)^{-1}$$

$$P_0 = .0449$$

The likelihood of no people being in the system is therefore .0449, or 4.49 percent.

The Probability of n Units in the System The probability of n units in the system can be computed in the following way:

$$P_n = P_0\frac{(\lambda/\mu)^n}{n!} \qquad \text{if } n \le S$$

$$P_n = P_0\frac{(\lambda/\mu)^n}{S!S^{n-S}} \qquad \text{if } n > S$$

Returning to our example, we will compute the probability of $n = 1$, 2, and 3 people in the system. Since all these values of n are less than or equal to $S (S = 3)$, we can use the first formula.

For $n = 1$: $P_1 = P_0\frac{(\lambda/\mu)^n}{n!} = \frac{.0449(2.5)^1}{1} = .11225$

For $n = 2$: $P_2 = P_0\frac{(\lambda/\mu)^n}{n!} = \frac{.0449(2.5)^2}{2} = .1403$

For $n = 3$: $P_3 = P_0\frac{(\lambda/\mu)^n}{n!} = \frac{.0449(2.5)^3}{3(2)} = .1169$

Next we will compute the likelihood of $n = 4$, 5, and 6 people in the system. Since these values of n are greater than S $(S = 3)$, the second formula must be used.

For $n = 4$: $P_4 = P_0\frac{(\lambda/\mu)^n}{S!S^{n-S}} = \frac{.0449(2.5)^4}{3!(3)^1} = .0974$

For $n = 5$: $P_5 = P_0\frac{(\lambda/\mu)^n}{S!S^{n-S}} = \frac{.0449(2.5)^5}{3!(3)^2} = .08119$

For $n = 6$: $P_6 = P_0\frac{(\lambda/\mu)^n}{S!S^{n-S}} = \frac{.0449(2.5)^6}{3!(3)^3} = .06766$

The probabilities of seven or more people in the system can be calculated in the same way.

The Average Number in the Waiting Line The average number in the waiting line, L_q, can be calculated in the following way:

$$L_q = P_0 \frac{(\lambda/\mu)^{S+1}}{S(S!)[1-(\lambda/\mu)/S]^2}$$

Returning to our example, we have

$$L_q = \frac{.0449(2.5)^4}{3(3!)(1 - {}^{2.5}/_3)^2}$$

$$L_q = 3.507$$

Therefore the average number of people in the waiting line is 3.507.

The Average Waiting Time The average waiting time in line for an arrival can be computed in the following way:

$$W_q = \frac{L_q}{\lambda}$$

In our example we have

$$W_q = {}^{3.507}/_{100} = .035$$

and can therefore conclude that the average person waits .035 hours, or $.035 \times 60 = 2.10$ minutes.

COMPLEX WAITING-LINE CONFIGURATIONS

The two models presented in this chapter are simple but often useful. By no means do they represent all waiting-line configurations. For example, we have not discussed multiple-line, multiple-service systems, nor have we discussed the consequence of waiting-line disciplines other than first come, first served. These other disciplines might include priority customers who could enter the front of the line and thereby avoid long waiting periods. Clearly this change in discipline would change the average waiting time in the system.

Also not discussed was the possibility of arrivals being processed through a sequence of several service stations. In some health clinics for example, patients form a waiting line for the first test and, after this is completed, proceed to the next station to a waiting line for the next test. This sequence continues until all tests are completed.

Mathematicians have developed waiting-line models similar to the ones presented in this chapter for some of these other situations. But when the situations become so complex that it is very difficult, if not altogether impossible, to develop mathematical formulas for them, simulation models are used. Chapter 15, which is on simulation, includes several examples of waiting

lines. In fact it is reasonable to say that in most practical applications, simulation models are used to analyze waiting-line problems.

SUMMARY

Two queuing models were developed in this chapter. The first assumed a single-line arrival pattern and single-service facility; the second assumed a single-line arrival pattern but a multiple-service facility. Countless other models have also been developed but are beyond the scope of this book.

QUESTIONS

1 What are the opposing costs in a queuing system and how do they behave?

2 One way to reduce waiting time is by improving the efficiency of service personnel. Is this a possible alternative in the Mission Hill Clinic case?

3 On closer inspection it is found that the long waiting lines at the Mission Hill Clinic occur between 10 and 11 A.M. and 2 and 2:30 P.M. Do you think that there are other solutions to the problem in addition to increasing the size of the staff?

4 Outline the method that you would use in supporting Dr. Swanson's request with data.

5 Do you think that a reasonable objective in a health clinic is the minimization of physician idle time? What are the consequences of this objective? Should a supermarket schedule its check-out personnel with the objective of minimizing personnel idle time? What is the difference between these two examples?

PROBLEMS

13-1 The average number of arrivals at a downtown airline ticket office is 20 per hour. The average number of possible services in 1 hour is 25. Only one agent is on duty at any one time.

  a Compute the average number of customers in the system including the one being serviced.

  b Compute the average waiting time for a customer excluding service time (in minutes).

13-2 Return to problem 13-1 and compute the probability of no customers in the waiting line.

13-3 A repair facility for electronic calculators can repair on the average six calculators per hour. Only one technician is employed. Calculators arrive at the rate of five per hour.

If the technician is sent to repair school, the time it would take to repair the calculator can be reduced. By how much must the repair time be reduced to decrease the average number of calculators in the system, including the one being served, by 2 units.

13-4 A tollbooth at an exit ramp on the Maine Turnpike can serve but one car at a time in each direction. The average arrival rate for cars during the rush period is 100 per hour, and the average number of services which can be accommodated by the toll collector is 120. The arrival pattern and service pattern can be represented by a Poisson probability distribution.

a Is it appropriate to represent this system by a model with one line, one service station, and infinite source?

b Find the average number in the system, the average number in the waiting line, the average waiting time, and the probability of eight cars in the system.

c Is this a reasonable system, or should an additional tollbooth be constructed? What factors must be considered?

d Assume that the arrival rate during late morning and early afternoon hours drops to 80 per hour. Find the average number in the system, the average number in the waiting line, the average waiting time, and the probability that five automobiles will be in the system.

13-5 A major manufacturer of electronic medical instruments maintains repair facilities in several locations throughout the United States and abroad. One such facility, in Florence, South Carolina, includes five repair departments, each specializing according to instrument type. The blood-analyzer repair department has only one repairman. Lately he has been swamped with work, and management is considering the addition of another person.

a The arrival and service patterns of the blood-analyzer units can be approximated by the Poisson probability distribution. On the average, a unit arrives every day during the 5-day workweek, and it is possible to service six per week. Find the average number in the system, the average number in the waiting line, and the probability of four in the system at any time.

b Assume that the electronics manufacturer has improved his quality control and that the consequence of this is an arrival every other day at the repair facility. Find the average number in the system, the average number in the waiting line, and the probability of three instruments in the system.

13-6 The First National Bank of Denver has just changed the physical layout of its lobby. A series of rails has been installed which force depositors to form a single line. A bank guard stands at the head of this line and directs depositors to the first available teller.

a During peak hours the bank opens four teller windows. The average rate of arrivals during this period is 100 per hour. Each teller requires an average of 2 minutes to service each customer. Arrival and service patterns can be approximated by the Poisson probability distribution. Find the average number in the waiting line and the average waiting time.

b What would happen to the average waiting time if an additional teller were added during the peak period?

c What would happen to average waiting time if the arrival rate for customers increased to 150 per hour and the bank maintained four teller windows?

13-7 One of the earliest applications of complex mathematical theory to management problems was the use of waiting-line theory by AT&T to determine the number of operators required to service subscribers' demands adequately.

Let us assume that calls come into an exchange in single-channel fashion. The first calls in are the first ones serviced by an operator. If all operators are busy, the person placing the call lets the phone ring until it is answered by an operator. Substantial overloads on the exchange, however, result in busy signals, and the caller must hang up and try again.

a It has been determined that arrival and service patterns can be represented by the Poisson distribution. If the average arrival rate during one part of the day is 100 calls per hour, and if the average service rate is 30 calls per hour, what will be the average waiting time if four operators are used?

b Is this mathematical model an accurate representation of the telephone exchange process?

13-8 Show that the average number in the waiting line for the case with one line and one service station

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

is equal to the average number in the waiting line for the case with one line and multiple service, when $S = 1$.

$$L_q = P_0 \frac{(\lambda/\mu)^{S+1}}{S(S!)[1-(\lambda/\mu)/S]^2}$$

CASE STUDY: Gold Star Company

The Gold Star Company of Madison, Wisconsin, produces potato chips, corn chips, cheese twists, and popcorn. These products are made in Madison and then shipped to company warehouses and distribution centers located throughout the Midwest.

In the past the company has leased trucks or used common carriers to ship its product, but in 1970 a decision was made to purchase a fleet of trucks and to avoid the use of other more costly means of transportation.

The trucks are kept in a company garage on Langdon Street, and it is there that dispatcher Harold Bascom assigns drivers, loads, and destinations. It is not unusual for a truck to be sent to several destinations, dropping off a part of its load at each stop. In these situations Mr. Bascom determines which combination of destinations will be assigned, totals the orders placed by each distribution center, and loads the trucks with these quantities. Seldom does this quantity completely fill a truck. On average, trucks are 80 percent filled. Occasionally, they have been only half filled.

The product is shipped in large cardboard drum containers. At each stop along the route, the driver must first unload the requested number of drums and then load as many empty drums as possible onto the truck. When the truck returns to Langdon Street, the empty drums are unloaded and stored for reuse.

Recently the shipping department has been unable to keep up with production and warehouse demand. Trucks are in operation for three shifts daily during the 250-day work year, but more trucks are still needed. Consequently Mr. Porter, manager of inventory and transportation, has requested that the company purchase three new trucks. Since business has been unusually strong, Mr. Porter expects quick approval for the vehicles.

Upon receiving this request, Mr. Bond, the company president, requested that the accounting department prepare an analysis of the problem and submit a final recommendation within 2 weeks.

To gain some insight into the problem, accounting representatives spent several hours observing the shipping department at its Langdon Street location. It became apparent that trucks often had long waits before they were unloaded and loaded. One of the accounting staff asked Mr. Bascom if this was normal. Mr. Bascom replied that on some occasions all the trucks were out and his shippers were idle. On other occasions, he continued, there were several trucks waiting to be unloaded and loaded. "If we only had trucks during our idle period," he said, "we could increase our shipments."

Upon closer analysis it was determined that the average time between truck arrivals was 4 hours and the average time to unload and load a truck was 3 hours. The company maintained one shipping platform, and it was in operation 24 hours a day.

But adding more trucks was not the only solution. The accounting department suspected that the construction of a new shipping platform might be

far more effective and economical. In fact, *if* the trucks were purchased and no new platform were built, the trucks could only add to the bottleneck at the unloading and loading platform. And waiting time was indeed expensive. They estimated the opportunity cost of waiting at a minimum of $10 per hour. This cost included the lost shipments and subsequent lost sales of Gold Star Products due to stockouts on warehouse shelves.

The accounting staff learned from the building and grounds department that it would be possible to construct another shipping platform identical to the present one. The cost would be $25,000 and would be depreciated on a straight-line basis over 10 years. If the new platform were open three shifts each day, the labor costs would total $20,000 per year, including fringe benefits.

QUESTIONS

1 What is the average number of trucks waiting at any time? What assumptions were necessary to make this computation?
2 What is the yearly cost of this waiting time?
3 If a second loading-unloading platform is built, what will be the average number of trucks waiting at any time?
4 What is the yearly cost of waiting for two loading-unloading platforms?
5 Compare the costs of the two alternatives. Which one would you recommend?
6 What other factors could affect the arrival and service time of the trucks and thereby speed the process?

APPENDIX A: Derivations

POISSON DISTRIBUTION

The probability of the number of arrivals (or services) that occur during an interval of time can be expressed as a Poisson distribution in the following way:

$$P_n(\Delta t) = \frac{(\lambda\ \Delta t)^n e^{-\lambda \Delta t}}{n!}$$

where n = number of arrivals
Δt = small interval of time
λ = average number or arrivals in an interval of time

From this we can then determine the likelihood of one arrival ($n = 1$) in an interval of time Δt.

$$P_1(\Delta t) = (\lambda\ \Delta t)(e^{-\lambda \Delta t})$$

And if we let the interval of time become very small, $e^{-\lambda \Delta t}$ approaches 1 since $e^{-0} = 1$.

$$P_1(\Delta t) = \lambda\ \Delta t$$

This then expresses the probability of one arrival in a small interval of time Δt. We will make use of this relationship in the next section.

DERIVATION OF QUEUING FORMULA

First we must assume that the probability of more than one arrival during the small interval of time is so small that it can be ignored. Therefore the probability of no units entering the system during this small time period is

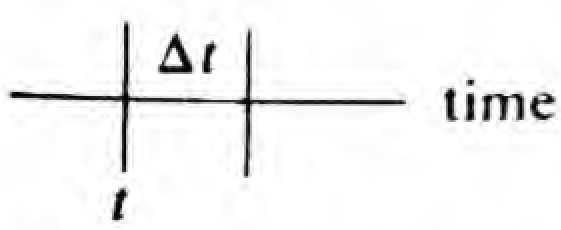
$$1 - \lambda\ \Delta t$$

while

$$1 - \mu\ \Delta t$$

is the probability of no service taking place.

Let us now take a look at our system at some point in time t including the next instant in time Δt.



Exactly n units or individuals can be found in the system during this interval in any one of the following ways:

$a = n$ units at t with no arrivals and no service during Δt
$b = n + 1$ units at t with no arrivals and 1 unit serviced during Δt
$c = n - 1$ units at t with 1 arrival and no service during Δt
$d = n$ units at t with 1 arrival and 1 service during Δt

We can express the probabilities of these events in the following way:

$$P(a) = P_n(t)(1 - \lambda\, \Delta t)(1 - \mu \Delta t)$$
$$P(b) = P_{n+1}(t)(1 - \lambda \Delta t)(\mu \Delta t)$$
$$P(c) = P_{n-1}(t)(\lambda \Delta t)(1 - \mu\, \Delta t)$$
$$P(d) = P_n(t)(\lambda\, \Delta t)(\mu\, \Delta t)$$

Since these events are indipendent, we can express the probability of n units in the system at time $t + \Delta t$ in the following way:

$$P_n(t + \Delta t) = P(a) + P(b) + P(c) + P(d)$$

$$\begin{aligned} P_n(t + \Delta t) = P_n(t)(1 - \lambda\, \Delta t)(1 - \mu\, \Delta t) \\ + P_{n+1}(t)(1 - \lambda\, \Delta t)(\mu\, \Delta t) \\ + P_{n-1}(t)(\lambda\, \Delta t)(1 - \mu\, \Delta t) \\ + P_n(t)(\lambda\, \Delta t)(\mu\, \Delta t) \end{aligned}$$

Multiplying, we have

$$\begin{aligned} P_n(t + \Delta t) = P_n(t)[1 - \lambda \Delta t - \mu \Delta t + 2\mu\lambda(\Delta t)^2] \\ + P_{n+1}(t)[\mu \Delta t - \mu\lambda(\Delta t)^2] \\ + P_{n-1}(t)[\lambda \Delta t - \mu\lambda(\Delta t)^2] \end{aligned}$$

Rearranging,

$$\begin{aligned} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \mu P_{n+1}(t) - (\lambda + \mu)P_n(t) \\ + \lambda P_{n-1}(t) \\ + \mu\lambda\, \Delta t[-P_{n+1}(t) + 2P_n(t) - P_{n-1}(t)] \end{aligned}$$

As the interval Δt becomes smaller and smaller, the *last* term approaches zero. In addition, as Δt becomes smaller, $P_n(t)$ will become equal to $P_n(t + \Delta t)$, and therefore $P_n(t + \Delta t) - P_n(t) = 0$. We can now write the above equation in the following way:

$$\mu P_{n+1}(t) - (\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) = 0 \tag{13A-1}$$

From this equation we can now determine the probability of $n = 0, 1, 2, 3$, etc., units in the system. First we will determine the probability of $n = 0$ units in the system.

$$\mu P_1(t) - (\lambda + \mu)P_0(t) + \lambda P_{-1}(t) = 0$$

We can immediately drop the last term, $\lambda P_{-1}(t)$, since negative arrivals are not

possible. We then have

$$\mu P_1(t) - (\lambda + \mu)P_0(t) = 0$$

or

$$\mu P_1(t) - \lambda P_0(t) - \mu P_0(t) = 0$$

Since a unit cannot be serviced if there are zero units in the system, then

$$\mu P_0(t) = 0$$

and

$$\mu P_1(t) - \lambda P_0(t) = 0$$

or

$$P_1(t) = \frac{\lambda}{\mu} P_0(t) \tag{13A-2}$$

Next we will determine the probability of $n = 1$ units in the system. Returning to Equation 13A-1, we have

$$\mu P_2(t) - (\lambda + \mu)P_1(t) + \lambda P_0(t) = 0$$
$$\mu P_2(t) = (\lambda + \mu)P_1(t) - \lambda P_0(t)$$

Using Equation 13A-2,

$$\mu P_2(t) = (\lambda + \mu)\frac{\lambda}{\mu} P_0(t) - \lambda P_0(t)$$
$$P_2(t) = \left[(\lambda + \mu)\left(\frac{\lambda}{\mu}\right)\left(\frac{1}{\mu}\right) - \frac{\lambda}{\mu}\right] P_0(t)$$
$$= \left(\frac{\lambda^2 + \lambda\mu - \lambda\mu}{\mu^2}\right) P_0(t)$$

and

$$P_2(t) = \left(\frac{\lambda}{\mu}\right)^2 P_0(t) \tag{13A-3}$$

If we went through the same process for $n = 3$, we would find

$$P_3(t) = \left(\frac{\lambda}{\mu}\right)^3 P_0(t)$$

and clearly the general form is

$$P_n(t) = \left(\frac{\lambda}{\mu}\right)^n P_0(t) \tag{13A-4}$$

What remains is to find $P_0(t)$. Since all the terms in a probability distribution must equal 1, we can write

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

and from Equation 13A-4,

$$P_0(t) + P_0(t)\left(\frac{\lambda}{\mu}\right) + P_0(t)\left(\frac{\lambda}{\mu}\right)^2 + \cdots + P_0(t)\left(\frac{\lambda}{\mu}\right)^n + \cdots = 1$$

or

$$P_0(t)\left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\lambda}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^n + \cdots\right] = 1$$

The terms in the brackets are in the form of the following geometric progression:

$$1 + a + a^2 + a^3 + a^4 + \cdots + a^n + \cdots$$

Their sum is

$$\frac{1}{1 - a}$$

If we let $a = \lambda/\mu$, the terms within the brackets can be written as

$$\frac{1}{1 - \lambda/\mu}$$

We then have

$$P_0(t)\frac{1}{1 - \lambda/\mu} = 1$$

Rearranging,

$$P_0(t) = 1 - \frac{\lambda}{\mu}$$

And from Equation 13A-4 we can now write

$$P_n(t) = \left(\frac{\lambda}{\mu}\right)^n\left(1 - \frac{\lambda}{\mu}\right)$$

This, then, is the general expression for determining the probability of n units in the system at any time t.

Chapter 14

Simulation

INTRODUCTION

In the preceding chapters the emphasis was on the formulation and analysis of mathematical models. In those models the relevant elements from the real-world system and their interrelationships were expressed in mathematical terms. Then these models were analyzed and solved.

There are many situations, however, where the system that must be modeled is so complex that it is not possible to express the elements and their interrelationships as a mathematical model and then to solve this model. In those situations the only practical alternative is to simulate the system.

When a simulation model is developed, the focus is first directed at the individual elements of the system. Often the behavior of these elements is described by a probability distribution.

After the elements have been specified, a trial alternative to the problem is tested by combining these elements in their natural order. This process generally continues as several alternatives are tested.

After the trials have been completed, the results are compared. The trial alternative with the best outcome is generally chosen as the solution to the problem.

Since simulation methods are often limited to the comparison of a

relatively small number out of all possible alternatives, it is not necessarily true that the best possible alternative will be uncovered. In spite of this and the fact that simulation models are often expensive to develop, they are becoming a common decision tool at all levels in the organization.

Simulation methods, however, have not been limited to management decision problems. They have been used in engineering for several decades. In the design of aircraft, for example, it is routine procedure to simulate the consequence of design alternatives by constructing a small scale replica of an aircraft and observing its behavior in a wind tunnel. Although the content of the simulation models developed by engineers and managers is different, the general concepts behind these models are similar.

To introduce the methods of simulation, we turn to the first of three case studies.

CASE STUDY: St. Louis Safe and Trust Company

The St. Louis Safe and Trust Company is a commercial bank which offers a broad range of banking services to its customers. At the present time it is considering the addition of a drive-in teller window at one of its branch banks. The bank's concern, however, is that this single drive-in window will be inadequate to meet demand. The consequence of this would be long waiting lines which could back traffic into a busy street. Unfortunately there is inadequate space to construct two windows.

Before approval is given to proceed with construction, the vice president of the bank, Sandra Murray, has asked that a study be undertaken to determine if this problem will in fact exist.

Simulation of the Drive-in Teller System

The system described in the St. Louis Safe and Trust Company case can be classified as a queuing system much like the ones discussed in the last chapter. This system is illustrated in Figure 14-1. Automobiles arrive at the window, are serviced by the teller, and exit from the system. If several arrivals occur within a short span of time, a line, or queue, will form. If, on the other hand, few arrivals occur, little or no line will form. The problem is to determine if a single teller will be adequate to service the bank's customers. Although the queuing methods developed in the last chapter might be used in this simple example, they become inadequate in more complex queuing

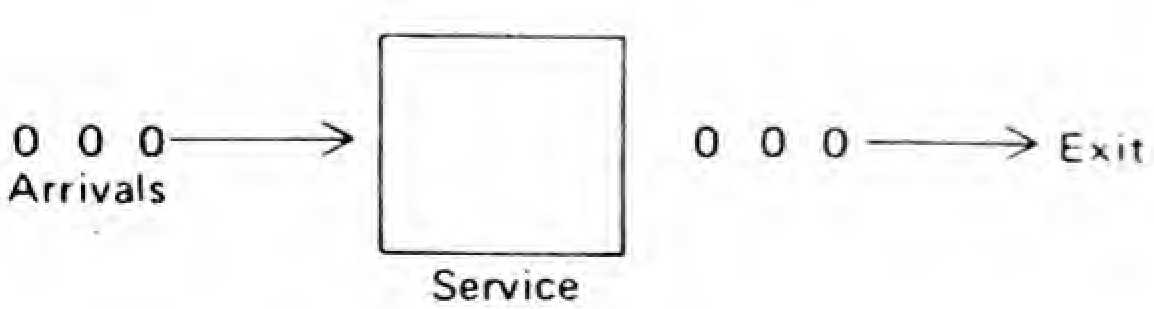


Figure 14-1 Drive-in teller represented as a queuing system.

situations. The simulation model which will now be developed can be used to analyze both simple and complex queuing systems.

But simulation is not limited to the analysis of queuing systems. Later in the chapter two other cases will be analyzed using this technique.

The first step in formulating a simulation model for the St. Louis Safe and Trust Company case is to estimate arrival and service patterns.

Arrival Pattern

Since it is not possible to obtain an arrival pattern for a facility which does not yet exist, one must be approximated. It has been determined that the arrival pattern at a nearby branch bank which maintains two drive-in windows will be closely approximated by that of the new system. The time between arrivals, then, was observed at that facility during a typical hour of the day for 100 automobiles. The results are found in Table 14-1. It can be seen that for 10 of the 100 observations a time of 1 minute between arrivals was observed; for 20 of the 100 observations a time of 2 minutes between arrivals was observed; for 50 of the 100 observations a time of 3 minutes between arrivals was observed; and so on. In the last column of this table the number of observations are converted to probabilities.

If we can assume that this sample of time between arrivals is representative of the system from which this sample was drawn, we can conclude that the probability of a time of 1 minute between arrivals is .10; of a time of 2 minutes between arrivals, .20; of a time of 3 minutes between arrivals, .50; and so on. Shortly this distribution will be used to simulate this system, but first the service-time distribution must be developed.

Service Pattern

The service pattern for this proposed drive-in teller was also approximated at the same nearby branch by observing one of its tellers. The length of time to service 100 automobiles was recorded; this is presented in Table 14-2. For 20 of

Table 14-1 Time between Arrivals at Nearby Branch Bank Drive-in Window

Time between arrivals, minutes	Number of observations	Probability of this time between arrivals
1	10	.10
2	20	.20
3	50	.50
4	10	.10
5	5	.05
6	5	.05
	100	1.00

Table 14-2 Service Times at Nearby Branch Bank Drive-in Window

Service time, minutes	Number of observations	Probability of this service time
1	20	.20
2	30	.30
3	30	.30
4	10	.10
5	6	.06
6	4	.04
	100	1.00

the 100 observations it took 1 minute to service the customer, for 30 of the 100 observations it took 2 minutes, and so on. In the last column of the table the number of observations are converted to probabilities.

If we can assume that this sample of service times is also representative of the system from which it came, we can conclude that the probability of a 1-minute service time is .20, of a 2-minute service time is .30, and so on.

An Intuitive Development of the Simulation Process

Now that we have the probability distributions for the time between arrivals and service time, the system can be simulated.

In this section an intuitive method for simulating this process is developed. A more formal method using random numbers will be presented in the next section.

Suppose that we have two containers, one of which is labeled "time between arrivals" and the other "service times." In each container will be placed 100 chips. First the chips in the "time between arrivals" container will be labeled. According to Table 14-1, 10 percent, or 10 chips will be labeled "1 minute"; 20 percent, or 20 chips, will be labeled "2 minutes"; 50 percent, or 50 chips, will be labeled "3 minutes." The labeling process will continue until all chips are labeled to correspond with the arrival times and probabilities found in Table 14-1.

The chips in the second container are marked to correspond with the service times and probabilities found in Table 14-2. Twenty percent, or 20 chips, are marked "1 minute"; 30 percent, or 30 chips, are marked "2 minutes"; 30 percent, or 30 chips, are marked "3 minutes"; and so on.

To simulate the system, a chip will be selected at random from the first container: this will generate a time between arrivals. Then a chip will be selected at random from the second container: this will generate a service time. As an example of this process suppose that the chip drawn from the first container is labeled "2 minutes." Furthermore we will assume that the drive-in teller window will open at 9 A.M. This means that the first arrival occurs at 9:02. Now suppose that the chip drawn from the second container is

labeled "3 minutes." This means that it takes 3 minutes to service the customer who arrived at 9:02. Therefore service is complete at 9:05, and the customer exits from the system. At 9:05 the drive-in teller is available for the next customer. From the first container a new time between arrivals is drawn. Suppose it is "2 minutes." This means that the second arrival occurs 2 minutes after the first one, or at 9:04. Since the drive-in teller is busy until 9:05, the second customer must wait 1 minute before being served. Next a service time is drawn from the second container. Suppose it is "3 minutes." Service is therefore complete at 9:08.

This process could continue until an 8-hour day is simulated. In fact it could be continued until several 8-hour days are simulated. Then by evaluating the waiting time and the length of the waiting line, it could be determined whether they would be excessive and whether a single drive-in window would be adequate.

Although the concepts behind the process just presented represent the essence of simulation, the method used was cumbersome. Instead of containers and chips more formal methods use random numbers and computers.

Simulation with Random Numbers

Random numbers are used to select times between arrivals and service times. To do this, it is necessary to make some additions to Tables 14-1 and 14-2.

A series of 100 two-digit numbers ranging from 00 to 99 will be added to both tables. In Table 14-1 this series will be assigned to each time between arrivals in a way that will correspond to the probability of occurrence for that time between arrivals. For example, the numbers 00 to 09 will be assigned to the category of 1 minute between arrivals. This is done in Table 14-3. By assigning 10 of the 100 numbers (10 percent of the numbers) to this category, we have maintained the correspondence between the probability of that particular time between arrival and the percentage of two-digit numbers assigned. Now we turn to the next category, 2 minutes between arrivals. Since

Table 14-3 The Assignment of Two-Digit Numbers to the Time-between-Arrivals Distribution

Time between arrivals, minutes	Number of observations	Probability of this time between arrivals	Two-digit number
1	10	.10	00–09
2	20	.20	10–29
3	50	.50	30–79
4	10	.10	80–89
5	5	.05	90–94
6	5	.05	95–99
	100	1.00	

the probability of this event is 20 percent, we will assign the next 20 two-digit numbers here. This series will therefore include the numbers 10–29. The next category is 3 minutes, and 50 percent of the numbers, or 30–79, will be assigned here. The remaining two-digit numbers are assigned in a similar fashion, and the results are shown in Table 14-3.

The assignment of two-digit numbers to Table 14-4 proceeds in the same way. In the first category 20 percent of the numbers, or 00–19, are assigned. Thirty percent of the numbers, or 20–49, are assigned to the second category. Thirty percent, or 50–79, are assigned to the third category, and so on until all the assignments have been completed.

Once the two-digit numbers have been assigned, the simulation process can begin. The first step involves the generation of arrival and service times using random numbers which can be found in Table 2 at the back of the book. To generate a service time, for example, a random number is read from the table. Then the interval of two-digit numbers into which the random number falls is identified. That category becomes the simulated time between arrivals. Before we apply this method to our problem, the concepts behind, and the use of, the random number table will be covered more fully.

The random number table at the back of the book was compiled by the random selection of numbers between 00000 and 99999. This was done in such a way that each number in this interval had an equally likely chance of being selected. The table can be read in any way: one way would be to read down each column starting with the first, and another would be to read across the rows starting with the first.

Since our example has two-digit numbers assigned to each category, we need only two-digit random numbers. The table, however, includes five-digit random numbers. One way to modify the table, then, would be to read only the first two digits of the five-digit random number. Therefore if we read down the first column, the first few random numbers would be 53, 67, 11, 80, and so on.

Proceeding with the simulation, we read the first random number from

Table 14-4 The Assignment of Two-Digit Numbers to the Service-Time Distribution

Service time, minutes	Number of observations	Probability of this service time	Two-digit number
1	20	.20	00–19
2	30	.30	20–49
3	30	.30	50–79
4	10	.10	80–89
5	6	.06	90–95
6	4	.04	96–99
	100	1.00	

Table 2: it is 53. Reference to Table 14-3 shows that this random number falls within the interval 30–79 and is therefore associated with a time of 3 minutes between arrivals. If the drive-in teller opens at 9 A.M., the first arrival is therefore at 9:03. To continue, a second random number is selected from Table 2: it is 67. Table 14-4 shows that this is associated with a service time of 3 minutes since it falls in the interval 50–79. Therefore the customer who arrived at 9:03 is served by 9:06.

Before continuing with the simulation, perhaps a few more words ought to be said about the way these arrivals and service times were simulated. First, two-digit numbers were assigned to each time between arrivals and each service time. They were assigned in such a way that the percentage of two-digit numbers corresponded to the probability of each category. For example, two-digit numbers from 00 to 19 were assigned to "1-minute service time." When read from the random number table, the numbers 00 to 19 occur on a random basis and exactly 20 percent of the time. When, in fact, these numbers are read, this category will be identified as the service time. We can therefore conclude that this process will *randomly* generate a 1-minute service time on an *average* of 20 percent of the occasions.

Now we will continue with the simulation example. To facilitate record keeping, all the simulation data will be recorded in Table 14-5. The first row in the table includes the data that have already been generated. The first random number selected was 53, and this corresponded to a time of 3 minutes

Table 14-5 Simulation of Drive-in Teller

Random number	Time between arrivals	Clock time at arrival	Time at which service can begin	Random number	Service time	Service ends	Waiting time
53	3	9:03	9:03	67	3	9:06	0
11	2	9:05	9:06	80	4	9:10	1
18	2	9:07	9:10	74	3	9:13	3
10	2	9:09	9:13	81	4	9:17	4
43	3	9:12	9:17	76	3	9:20	5
89	4	9:16	9:20	32	2	9:22	4
43	3	9:19	9:22	92	5	9:27	3
06	1	9:20	9:27	86	4	9:31	7
87	4	9:24	9:31	24	2	9:33	7
69	3	9:27	9:33	19	1	9:34	6
98	6	9:33	9:34	52	3	9:37	1
93	5	9:38	9:38	53	3	9:41	0
72	3	9:41	9:41	20	2	9:43	0
18	2	9:43	9:43	32	2	9:45	0
32	3	9:46	0:46	42	2	9:48	0
45	3	9:49	9:49	15	1	9:50	0
56	3	9:52	9:52	89	4	9:56	0
79	3	9:55	9:56	22	2	9:58	1
62	3	9:58	9:58	68	3	10:01	0
77	3			45			
							42

between arrivals. Since the drive-in teller opened at 9 A.M., the first arrival occurred at 9:03. The second random number was 67, and this corresponded to a service time of 3 minutes. Service for the first arrival was therefore complete at 9:06. Waiting time is recorded in the last column of the table. The first customer did not have to wait for service; therefore waiting time is recorded as zero.

The next arrival is generated by selecting the next random number: it is 11. This number corresponds to a time of 2 minutes between arrivals. Therefore the second arrival occurs at 9:05. The drive-in teller is not available, however, until service has been completed for the previous customer; this will occur at 9:06. Consequently service cannot begin until 9:06, and 1 minute of waiting time must be incurred. The next random number, 80, is used to select a service time of 4 minutes. Therefore service for this second arrival is complete at 9:10.

As we continue with what now should be a familiar process, the next arrival is generated by selecting the next random number; this arrival occurs at 9:07 and must wait until 9:10 to be served. Waiting time is therefore 3 minutes. The next random number is used to select a service time, and service is complete at 9:13.

This process is continued in Table 14-5 until 10 A.M. We can therefore conclude that in the first hour of operation customers had to wait a total of 42 minutes. No one customer had to wait more than 7 minutes, and the average waiting time for all 19 customers was 42/19, or 2.2 minutes. In addition no more than two automobiles were waiting at any one time, and it can tentatively be concluded that waiting lines will probably not extend into the street.

THE NUMBER OF REPLICATIONS

On the basis of a 1-hour simulation we concluded that the average waiting time per customer was 2.2 minutes. Now let's stand back and consider what confidence we can place in this estimate.

The particular arrival and service times just used were based on a series of random numbers selected from the random number table. Suppose that the process was repeated once again. If the random numbers were selected from a different place in the random number table, the arrival and service times would be somewhat different from the ones just determined. This would also lead to a slightly different average waiting time for the half-hour period. With two different average waiting times the question is: Which one should be used? The answer is: both. The reason behind this is that each run or *replication* will lead to different results, and it is the average of these results that will prove to be more useful since extreme results from any one replication will be smoothed out in the averaging process. If two replications are better than one, why not undertake several? In fact most simulations involve several replications, sometimes numbering in the thousands. The more replications, the more the credibility that can be placed in the average value as being a true representation of the process average.

There is no hard and fast rule to determine the number of replications. However, if it can be seen from one replication to the next that the outcomes are quite close to one another, a fairly small number of replications may be enough. On the other hand, if the outcomes are quite different, many more replications may be needed.

Performing many replications by hand can be very time-consuming, and in the case of large-scale simulation problems it can be impossible. The computer in most cases is absolutely necessary. Once the computer program is written for the first replication, it can be repeated over and over again very efficiently until the desired number of replications is achieved.

Sample Size

The data that were used for the simulation were based on a sample of 100 arrival and service times. Is it enough to simulate this process from this small sample, or would a larger sample have been more representative of the population from which these statistical patterns were drawn? The answer to this question is quite complex and is beyond the scope of this introductory chapter. At the very least the determination of sample size requires careful consideration. If the variability in the population is large, a larger sample size will be needed than if this variability is small.

Quite related to this is the fact that these statistical patterns may change over time. In our example it was implied that the arrival pattern was the same at 9:00, 9:15, 9:30, perhaps even at 10, 11, 12, 1, 2, and 3 o'clock. The assumption makes the data collection phase simpler, but the resultant simulation may not be realistic. Indeed arrival patterns change throughout the day and even between the days of the week. A more realistic simulation might require several arrival patterns, each one determined for different days and different time periods.

Independence

One of the basic assumptions of the simulation model just developed was the independence between arrival and service times. That is, a service time was generated independently of the arrival time. For example, if several arrivals occurred in quick succession and a waiting line developed, this in no way influenced service time. Our model, therefore, had no provision for a teller to increase service speed in response to a long line. Although this may be a perfectly reasonable assumption to make in this simulation, it may not always be the case. When it is not, a more complex model must be formulated.

CASE STUDY: Chester Electrical Company

The Chester Electrical Company is a wholesaler of replacement parts used in household appliances. Recently it has been experiencing an increasing number of stockouts. The president of the company, Mike Chester, has asked the inventory control department to study the present inventory control system and make any recommendations that could improve the system.

The manager of inventory control, Jack Salem, decided to choose one product, a replacement motor used in a well-known washing machine, and determine if the inventory control methods used for this product could be improved. If the result was successful, he planned to extend this study to other products.

Jack recognized that two decisions had to be made for each item in the stockroom: when to reorder and how many to reorder. He was also aware that neither lead time nor demand were known with certainty and that this complicated the decision.

To obtain some additional information he collected the demand data for this product over the last 200 weeks and lead time data over the last 100 reorders. These data are presented in Tables 14-6 and 14-7.

As he examined these data carefully he wondered if it would be possible to simulate several different reorder point and reorder quantity strategies. If this could be done, the results could be compared and the best one chosen.

Simulation of the Inventory System

Given demand and lead-time distributions, it is possible to simulate any number of order quantity and reorder point strategies. In simulation, the model cannot be used to determine the *best* reorder point and order quantity strategies. All that can be done is to test several different ones by "running" the simulation model. In this way a good but not the best strategy can be uncovered.

Table 14-6 Weekly Demand Data for Washing-Machine Motor

Demand	Number of weeks in which this level of demand was observed	Probability of this demand	Two-digit numbers
1	10	.05	00–04
2	60	.30	05–34
3	100	.50	35–84
4	20	.10	85–94
5	10	.05	95–99
	200	1.00	

Table 14-7 Lead-Time Data for Washing-Machine Motor

Lead time, week	Number of occasions on which this lead time was observed	Probability of this lead time	Two-digit numbers
1	40	.40	00–39
2	40	.40	40–79
3	20	.20	80–99
	100	1.00	

For illustrative purposes only one strategy will be tested. If a computer program were available, countless others could also be tested and the results compared. The strategy which will be tested requires that a reorder be placed for $Q = 10$ motors when inventory is depleted to $r = 5$.

The simulation of a 20-week period is presented in Table 14-8. In week 1 it is assumed that no units have been carried over from previous periods and that 10 units arrive at the beginning of that first week. The beginning inventory, found in the third column, is therefore 10 units. Next a random number is read from the random number table at the back of the book for the purpose of generating a demand from Table 14-6. The random number read was 53 and this is associated with a demand of 3 units. Ending inventory is therefore $10 - 3$, or 7 units. This is entered in column 6 of the table. Since no stockouts occurred, the entry in column 7 is zero. In addition no reorder was placed during this week since demand was not depleted to $r = 5$ units.

In week 2 the beginning inventory is 7 units. The next random number, 67, is associated with a demand of 3 units, and this therefore depletes the ending inventory to 4 units. Since the reorder point was passed ($r = 4$), a reorder for $Q = 10$ units is issued. Now it is necessary to read another random number so that the lead time for this order can be established from Table 14-7. The random number is 98 and is associated with a lead time of 3 weeks. The order will therefore arrive at the beginning of the fifth week. This expected delivery is entered in the second column of the fifth week.

In week 3, beginning inventory is 4 units, demand is determined to be 2 units, and ending inventory is therefore 2 units.

In week 4, beginning inventory is 2 units, and demand is determined to be 3 units. Since demand exceeds available supply, a stockout is incurred. It is entered in column 7. Because of the nature of the product it is unlikely that the customer will wait until the next shipment is received. Therefore all stockouts in this problem will be considered to be lost sales and the order will not have to be filled when this shipment arrives.

The simulation process is continued in the same way for the next 16 weeks. On the basis of this 20-week simulation it can be seen that the current

Table 14-8 Simulation of Inventory Problem When $Q = 10$ and $r = 5$

(1) Week	(2) Order receipts	(3) Beginning inventory	(4) Random number	(5) Demand	(6) Ending inventory	(7) Stockouts	(8) Order placed	(9) Random number	(10) Number of weeks in which order will arrive
1	10	10	53	3	7	0			
2		7	67	3	4	0	10	98	3
3		4	11	2	2	0			
4		2	80	3	0	1			
5	10	10	18	2	8	0			
6		8	74	3	5	0	10	52	2
7		5	10	2	3	0			
8	10	13	81	3	10	0			
9		10	43	3	7	0			
10		7	76	3	4	0	10	93	3
11		4	89	4	0	0			
12		0	32	2	0	2			
13	10	10	43	3	7	0			
14		7	92	4	3	0	10	53	2
15		3	06	2	1	0			
16	10	11	86	4	7	0			
17		7	87	4	3	0	10	72	2
18		3	24	2	1	0			
19	10	11	69	3	8	0			
20		8	19	2	6	0			
						3			

strategy will result in the placement of five orders for 10 units each and the occurrence of three stockouts.

The economic consequence of this strategy can be determined if some cost estimates are made. Suppose that the cost of carrying inventory from the end of one week to the beginning of the next is \$1 per unit. In addition the costs of placing an order are \$20, and the cost of incurring a stockout is \$5 per unit.

The cost of carrying inventory for the 20-week simulation can be computed by multiplying the ending inventory for each week by \$1 and summing the results. Carrying costs for this period are therefore \$86.

Since five orders were placed, the fixed order costs were \$100, and with three stockouts these costs were \$15.

The total cost of this strategy is the following:

$$86 + 100 + 15 = \$201$$

Additional replications could be run and the average cost of these replications computed. Then these results could be compared with those of additional simulations in which other order point and order quantity strategies would be tested.

CASE STUDY: Giftware Company

The Giftware Company of Toledo, Ohio, manufactures a line of giftware products. At the present time it is deciding whether to add hanging planters to its line. If added the product will be manufactured in Giftware's Houston plant and will be distributed through eight wholesale showrooms located in major giftware markets across the United States. Before final approval is given to the project, the president has asked for a profitability estimate.

The investment in machinery necessary to manufacture this product has been estimated at \$25,000. There is still, however, a considerable amount of uncertainty associated with sales revenue and manufacturing cost estimates.

The marketing department has estimated that there is a 20 percent chance that sales in the first year will be \$40,000, a 30 percent chance that first-year sales will be \$50,000, a 40 percent chance that they will be \$60,000, and a 10 percent chance that they will be \$70,000. This subjective probability distribution is summarized in Table 14-9.

The machinery is capable of producing 15,000 units each year. Production personnel feel that there is a 10 percent chance that manufacturing expenses could be kept as low as \$20,000, an 80 percent chance that they could be \$30,000, and a 10 percent chance that they could be as high as \$40,000. This subjective probability distribution is repeated in Table 14-10.

Table 14-9 Subjective Probability Estimates of Project Revenues

Revenue	Subjective probability estimate	Two-digit number
$40,000	.20	00–19
50,000	.30	20–49
60,000	.40	50–89
70,000	.10	90–99

Table 14-10 Subjective Probability Estimates of Project Expenses

Expenses	Subjective probability estimate	Two-digit number
$20,000	.10	00–09
30,000	.80	10–89
40,000	.10	90–99

Simulation of an Investment Proposal

Given these subjective probability estimates, it will be possible to simulate this project and estimate its profitability. The measure of profitability that will be used is called the benefit-cost ratio and is computed by dividing the benefits derived from the investment by its investment cost. Benefits in our case will be the revenues less manufacturing expenses. To keep the analysis simple it will be assumed that this project will have an expected life of 1 year. This will avoid the need to take the present value of future flows into consideration. In fact, given the rapid change of consumer tastes in the giftware marketplace, this may not be an unrealistic assumption.

The simulation process for this problem requires that a revenue is generated from Table 14-9 and then an expense from Table 14-10. Finally the benefit-cost ratio is computed.

In the first replication a random number 98 is read from the random number table at the back of the book, thereby generating a revenue of $70,000. Then a second random number is read, and an expense of $20,000 is generated. Profit is therefore $70,000 – $20,000, or $50,000, and the benefit-cost ratio is 2.

$$^{50,000}/_{25,000} = 2$$

A ratio of 2 implies that the benefits from this investment are twice as large as

Table 14-11 Simulation of an Investment Proposal

(1) Replication	(2) Random number	(3) Revenue	(4) Random number	(5) Manufacturing expense	(6) Profit (3)–(5)	(7) Benefit-cost ratio (6)/25,000
1	98	70,000	01	20,000	50,000	2.0
2	66	60,000	36	30,000	30,000	1.2
3	77	60,000	36	30,000	30,000	1.2
4	52	60,000	07	20,000	40,000	1.6
5	40	50,000	80	30,000	20,000	.8
6	49	50,000	36	30,000	20,000	.8
7	39	50,000	39	30,000	20,000	.8
8	06	40,000	42	30,000	10,000	.4
9	21	50,000	32	30,000	20,000	.8
10	33	50,000	05	20,000	30,000	1.2
11	71	60,000	06	20,000	40,000	1.6
12	90	70,000	51	30,000	40,000	1.6
13	83	60,000	99	40,000	20,000	.8
14	11	40,000	49	30,000	10,000	.4
15	02	40,000	70	30,000	10,000	.4
16	36	50,000	10	30,000	20,000	.8
17	73	60,000	56	30,000	30,000	1.2
18	66	60,000	76	30,000	30,000	1.2
19	76	60,000	66	30,000	30,000	1.2
20	42	50,000	57	30,000	20,000	.8

its cost. The simulation process continues in this way; the results of 20 replications are presented in Table 14-11.

By examining column 7 it can be seen that in several replications the benefit-cost ratio was below 1. A ratio below 1 implies that the profits during the first year are not large enough to cover the investment cost. A ratio of exactly 1 implies that profits just cover the investment cost. There are, on the other hand, several ratios above 1, and this implies that there is a chance that the project will be profitable. A more effective analysis can be made if these results are displayed graphically. In Figure 14-2 a frequency distribution of the results is shown together with a histogram. The frequency distribution shows that for three of the replications the ratio was .4, for seven of the replications the ratio was .8, and so on. The histogram presents these results graphically.

The histogram displays clearly the risk associated with the project. For example, it can be seen that there is a 50 percent chance (.35 + .15) that the project will not be profitable (benefit-cost ratio of .4 or .8). But there is a slight

Benefit-cost ratio	Frequency	Relative frequency
.4	3	.15
.8	7	.35
1.2	6	.30
1.6	3	.15
2.0	1	.05
	20	1.00

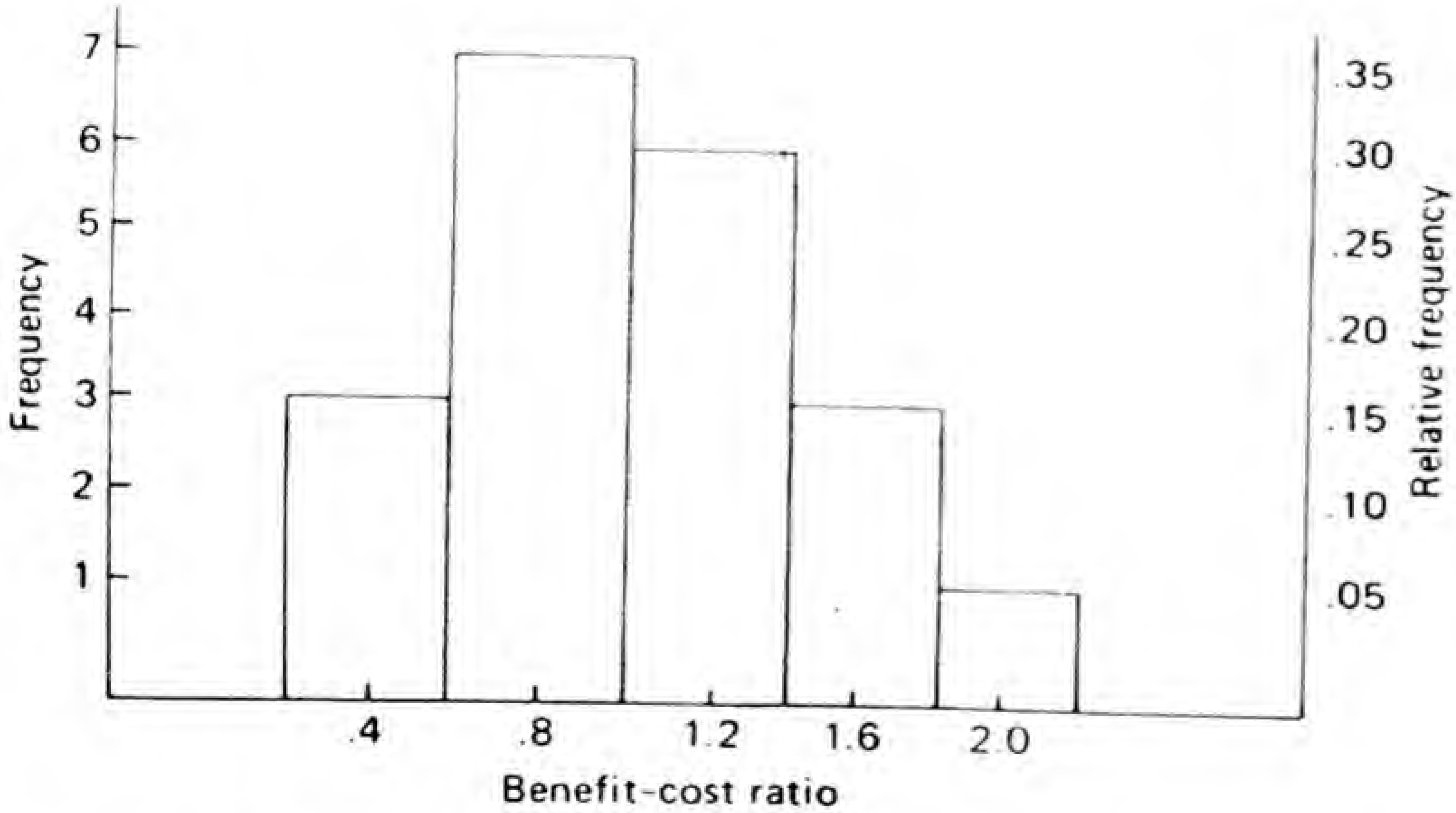


Figure 14-2 Simulation results presented as a frequency distribution and histogram.

chance (5 percent) that the benefits will be twice as large as the costs. The average or expected value can also be computed from the frequency distribution; it is 1.04. Therefore the conclusion drawn from the expected value alone is that the project is slightly profitable. But when the full range of outcomes is observed, it can be seen that the project does involve a considerable degree of risk. Depending on the decision makers' attitude toward this risk the project will either be accepted or rejected.

The traditional approach to investment analysis completely ignores risk. For example, if this problem were solved by those methods, the expected value of revenue determined from Table 14-9 would be subtracted from the expected value of costs determined from Table 14-10. The result would be an expected profit of $24,000.

$$54{,}000 - 30{,}000 = \$24{,}000$$

The benefit-cost ratio would be .96:

$$^{24{,}000}/_{25{,}000} = .96$$

The traditional analysis, then, would ignore the riskiness of the project and summarize its attractiveness with *one* single benefit-cost ratio. The result of the simulation analysis was not just the expected benefit-cost ratio but the full range of ratios which expressed the *riskiness* of the project. Most practitioners feel that this is a valuable dimension to the problem which should *not* be ignored.

SUMMARY

Simulation models capture the essence of complex systems and can be used to test alternative strategies in the search for solutions to decision problems. This technique has been applied in practice as much as, if not more than, any other quantitative method covered in this book. Its success assures it a secure place as a valuable management tool.

QUESTIONS

1 Why is a single replication generally inadequate in simulation analysis?
2 Determine the longest length of the waiting line during the 1-hour simulation in the St. Louis Safe and Trust Company case?
3 Why is it seldom possible to uncover the optimal solution using simulation methods?
4 Would you consider the project in the Giftware Company case risky?
5 The Pleasant Mountain Ski Area has just added a chairlift to its north slope. Management hopes that this new chairlift will eliminate the long waiting lines that formed last season.

Describe a simulation model that could be used to predict the length of the waiting lines at the Pleasant Mountain facilities, which now include three double chairlifts and two T bars.

PROBLEMS

14-1 Return to the St. Louis Safe and Trust Company case and continue the simulation presented in Table 14-5 until 12 A.M. What is the average waiting time per customer for the first 3 hours?

14-2 Starting from a different place in the random number table, return to the St. Louis Safe and Trust Company case and simulate the process for a 3-hour period. What is the average waiting time per customer?

14-3 Return to the St. Louis Safe and Trust Company case and compute the average time between arrivals from Table 14-1 and the average service time from Table 14-2. What conclusions can be drawn from these figures? If the simulation model were to be continued for a 7-hour business day, what would you expect to happen to the waiting line? Is it reasonable to assume that a second drive-in teller would completely eliminate the possibility of a waiting line?

What conclusion would you reach in this case? Should a drive-in window be constructed?

14-4 Continue the simulation of the Chester Company inventory system for another 10 weeks. How many stockouts are incurred over this 30-week period?

14-5 Return to the Chester Company case and test the following strategy for a 20-week period: $Q = 10$, $r = 3$. Assume that 8 units have been carried over from the previous period and that no orders are outstanding. What is the total cost of this strategy? What is the average cost per week?

14-6 Suppose that the investment required to manufacture hanging planters in the Giftware Company case can be reduced to \$15,000. What effect will this have on the risk of the project?

CASE STUDY: Atlanta Bank and Trust Company

The Atlanta Bank and Trust Company has recently purchased a parcel of land adjacent to its main downtown location for the purpose of offering a drive-in banking service to its customers. The problem which the bank is currently facing is this: Should one or two drive-in teller stalls be built?

At a recent meeting Jane Brown, vice president of operations, made the following comments. "Gentlemen, it seems clear to me that since our competition has been offering drive-in service for some time now, we ought not waste any more time. In fact I am sure that we have lost many customers just because this convenience has not been offered at our downtown branch. You know how banking customers are; they take their business to the bank that offers the most convenient service. When we do open our new drive-in facility, I am sure that most of these old customers will return. In my mind I am convinced that we therefore need two teller stalls. Even if they don't return, our business is expanding so fast that we will eventually need this capacity."

Gil Urban, senior vice president, was a little more cautious. "Given the traffic problems here in the downtown area and my skepticism about demand for this service, I would prefer to build one drive-in teller stall and see what happens."

Jane Brown responded, "I think the problem with your approach, Gil, is that a long queue of cars will turn away more customers than we gain. Imagine what would have happened if the New York Thruway opened with only one tollbooth in either direction."

"Just a minute, gentlemen," interrupted Bruce Dubins, senior analyst with the management science group. "I have collected some information which you might find useful. Last week I went to one of our large branches which already has drive-in banking and studied their arrival and service rates. I realize that these rates are not directly applicable to our situation, but given the volume of traffic, traffic flow patterns, and the proximity of competition, I feel that these data are close enough to be useful. Here are copies for each of you." Bruce then handed out Exhibits A and B.

After looking at the exhibits, Gil was the first to speak. "It looks like the average time between customer arrivals is about the same as the service time. Therefore one drive-in teller stall should be sufficient."

Jane replied, "I think it is premature to conclude that no waiting line will form. What happens if several autos arrive one right after the other? According to Exhibit A this is certainly a possibility."

"But, Jane," said Gil, "I don't think this means that we should spend the money to build a second drive-in facility whose capital costs will average \$1000 per year over the 15-year life of the structure and whose staffing costs will total \$10,000 per year including fringe benefits. Even if we assume that customer waiting time eventually costs the bank 10 cents per minute in loss of goodwill and eventual loss of customers, I think one drive-in facility will be sufficient."

Exhibit A Time between Arrivals for 100 Observations

Time between arrivals, minutes	Number of observations
1	15
2	20
3	35
4	15
5	10
6	5
	100

Exhibit B Service Time for 100 Customers

Service time, minutes	Number of observations
1	10
2	20
3	40
4	15
5	15

Gil continued, "Bruce, why don't you continue with your analysis and report back to the committee next week? Right now I have to run to another meeting."

QUESTIONS

1 Simulate a drive-in facility with one teller (limit the simulation to the time period 9–10 A.M.).
2 Simulate a drive-in facility with two teller stalls (limit the simulation to the time period 9–10 A.M.).
3 Compare the average yearly capital, operating, and waiting costs of the two alternatives. Assume that the bank is open 6 hours per day and 250 days per year.
4 Have any alternatives been left out?
5 Comment on the relevance of Exhibits A and B to the case.
6 Would additional replications of the simulations performed in questions 1 and 2 be helpful? Explain.

CASE STUDY: Somerville Fabrication Company

"Sorry," said the voice over the intercom system, "the overhead crane is backlogged with requests and will be unable to pick up your job for at least 15 minutes." Marie LeBlanc, a machinist for the Somerville Fabrication Company, lit up a cigarette and prepared for the long wait.

This condition was becoming more frequent lately. The shop was so busy that the crane was unable to keep pace with the demands for service.

The Problem

Somerville Fabrication Company specialize in the machining of large castings which generally weigh from 100 pounds to several tons. It is therefore impossible for these jobs to be moved or positioned by hand. Instead an overhead mobile crane is used.

The crane runs on tracks and is positioned 100 feet above the shop floor. The operator sits inside the cab and receives instructions through a mobile radio which keeps him in contact with the central dispatcher on the shop floor. The dispatcher, on the other hand, receives requests for crane service from the 135 machinists who are scattered throughout the 40,000-square-foot shop floor. Unless the machinists receive prompt service from the crane, they are unable to perform any additional work on their current job or other jobs. An unavailable crane always means lost production time.

The Alternatives

The manager of the machine shop, Dan Gurney, has been concerned about these delays for several months and has collected some information from two sales representatives who specialize in materials handling equipment. The first recommended that the use of the present crane be discontinued and that it be replaced by a new one capable of performing the jobs 50 percent faster. The present crane is leased for $5000 per year; the faster one could be leased for $12,000 per year. The present crane has an annual operating cost including fuel, maintenance, and labor of $30,000. It is expected that the new crane would incur the same costs. The shop is on a two-shift schedule, each one 8 hours long, and the company is open 200 days per year.

The second sales representative recommended that a second crane identical to the present one be used. He felt that this would meet the needs of the shop more than adequately. The cost of leasing the second crane would be the same as the present one.

Priority Calls

The demand for crane service is initiated when a machinist places a call to the dispatcher. The call is then put into one of two categories. The first is the routine category, and as calls arrive, they are scheduled on a first come, first served basis. The second is a priority category, consisting of all calls whose jobs are behind schedule. All priority jobs are taken ahead of routine jobs, but

a job which is already being serviced cannot be interrupted for a priority job. The priority job, however, would be next.

Relevant Cost Savings

Dan Gurney met with the accounting department for the purpose of determining the possible savings that could be accrued by the addition of crane capacity. Working together, they estimated that it cost the company $1 for every minute of waiting time on routine jobs and $2 for every minute on priority jobs.

Discussion of Problem

Dan Gurney was presenting this information to the vice president of finance, Betty Wilson. "Betty, it's very difficult to determine the total cost associated with either of these alternatives because there doesn't seem to be any pattern associated with calls for service. Sometimes calls arrive one right after the other and a large backlog occurs. Why, on Wednesday of last week one routine job had to wait 1 hour for the crane. On other occasions the crane is idle. I just can't seem to uncover any pattern."

Betty Wilson added, "It looks to me as if you already have enough crane capacity on the average, but I think this is one of those situations where the averages don't tell the whole story."

They were both silent for a moment, and then Betty Wilson continued, "Dan, I'll bet that our management science group can help us with some estimates. Why don't you familiarize them with this problem and ask them to report to us in 2 weeks with some preliminary comparisons?"

Data Collection

The management science group decided that the easiest way to compare the costs of the various alternatives was to simulate the process. To do this, they needed some data.

The crane operators on both shifts were asked to record the time it took to service each call. Early during this data collection process it was determined that service times seemed to be independent of arrival rates, the time of day, or the shift. The results of 200 calls for service are given in Exhibit A.

In addition, the dispatcher was asked to record the time between calls. After careful analysis it was determined that the pattern was the same throughout the day and on each shift. It was also discovered that approximately 30 percent of all calls were in the priority category. There was no pattern to these priority calls; they seemed to occur on a random basis. The results of the 200 calls which the dispatcher recorded are given in Exhibit B.

With the data collected, the next step was to simulate each of the alternatives and determine the cost of each of them.

Exhibit A Service Times

Service times, minutes	Number of observations
3	40
4	80
5	50
6	30
	200

Note: Service times include the length of time it takes for the crane to travel from one job to the next.

Exhibit B Time between Arrivals

Time between arrivals, minutes	Number of observations
2	4
3	20
4	100
5	40
6	20
7	10
8	6
	200

QUESTIONS

1 Simulate the crane process for 1 hour, given the current situation of one crane. (Assume that the plant opens at 8 A.M.)
2 Simulate the crane process using an additional crane identical to the present one. Simulate the process for 1 hour. (Assume that the plant opens at 8 A.M.)
3 Simulate the crane process using a crane which is 50 percent faster—one that will do the job in two-thirds of the time presently required. Simulate the process for 1 hour. (Assume that the plant opens at 8 A.M.)
4 Compute the total yearly costs associated with each of these alternatives. What will be the saving associated with a faster crane? With an additional crane?

CASE STUDY: New York Ambulance Service

It was the fifth time this week that the mayor's office had received a complaint about the quality of ambulance service. The sick and injured were waiting

what they thought was an unreasonably long time for an ambulance to arrive after a call had been placed. According to those who complained, the solution was very simple—add more ambulances.

City-operated Ambulance Service

The City of New York provides ambulance service to its approximately 8 million inhabitants with 109 ambulances stationed at 49 hospitals.

Ambulances are permanently assigned to each of these hospitals, and they respond to calls which originate only within their clearly defined hospital district. For example, seven ambulances are assigned to the Kings County Hospital in Brooklyn and respond only to calls originating in that district.

Most calls for emergency ambulance service are made through the police department. Upon receipt of a call the police officer on duty calls the nearest hospital and provides the details. The hospital dispatcher then determines whether an ambulance is available, and if one is not, the call is placed in a queue. When available, the ambulance is driven to the scene of the call, emergency treatment is provided if necessary, the patient is loaded into the ambulance, the ambulance is driven back to the hospital, the patient is unloaded, the ambulance is cleaned, and then it becomes available for another call.

The demand for ambulances had increased 50 percent in the last decade, and it is unlikely that this growth in demand would subside in the future. No new ambulances had been added in the last 3 years.

The Problem Discussed

The mayor called a meeting for the purpose of discussing this problem. Attending were the mayor's chief aide, the head of ambulance services, a budget department representative, a member of the mayor's management science department, and the head of data processing for the police department.

The representative from the budget department spoke early in the meeting. "I think it's clear that we need more ambulances, but I can assure you that we do not have enough money to add as many as we would like to. We should find the assignments that lead to highest payoffs and then try to get approval for those ambulances in the next fiscal budget. My guess is that we can get at most 10 new ambulances."

The head of ambulance services said, "It isn't very clear to me how we can determine the payoff associated with various assignments. For example, if we take the Kings County Hospital district, how could I determine the effectiveness of adding another ambulance? Perhaps we should just add the 10 ambulances evenly among the hospitals."

John Dennison of the management science staff interrupted. "I think our staff could provide some useful insight into this problem. Bill, what kind of records does the police department keep on ambulance calls?"

Bill Jones, head of data processing for the police department, thought for

a moment and replied, "Our records probably go back about 5 years and include a series of data for each ambulance call. These include the time the call was received, the location in the district from which the call originated, the time at which the ambulance was dispatched, the time it arrived at the scene, the time the ambulance started on its return to the hospital, the time it returned to the hospital, and the time it was available for the next call."

"If we can have access to those records," said John Dennison, "we can build a simulation model and determine the consequence of adding ambulances to these districts. Give me 3 weeks and we will simulate the Kings County Hospital district. Then I will be able to tell you just how effective the addition of new equipment to this district will be. After that we could simulate the other districts or just apply the insight that we gain from the Kings County Hospital district simulation to these other districts."

QUESTIONS

1 What are the alternative strategies which should be tested in the Kings County district simulation?
2 What criterion should be used as a basis for comparing the outcome of these alternatives?
3 Design a simulation model that can be used to test these alternatives. Show the sequence of steps that will be necessary to carry out the simulation. Identify the data needed, the resulting probability distributions, and the way in which these distributions will be used for each of these steps.
4 Would it be possible to run a simulation of this size by hand, or would a computer be necessary?
5 It is not absolutely necessary that available ambulances wait at the hospital.

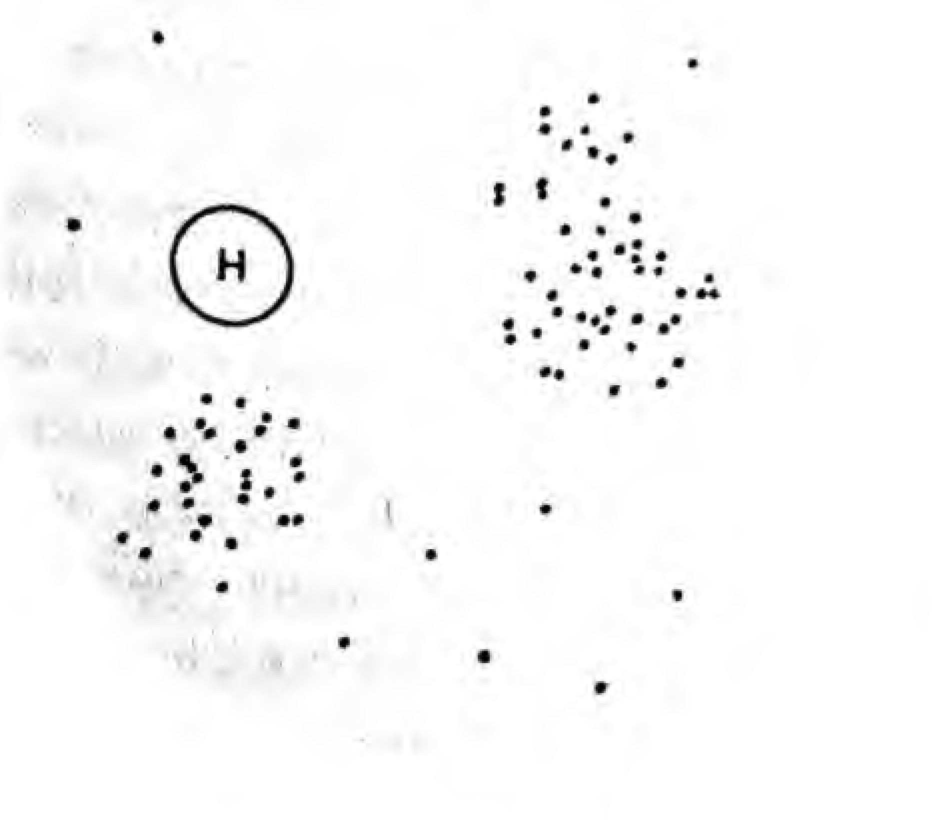


Exhibit A Kings County Hospital Region. Hospital represented as Ⓗ. The dots represent the approximate density of the calls.

Suppose, instead, that they were assigned to wait on street corners in areas of heavy demand (see Exhibit A). Could the simulation be used to test this strategy?
6 Discuss the difference between quantitative methods used by profit-maximizing firms and by government organizations.

CASE STUDY: Salant Sound Studio, Inc.

The vice president of finance, Ken Swenson, had just received a project proposal that looked better to him than anything he had seen in 3 years. Preliminary analysis of the project showed that it had a potential benefit-cost ratio of 4; that is, it could return four times as much as its investment. Surely this warranted a closer look.

Company Background

The Salant Sound Studio, Inc., was founded in 1968 by Sheldon Salant, Ken Swenson, and Dwight Edmonds. The company produces and markets a line of audio and video tapes for businesses, schools, and institutions. One of its most profitable packages is a combination of video cassettes and texts in the learning disabilities field. Most schools find the materials to be the best of their kind on the market. Other products include audio and video cassettes in such fields as reading, sex education, science, mathematics, and history.

In general the owners of the firm have done little script writing. Since the beginning they have made it a policy to hire the best people in the field, pay them well, and in addition offer them a royalty based on the quantity of tapes sold. Professionals have also been used in every phase of the production process: sound and video technicians, even the actors—all have been top-quality professionals.

Statistics by Video Cassette

In the past most of the packages developed at the Salant Sound Studio have been directed toward the elementary and high school markets. In fact, not many of Salant's competitors have ventured into the college market. The new proposal on Ken Swenson's desk was the first move of that kind ever suggested.

Dwight Edmonds was suggesting that the company develop a video cassette course in undergraduate statistics. The proposal went on to say that statistics was a required course in nearly all undergraduate curricula and that it was generally taught by graduate students who might lack the teaching experience and depth of knowledge to teach a good course. Furthermore, colleges and universities were under an ever-increasing financial squeeze, and they would welcome any opportunity to increase their productivity. With a well-designed video tape course, fewer professors would be needed to teach this course and the productivity of the teaching staff would thereby increase. The

report concluded with the fact that the return on this project could be about four times the magnitude of the $200,000 investment required.

The Risks Discussed

Ken Swenson admitted that this was a great idea, but he was concerned about the risks involved. He worried that this concept would not be accepted by academicians and that the company might lose nearly all its investment. With the hope of resolving this issue, he called a meeting with Dwight Edmonds and Sheldon Salant.

Dwight had already done some homework on the projected revenues and expenses. "If we look at the difference between revenue and direct expenses or gross margin, it seems to me that we have a good chance of a $200,000 gross margin in the first year," said Dwight.

Ken asked, "What do you mean by a good chance?"

Dwight answered, "I would say a 20 percent chance."

"What happens if we don't generate a gross margin of $200,000 the first year?" returned Ken.

"Well," said Dwight, "I think there is a 30 percent chance that the gross margin will be $150,000, a 20 percent chance that it will be $100,000, a 10 percent chance that it will be $50,000, a 10 percent chance that it will be $25,000, and a 10 percent chance that our margin will be zero.

"Furthermore," continued Dwight, "whatever our gross margin turns out to be in the first year, I would expect it to continue at that level over the life of the project."

"Exactly what do you think the life of this project will be?" asked Ken.

"That's hard to say," responded Dwight, "but I would expect it to be 4 years."

"Do you mean that it's unlikely to be longer or shorter?" asked Ken. "Because the profitability of this project surely depends upon its life."

"Of course not," answered Dwight. "There is only a 10 percent chance that our competition will bring this project to an end in 2 years; there is a 15 percent chance that it will last 3 years, a 40 percent chance that it will last 4 years, a 20 percent chance that it will last 5 years, and a 15 percent chance that it will last as many as 6 years."

Ken then turned to Sheldon Salant. "Shelly, what kind of investment will it take to come out with this package?"

"If we include a programmed text along with the cassette package, I think that our costs will run about $200,000. There is, however, a 10 percent chance that I could keep these costs to $150,000, but on the other hand there is also a 10 percent chance that costs could go as high as $250,000. But I think there is an 80 percent chance that I can hold to that $200,000 figure."

The room was silent as Ken did some figuring; then he said, "If I use average values for the gross profit per year, the life of the project and its costs, I get the following figures." Ken went to the blackboard and started to write the figures found in Exhibit A. "If these are right, the benefits are 1.6065

Exhibit A

Average profits = .20(200,000) + .30(150,000) + .20(100,000) + .10(50,000) + .10(25,000) = $112,500

Average life = .10(2) + .15(3) + .40(4) + .20(5) + .15(6) = 4.15 years

Average cost = .10(150,000) + .80(200,000) + .10(250,000) = $200,000

Present Value of Profits over a 4-Year Life and a Cost of Financing at 15 Percent

Year	(*A*) Profit	(*B*) Discount factor		(*A*) × (*B*)
1	112,500	.870		97,875
2	112,500	.756		85,050
3	112,500	.658		74,025
4	112,500	.572		64,350
			PV =	321,300

Benefit-cost ratio = R = 321,300/200,000 = 1.6065

times as great as the costs. So even if I use your *average* figures, the project still looks good."

"We all know our policy," continued Ken, "that if a project has more than a 15 percent chance of earning less than its investment cost, we forget it. On this basis, should we still be considering this project?"

QUESTIONS

1 Is Ken Swenson's analysis adequate? Why?
2 Formulate a simulation model for evaluating this investment.
3 Perform 20 replications of this model.
4 Construct a histogram of the results.
5 What other factors should be considered before the investment decision is made?
6 Determine whether or not the project should be undertaken.

Chapter 15

Network Analysis

INTRODUCTION

Organizations frequently undertake large-scale projects. Some examples include the construction of a new facility, the development of a new product, the preparation of a year-end stockholder's report, the development of a new production process, or the installation and test of a new computer-based accounting system.

Because most projects share several characteristics in common, it has been possible in the past 20 years to develop several models that have been used in the management and control of a wide variety of these projects.

One of the most successful applications occurred during the 1950s in the Polaris missile program. Never before had a project of this magnitude been planned, scheduled, managed, and controlled more effectively. For many in business and government, this was enough proof that these models were indeed a valuable aid in the management of large projects.

Before these models are covered, a case study will be used to introduce the concept of a project and the common characteristics which they all share.

CASE STUDY: ICM Corporation

The ICM Corporation, manufacturer of home heating equipment, has recently decided to redesign a burner used in one of its oil-fired heating systems. The new burner will operate more efficiently than the one currently in use and should save homeowners 15 percent on their heating bills.

The project involves several activities which are listed in Table 15-1. First, the engineering staff must finish the design of the burner. Second, the marketing program for promoting the product must be developed. Third, a new manufacturing process must be designed. Fourth, advertising media must be selected. Fifth, an initial production run must be successfully completed. And sixth, the product must be released to the market.

The president of the company has asked that the product be ready for delivery in 24 weeks. Otherwise the winter selling season will be lost.

THE DEFINITION OF A PROJECT

A project can be defined as a task of considerable magnitude involving the expenditure of both time and money that is usually, but not always, carried out once. ICM's burner project falls into this category.

Projects share certain characteristics in common. First, they all have a *starting* and a *finishing* point. Second, some activities can be carried out *independently* of others. For example, in house construction the installation of electrical wiring can be carried out independently of the installation of the plumbing. Other activities, however, can be initiated only after the completion of earlier activities. For example, the roof can be shingled only after the framing has been completed. In respect to sequence, these activities are said to be *dependent* on one another. The third characteristic that projects share is that each activity requires the expenditure of time and money.

NETWORK DESIGN

Projects can be described in a very formal way by structuring them as a network of activities. In the next two sections, activities and events are defined, and then, in the third section, the concept of a network is developed.

Table 15-1 New-Product Project

Activity	Description of activity	Precedent relationship
A	Finish product development	
B	Design marketing program	A
C	Design production system	A
D	Select advertising media	B
E	Initial production run	C
F	Release product to market	D, E

Activities

Activities are the basic jobs, tasks, or operations that must be performed. Occasionally an activity can be initiated with complete independence of other activities. That is, it can be performed at any time during the project life. The usual case, however, is that it can begin only after one or more other activities are completed. These are called *predecessors*.

The activities and their predecessors for the ICM case are shown in Table 15-1. From this table it can be seen that the design of the marketing program and production system must be preceded by the completion of product development. The selection of advertising media must be preceded by the design of the marketing program. The initial production run must be preceded by the design of the production system. The release of the product to the marketplace must be preceded by both the selection of advertising media and the initial production run.

Events

An *event* marks the beginning or end of an activity. Frequently an event will be used to mark the end of one *or more* activities and the beginning of one *or more*. Therefore the major difference between activities and events is that activities represent the passage of time whereas events are points in time.

Networks

Now we are ready to combine activities, events, and precedent relationships into a network. In this network an activity will be represented by an arrow and an event will be represented by a circle. Activity A from Table 15-1 is depicted in Figure 15-1. The circle to the left of the arrow represents the beginning of activity A, and the circle to the right represents the completion of activity A.

All the activities associated with the project presented in Table 15-1 can be combined into an integrated network of events and activities. This is done in Figure 15-2. Notice that the event circle found at the end of the activity A arrow denotes not only the completion of activity A but also the beginning of activities B and C.

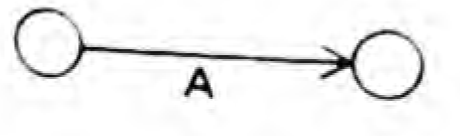


Figure 15-1 Activities and events.

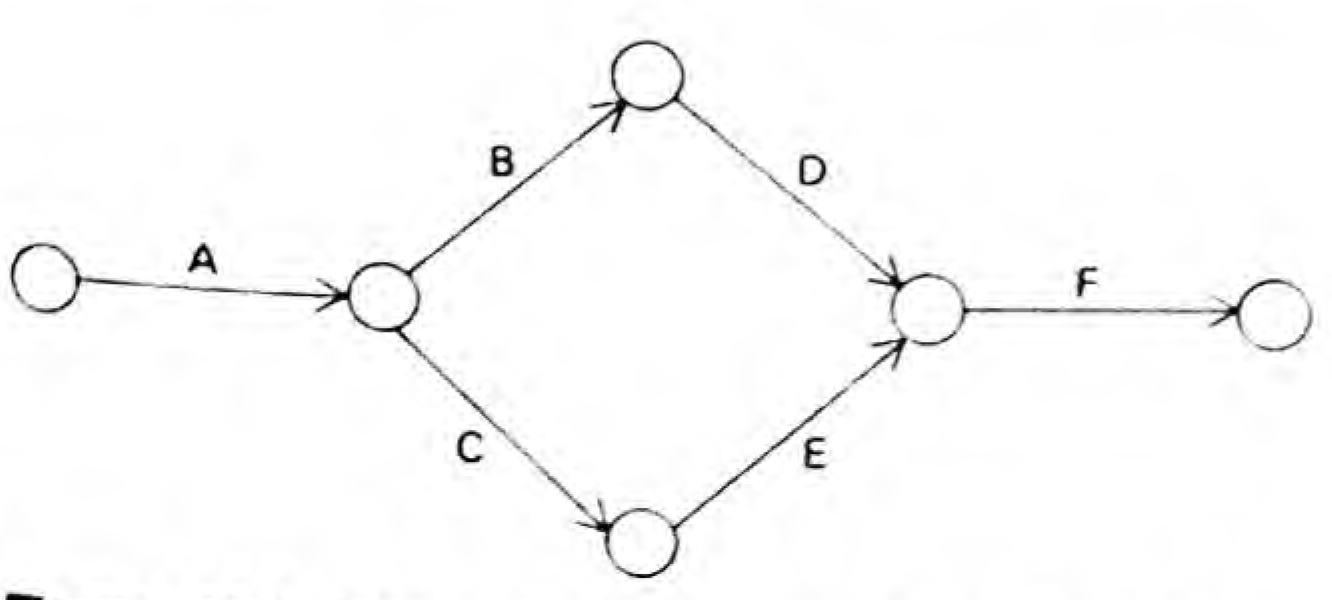


Figure 15-2 Network for new-product project.

Table 15-2 Network and Time Estimates for New-Product Project

Activity	Precedent relationship	Time estimate, weeks
A	—	5
B	A	4
C	A	7
D	B	8
E	C	9
F	D, E	4

If you carefully compare the network in Figure 15-2 with the description of the project in Table 15-1, you should come to the conclusion that Figure 15-2 is indeed an accurate representation of the activities and their precedent relationships. For example, we find in Table 15-1 that activity F has as its predecessors activities D and E. Returning to Figure 15-2, we find that indeed the arrows imply that activities D and E must be completed before activity F can begin. Checking the other precedent relationships, we find that they also are met.

Activity Times

The completion of an activity generally requires the expenditure of the scarce resource time. A time estimate for each of the activities in our problem is given in Table 15-2.

ANALYSIS OF THE NETWORK

The length of time to complete a project is an important piece of information. Given the time estimates presented in Table 15-2, it is possible to calculate the earliest time at which the project can be completed. To calculate this systematically, we must first determine the earliest start and earliest finish times for each activity.

Early Start and Early Finish Times

If we start at the beginning of the network and work toward the end, it is possible to compute the earliest start and earliest finish times for each activity. Starting with activity A, its earliest start time is zero. We enter this in the left-hand side of the bracket over activity A in Figure 15-3. Next we must calculate the earliest time at which activity A can be completed. If activity A starts at zero and takes 5 weeks, the earliest time at which it can be completed is week 5. This earliest finish time is entered in the right-hand side of the bracket above activity A. Next we turn to activity B.

The earliest that activity B can begin is week 5, since its predecessor, activity A, is completed at that time. Activity B requires 4 weeks; therefore

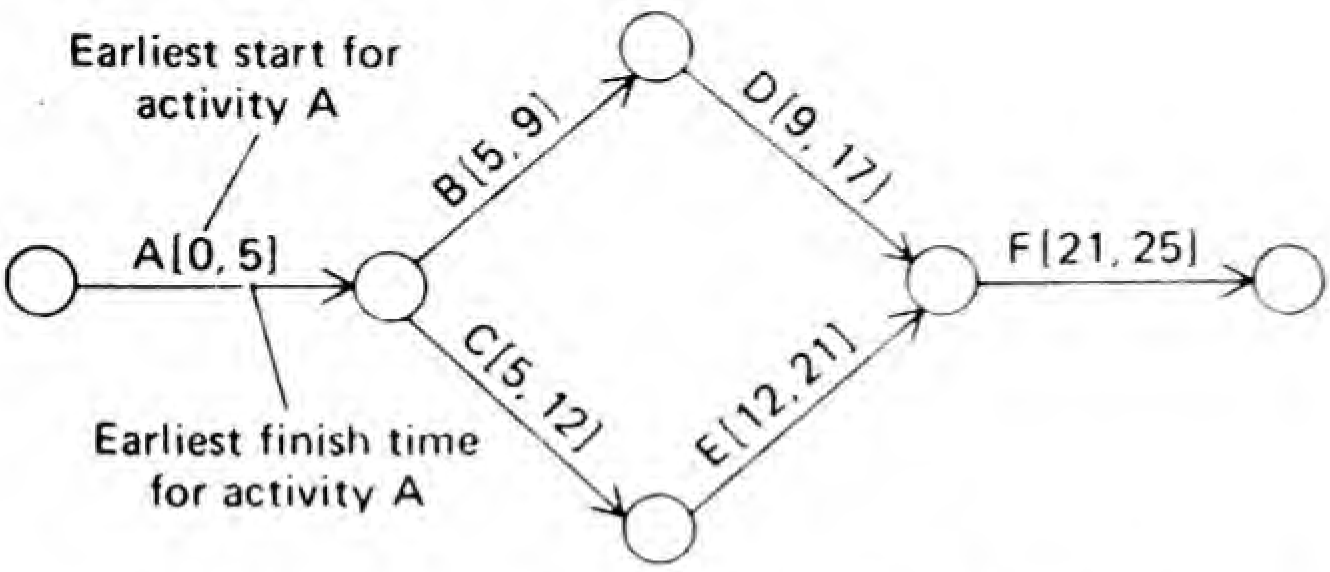


Figure 15-3 Earliest start and earliest finish times.

the earliest time at which it can be completed is 5 + 4, or 9 weeks. These early start and early finish times are entered in Figure 15-3.

Turning to activity D, we find that its immediate predecessor, activity B, can be completed—at the earliest—in week 9, and therefore the earliest that activity D can begin is week 9. Since D requires 8 weeks, the earliest time at which it can be completed is 9 + 8, or 17 weeks. These times are entered in Figure 15-3.

The earliest activity C can be started is week 5. Since this activity takes 7 weeks, the earliest it can be completed is 5 + 7, or 12 weeks.

The earliest activity E can begin is week 12. Since this activity takes 9 weeks, the earliest it can be completed is 12 + 9, or 21 weeks.

Activity F requires that *both* activity D and activity E be completed *before* it can be started. Activity D is completed in week 17, and activity E is completed in week 21. Therefore, activity F cannot be started before week 21 because *both* of these precedent activities must be completed before activity F can begin. We can generalize that whenever two or more activities merge *into* an event circle, the activity which follows can be started only after all the predecessors are finished.

In our case, all the predecessors are finished by week 21. Therefore the earliest that activity F can begin is week 21, and the earliest it can be completed is 21 + 4, or 25 weeks.

It can also be concluded that the earliest time the project can be completed is week 25.

The path through the network which includes activities A, C, E, and F is called the *critical path*. A critical path can be defined as a path of activities such that a delay for any activity on this path will delay the completion of the project. In our example any delay along the path ACEF will delay the project completion beyond week 25. To illustrate this, suppose that activity E is delayed and requires 10 weeks rather than 9; then the earliest that this activity can be completed is week 22. Activity F can then begin in week 22 and can be finished in week 26. This increase therefore delays the completion of the project from week 25 to week 26. Clearly all activities along this path are critical to the timely completion of the project.

Activities B and D are not critical to the completion of the project in 25

weeks. They can be delayed by a few weeks with no effect on project completion. Because of this, activities B and D are said to have slack. In the next section, we will develop a formal method for identifying the critical path and determining the slack associated with those activities which do not lie along this critical path.

Late Start and Late Finish Times

The formal method for identifying the critical path and determining the slack associated with those activities which do not lie along this path requires that we calculate the latest times at which an activity can be started and finished. This is accomplished by working from the end of the network backward to the beginning.

Turning to the end of the network, we must determine the latest time at which activity F can be finished and still remain on schedule. If we would like the project to be completed by week 25, this becomes the latest finish time for activity F. It is entered in the right-hand side of the bracket which is found below the activity arrow in Figure 15-4. Since activity F requires 4 weeks, the latest it can be started is $25-4$, or week 21. It also is entered in Figure 15-4.

To turn to activity E, the latest it can be finished is week 21; otherwise it would interfere with the latest start for activity F. The latest it can be started is $21-9$, or week 12.

The latest that activity C can be finished is week 12; otherwise it would interfere with the latest start of activity E. The latest it can be started is $12-7$, or week 5.

The latest that activity D can be finished is week 21; otherwise it would interfere with the latest possible start of activity F. The latest activity D can start is $21-8$, or week 13.

The latest that activity B can be finished is week 13, and the latest it can be started is $13-4$, or week 9.

Next we turn to activity A. From Figure 15-4 we can see that the latest activity B can be started is week 9, and the latest activity C can be started is week 5. It can be concluded that the latest activity A can be finished is week 5 if activity C is to proceed on schedule. We can generalize that when an activity is followed by one or more activities, the latest finish time for the activity being analyzed is

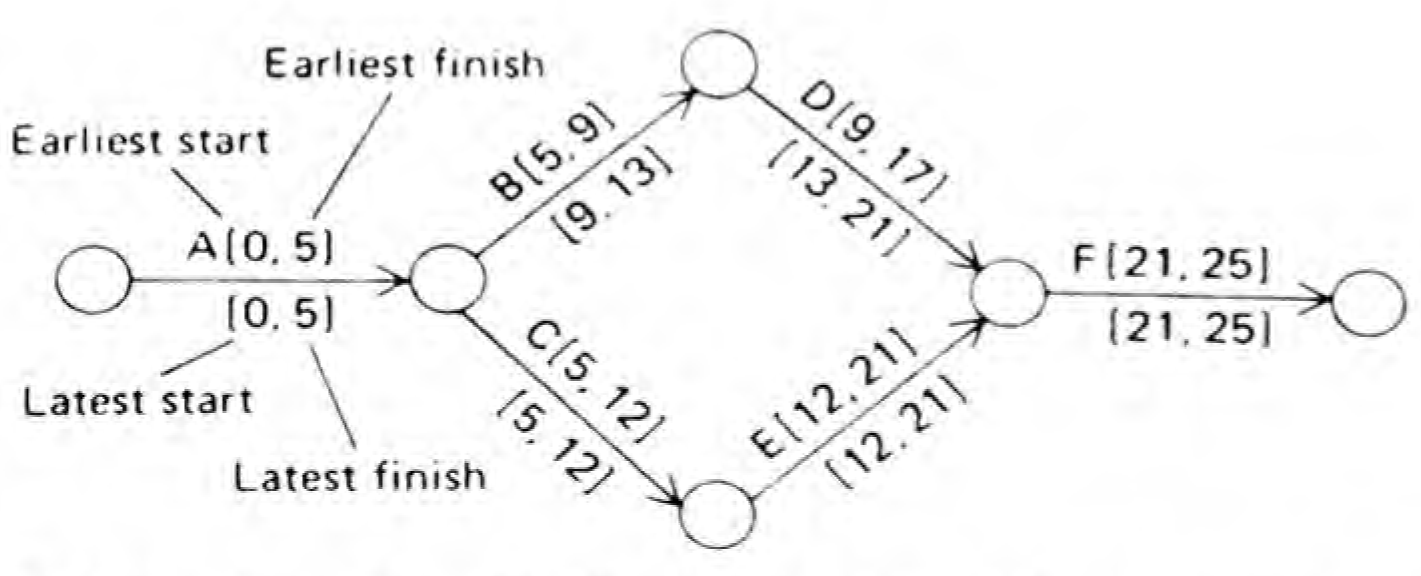


Figure 15-4 Earliest start and earliest finish times. Latest start and latest finish times.

Table 15-3 Computation of Slack for Each Activity

Activity	Latest start less earliest start	Slack
A	0 − 0	0
B	9 − 5	4
C	5 − 5	0
D	13 − 9	4
E	12 − 12	0
F	21 − 21	0

the smallest of the latest start times associated with those activities which follow.

To continue with activity A, the latest that it can be started is 5 − 5 weeks, or week 0. This completes the computation for latest start and latest finish times.

When the delay of an activity will not delay the completion of the project, we say that the activity has slack. The exact amount of slack associated with each of these activities can be computed by taking the difference between the latest start times and the earliest start times. This is done in Table 15-3. Notice that activities A, C, E, and F all have zero slack. These activities therefore define the *critical path* since a delay in any of them would delay the completion of the project. Activities B and D have 4 weeks of slack. This means that if either activity B or D were delayed for up to 4 weeks, the completion of the project in 25 weeks would not be jeopardized. Notice also that if activity B used 2 weeks of slack, only 2 weeks of slack would remain for activity D. When these slack times are violated, a new critical path is born.

PROJECT MANAGEMENT

The management of a complex project requires more than the design and analysis of a network. It requires that all the following steps be undertaken either on a formal or an informal basis. First, the establishment of project objectives. Second, the design of the network. Third, the analysis of the network. Fourth, the initiation of the project. Fifth, the control of the project. And sixth, the completion of the project. These steps are summarized in Table 15-4 and will be discussed in greater detail in the following paragraphs.

Objectives

A project is carried out for the purpose of achieving a set of objectives. For example, the objectives of the project presented in Table 15-1 were to develop, market, and deliver a new oil-fired burner. The project was to take

Table 15-4 The Management of Projects

1	Project objectives
2	Design of the network
3	Analysis of the network
4	Project initiation
5	Project control
6	Project completion

24 weeks or less, ensure a high likelihood of product success, and require an investment of no more than \$100,000.

Time and money are two objectives which are common to most projects. That is why network analysis with its primary emphasis on time has such a broad range of applicability.

Design of the Network

Once the objectives have been established, the network can be designed. This involves the following steps: the identification of those activities which make up the project, their predecessors, the collection of time and cost estimates for these activities, and, finally, the formal construction of the project network.

Identification of Activities and Predecessors Each project must be broken down to its basic activities. The question that always arises is: What is a basic activity? Suppose the project under study is the construction of a house. Certainly the installation of plumbing is an activity which must appear in the network. But is it enough to include "plumbing" as an activity, or should more detail be incorporated in the network by including the 85 activities which the plumber must carry out to complete the job?

If every detail is included, the network may become too large and cumbersome. One remedy, if this detail is needed, is to have a master network which incorporates little detail but, for certain activities, refers to subnetworks which elaborate all the detail. For example, the master network for the house project could include an activity called "plumbing" and a note alongside this activity would refer the reader to a subnetwork for the detailed development of the plumbing network.

Collection of Time and Cost Estimates It is essential that time and cost estimates be as realistic as possible. Consequently, it seems only reasonable that these estimates be collected from the individuals, groups, or managers responsible for undertaking each activity. If done in this way, these estimates will be more useful in the control phase of the project since they represent a commitment from the group responsible for doing the work.

Care must of course be exercised in collecting these estimates. To protect

their group, many managers might quote high estimates. For example, even if a manager feels he may be able to complete an activity in 4 weeks, he might estimate that it will take 5 weeks. The extra week buys some "insurance." But these biased estimates must not be allowed to appear in the final network. Perhaps past experience with certain groups might lead to readjustments of the estimates, or a careful explanation to those groups of the importance of the estimates might help eliminate these biases.

Formal Construction of the Network Now the formal network can be drawn. Since this is a mechanical step, it is not surprising to find that computers are often employed to do this job. Into the computer go the activities and their predecessors, and out of the computer comes a completed network.

Analysis of the Network

Analysis of the network can begin once the construction of the network has been completed. Analysis may include one or more of the following steps: the determination of the critical path and slack for each activity, the reassignment of resources, the increase of resources, and the decrease of activity times. We will now take a closer look at these steps.

Critical Path The critical path is a crucial point of focus in the management of the project. Along this critical path are those activities which may increase or decrease the project length if their completion times differ from the estimates. In the ICM case, the critical path includes activities A, C, E, and F. Activities B and D have 4 weeks of slack, and therefore a delay of 4 weeks or less in either one will have no effect on project completion. At this stage, then, management should prepare to give activities A, C, E, and F special attention: they must be watched very carefully during the execution of the project.

Reassigning Resources In some situations it may be possible to shorten the length of time needed to complete a project by reassigning resources from slack activities to those activities which fall along the critical path. Consider the ICM case. Suppose that the workforce allocated to activity D is reduced from 5 to 3 workers and these extra workers are reallocated to activity C. It has been estimated that this would increase the time necessary to complete activity D from 8 to 10 weeks but decrease the time necessary to complete activity C from 7 to 6 weeks. The new network is drawn in Figure 15-5, and it can be seen that the project can now be completed in week 24.

An advantage of the completed network, then, is that it provides management with the relevant information necessary to make readjustments in resources wherever possible.

Increasing Resources Another advantage of the completed network is that it provides the basis for the efficient assignment of additional resources if

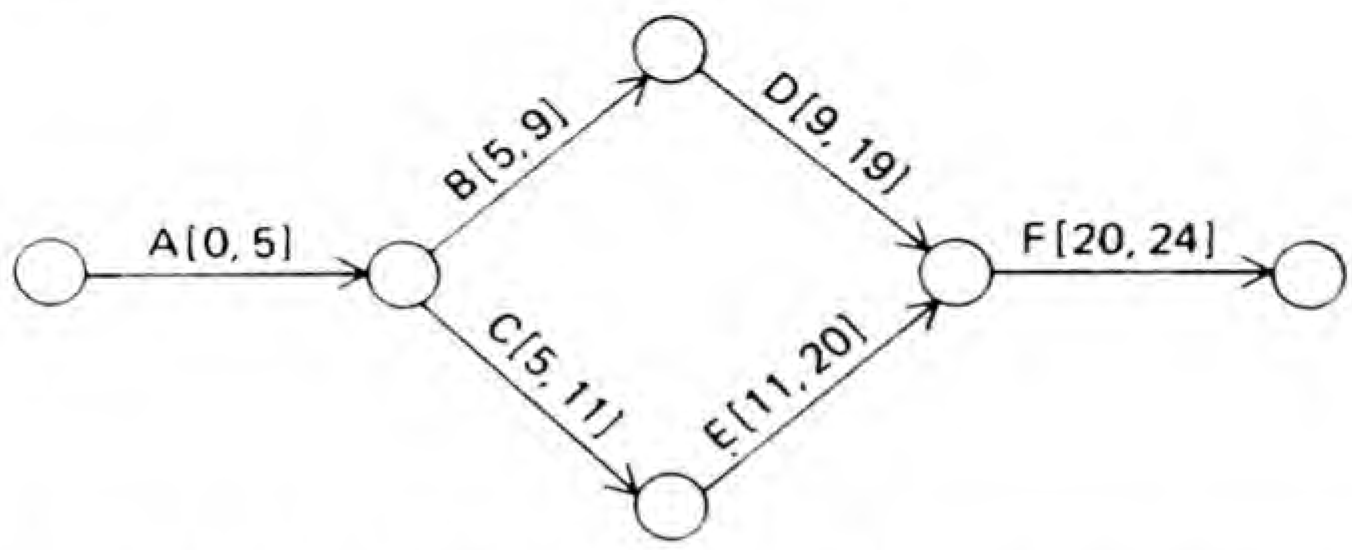


Figure 15-5 The consequence of reassigning resources.

needed. Clearly if workers and machines are to be added in order to reduce the length of time necessary to complete the project, they should be added to activities along the critical path.

Decreasing Times After the network has been completed and the critical path identified, it might be possible to reduce the time required for one or more of the activities along this path. For example, consider the following situation. A large firm was increasing its capacity by building a new production facility in Des Moines, Iowa. The network that was drawn for the project revealed that it would take more than 1 year to complete. In the interest of completing the project earlier, the project manager took a careful look at the critical path. One of the activities along this path involved the purchase of new production equipment. The purchasing department at corporate headquarters had quoted standard times for the components of this activity. It allowed 2 weeks to issue purchase orders, 1 week for the orders to arrive at the suppliers, 5 weeks for the suppliers to ship the orders, 3 weeks for shipping, and 1 week to process the machines through incoming paperwork and inspection. Total time for this activity was 12 weeks. The project manager arranged to reduce this to 7 weeks at no increase in project cost by alerting the purchasing department and suppliers to the urgency of the project. They both agreed to give these orders top priority.

Project Control

After the project has been started, the formal network remains a valuable tool; it identifies the critical path and therefore those critical activities which must be watched very closely, for if these activities are delayed, the entire project is delayed. Slack activities do not require this close attention.

Periodic Updating Since it is likely that some activities proceed faster than expected and some proceed slower than expected, the network must be periodically updated. It is possible that the critical path may change over time.

In order to update the network, it is necessary to collect new information on the status of each activity including its revised activity time. Depending on

the project, this might be done daily, weekly, or monthly. After the data are collected, the network is updated. The critical path may have changed, and activities which had slack before may now be critical.

New Analysis With this revised network, the process of reanalysis must begin. Resources may be shifted, additional resources may be required, or it may be possible to decrease the activity time for some activities by giving them special emphasis.

Perhaps the most valuable aspect of the updated network is that it serves as a very real focal point for planning and controlling the remainder of the project. Without the network all those involved in the project might have different ideas as to what is critical and what is not. The network eliminates this confusion and provides a common ground for discussing the alternative strategies which can be employed to ensure efficient and timely project completion.

THE PROBLEM OF UNCERTAINTY IN TIME ESTIMATES

It has been assumed that activity times are known with certainty. In Table 15-2, for example, it was assumed that activity B required exactly 4 weeks to complete—no more, no less. We have been developing, then, a decision model under certainty.

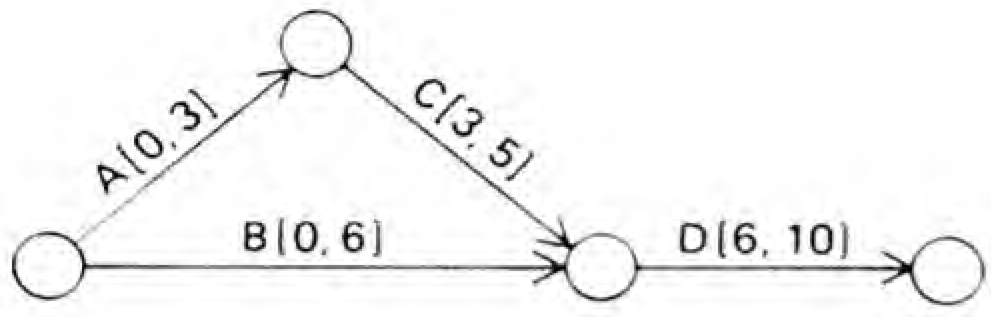
Since the length of time it takes to complete an activity is seldom known with certainty, it would seem more realistic to incorporate risk in the analysis. One way to accomplish this is to express the length of time it would take to complete an activity in three estimates. The first is an optimistic estimate based on encountering few if any difficulties. The second is a most likely estimate, and the third is a pessimistic estimate based on encountering several difficulties.

These three estimates express the risk associated with an activity time. Naturally as the risk gets greater, the spread between the optimistic and pessimistic times will increase. As the risk gets smaller, the spread between these values will decrease, and when the activity times are known with certainty, all three estimates will be the same.

This method has been used for some time. It was first developed in the 1950s for the Polaris missile project, and was known as PERT, an acronym for program evaluation and review technique. Every PERT network, then, includes three estimates for each activity time, and is therefore a network model under risk. A complete development of this model is presented in Appendix A.

SCHEDULING WITH SCARCE RESOURCES

It was also assumed in our network model that resources were available when needed. However, it often happens that a scarcity of resources can interfere with the project schedule.



Activity	Time
A	3 weeks
B	6 weeks
C	2 weeks
D	4 weeks

Figure 15-6 Network for a construction project.

Consider the construction project shown in Figure 15-6. The critical path is BD, and from the information given, the project can be completed in 10 weeks.

Suppose that a new dimension is introduced into the problem: resources. The company responsible for the entire project has a limited quantity of heavy-duty construction equipment. Activity A requires five pieces of this equipment, and activity B requires eight pieces. The company, however, has only seven pieces. This means that it will be impossible to complete the project in 10 weeks. Some work can be started on activity B, but the bulk of the work will have to wait until activity A is completed. The result of this will be the delay of the project.

We can conclude that when resources act as constraints, the network problem and the subsequent scheduling of activities become a difficult job. Models have been developed to incorporate this added dimension, but they are beyond the scope of this book. As long as the availability of resources is not a dominant problem, the models which are presented in this chapter are perfectly adequate.

SUMMARY

The use of network models tends to structure the design and management of projects in several ways. First, it requires that estimates be made of the length of time necessary to complete an activity; second, it requires that precedent relationships be clearly established; and third, it focuses attention on paths which are critical to the completion of the project. Already in widespread use, network models are a proven management tool.

QUESTIONS

1 When does a slack activity become critical?
2 If the "latest finish" of a slack activity is exceeded, will this have any effect upon the completion date of the project?
3 What action can management take to shorten the critical path?
4 Why is it necessary to update a network? How often should this be done?
5 What are the advantages of using network models for the management of large projects?

PROBLEMS

15-1 Return to the ICM case and determine the consequence of a delay in the design of the marketing program by 7 weeks.
15-2 The design of the production system in the ICM case has been delayed by 3 weeks. What effect will this have on the slack time for each activity?
15-3 Draw a network for the construction of a house.
15-4 Draw a network which includes all the activities that would be undertaken by a student applying for admission at a university. The final activity in the network should be the student's arrival on the university campus.
15-5 The activity times and precedent relations are given below for a project. Find the following:
a Critical path
b Earliest start and earliest finish times
c Latest start and latest finish times
d Slack for each activity

Activity	Precedent relation	Time
A	—	4
B	A	3
C	A	7
D	A	9
E	B	11
F	E	4
G	C	6
H	D	3
I	G, H	7
J	F, I	11

15-6 The activity times and precedent relations are given below for a project. Find the following:
a Critical path
b Earliest start and earliest finish times
c Latest start and latest finish times
d Slack for each activity

Activity	Precedent relation	Time
A	—	4
B	—	7
C	—	11
D	A	9
E	B	8
F	C	4
G	F	7
H	D, E	8
I	C	4
J	I	2
K	H, G, J	5

15-7 The activity times and precedent relations are given below for a project. Find the following:

a Critical path
b Earliest start and earliest finish times
c Latest start and latest finish times
d Slack for each activity

Activity	Precedent relation	Time
A	—	4
B	A	7
C	B	6
D	A	3
E	B	2
F	C	5
G	D	6
H	E	11
J	E	5
K	J, G	10
M	F, H	2
N	M, K	5

15-8 The Ace Construction Company has just been awarded a contract to construct a large warehouse. When the project is started in 3 months, it will be the only project on which the company will be working.

Ace owns six pieces of heavy-duty construction equipment. It is unlikely that the company will be able to rent any more on such short notice.

In the table below, the activities, their predecessors, and the activity times are given. In the last column of the table can be found the number of pieces of heavy-duty construction equipment that will be needed during the activity.

Activity	Predecessors	Time	Equipment requirements
A	—	3	2
B	A	4	4
C	A	6	3
D	B	7	5
E	B	3	2
F	C	2	1
G	E	2	6
H	F	4	3
I	G, H	3	1
J	D	7	4
K	J, I	5	2

Under the present circumstances, when can the project be completed? Develop a schedule for the activities.

If Ace could rent as many pieces of equipment as it needed, when would the project be completed?

CASE STUDY: Electrodyne Company

The Electrodyne Company of Palo Alto, California, was founded in 1942 by its current chairman of the board. In those early years it produced only one piece of electronic equipment: a frequency generator. The demand for such generators was so great that the owner soon moved the operation from his garage to more spacious quarters.

Over its 35-year life, the company has shown spectacular growth. It is now a Fortune 500 company with a product line including hand calculators, small computers, electronic medical equipment, and test equipment. Amazingly enough, Electrodyne still manufactures the frequency generator that got the company started, although it has undergone some minor electronic changes.

The company includes twelve divisions located on the East and West Coasts of the United States. Ten of these divisions occupy separate facilities in ten different locations. Two divisions—the electronic test equipment and calculator divisions—occupy the same facility in Palo Alto.

The calculator division has recently undergone a period of substantial growth. As a result of this, the corporate manager of manufacturing, Ralph Leed, has made the decision to separate these divisions. He felt that the calculator division should remain in Palo Alto, but the electronic test equipment division should be located elsewhere.

Shortly after making this decision, Mr. Leed assigned a task force to undertake the project of finding this division a new home. The task force was to manage the project from the search for land to the successful completion of a prototype production run. In fact, the prototype run was to consist mainly of the manufacture of the division's most popular product, the frequency generator.

During the first few task force meetings, the activities which made up the project were discussed and analyzed. It was then decided to divide these responsibilities among the three members of the task force. One member was to be responsible for site selection through plant construction, another member was to be responsible for workforce selection and training, and the third member was to be responsible for designing the production system. The details for these three areas are given in Exhibit A.

When Mr. Leed first met with the task force, he asked that they report to him after preparing a preliminary plan and again halfway through the project.

After the preliminary plan was completed, the group held their meeting. They estimated that the project would cost $471,850 and that it would be completed in 50 weeks. Approval was given to proceed with the project, and it was agreed that the group would meet again at the end of the project's twenty-fifth week.

Last Friday was the twenty-fifth week of the project. Exhibit B represents the report filed by the task force.

Exhibit A

Site Selection through Plant Construction

The first activity which must be undertaken is the search for available land. This activity is estimated to take 6 weeks and cost $10,000. Once this is completed, site evaluation and selection can take place. If the land cost is included in this activity, its total cost will be $64,000. This activity will take 3 weeks.

Once the site is selected, the architects can begin to design the building. The architectural fee will be $11,000, and the design will be completed in 6 weeks.

After the building has been designed, construction bids will be received and the contract awarded. This activity will require 3 weeks and will cost an estimated $1600.

Construction will begin immediately after the contract has been awarded. It will take 28 weeks to complete. Total construction costs are estimated to be $240,000.

Workforce Selection and Training

Once the site for the new facility has been selected, it will be possible to identify those key people who will be willing to move to the new location. These key people, including a division manager, will then become actively involved in the project. It will take an estimated 4 weeks and $2000 to accomplish this activity.

Once the key people have been identified, and the production system has been designed, the next step will include the recruitment of a labor force. This will require newspaper ads and interviews which can be undertaken from a suite of rooms in a local motel or hotel. The estimated cost for this activity is $15,000, and the process will take 4 weeks.

After the labor force has been recruited and after the arrival of the necessary production fixtures and equipment, a training program will be started. It is expected that adequate facilities will be rented for this training program and that it will not be necessary for the new plant to be completed before training can begin. It is estimated that training costs, including rental of training facilities, will total $16,500 and will require 4 weeks to complete.

After the key people have been identified and the production system has been designed, it will be necessary to establish production standards for the new plant. This will take 5 weeks and cost $5000.

The Production System

The design of the production system can begin after the building design has been finalized. The activity will take an estimated 12 weeks and cost $10,000.

After the process has been designed and the key people have been selected, it will be necessary to order the materials handling equipment. This activity, including the cost of the equipment, will total $11,400 and require 3 weeks.

After the materials handling equipment is ordered, it will take 15 weeks to receive delivery. Delivery costs will be $450.

After receipt of the materials handling equipment and after completion of the construction, the materials handling equipment can be installed. Installation will take 3 weeks and cost $3500.

The production fixtures and equipment can be ordered after the production system is designed and the key people are selected. This activity will take 2 weeks, and the cost, including the equipment, will total $61,800.

Delivery of the production fixtures and equipment will take 6 weeks and cost $600.

All parts and subassemblies must be ordered before the prototype phase can begin. During this time it should become apparent whether any new suppliers will be necessary. This activity can start at any time during the project, provided that the parts and subassemblies are ready for the prototype run. Cost of this activity is $6000, and it will take 12 weeks.

Exhibit A (continued)

After training has been completed, and after the materials handling equipment has been installed, the fixtures must be moved and installed in the new plant. This activity will take 1 week and will cost $500.

The last step in the project is the prototype run. This will take 4 weeks and cost $12,500. Before it can be started, however, the following activities must be completed: parts and subassemblies ordered, production standards available, and fixtures moved and installed in the new plant.

Exhibit B Status Report, Twenty-fifth Week

The following activities have been completed:

Activity	Cost incurred
Land search	$ 7,600
Site evaluation and selection	73,200
Building design	10,100
Key personnel selected	1,100
Order parts and subassemblies	8,000
Contract awarded	2,000

The following activity has been started but not completed. The figure in the parentheses represents the portion of the activity completed. The cost figure represents the cost already incurred.

Activity	Cost incurred
Construction $\frac{1}{7}$	$80,000*

The activities which remain have not been started.

*Assume that costs are spread evenly over the construction period.

QUESTIONS

1 Was the original target of 50 weeks a reasonable one?
2 Would it be reasonable to place all the emphasis on the critical path?
3 On the basis of the progress report, will the project be completed on time? If not, what could be done to reduce the time necessary to complete the project?
4 Is the project over or under its budget?
5 (Optional; see Apendix D.) Suppose that the ordering of fixtures and materials handling equipment required that only production system design be completed before they could be started. How would this change the original network, and would this affect the critical path?

CASE STUDY: Ronald's

Ronald's is a fast-food chain with 150 take-out locations throughout the Midwest. Its menu includes hamburgers, french fries, and soft drinks.

At the present time Ronald's is preparing a construction plan for a new location at the Wisconsin Dells, a summer resort area in central Wisconsin.

The site was purchased 4 months ago, and preliminary blueprints of the facility were prepared at that time. On the basis of these plans Ronald's asked three construction companies to bid on the job. Last week the job was awarded to the Wapon Construction Company.

The project has been scheduled to begin on February 1 and to be

Exhibit A Activities and Predecessors

In the first week of the project the final blueprints can be started and the excavation for the foundation can begin.

After the blueprints are drawn, the lumber and building materials can be ordered. At that same time it will also be possible to order the kitchen equipment.

Once the excavation for the foundation has been completed, the foundation can be poured.

The delivery of the lumber and building materials is an activity which follows the ordering of these materials. After the lumber is delivered and the foundation poured, the frame construction can begin.

After the frame is constructed, two activities can begin. First, plumbing and electrical work and, second, exterior finishing.

Following the placement of the order for kitchen equipment, the delivery of the equipment must take place. After the equipment is delivered and both the plumbing and electrical work are completed, the kitchen equipment can be installed.

The interior finishing must wait until the kitchen equipment is installed.

Landscaping and parking facilities must wait until the exterior finish work is completed.

Once the landscape and interior finish work is completed, the project is finished.

These activities and their estimated completion times are given below.

Activity	Estimated completion time in weeks
A. Final blueprints	3
B. Foundation excavation	2
C. Order lumber and building materials	1
D. Order kitchen equipment	3
E. Delivery of lumber and materials	2
F. Pour foundation	1
G. Frame construction	8
H. Exterior finish	5
I. Electrical and plumbing	3
J. Install kitchen	3
K. Delivery of kitchen equipment	5
L. Landscape and parking	3
M. Interior finish	4

completed in no more than 20 weeks from this date. Any delay of the project beyond this 20-week deadline would mean that revenue from the height of the tourist season would be lost.

At a recent meeting with Wapon representatives, the various activities, their time estimates, and precedent relationships were discussed. A summary of these data is presented in Exhibit A.

At the close of the meeting it became clear that it would be very difficult to finish the project in 20 weeks and at the cost which Wapon had originally quoted. Ronald's management then asked Wapon's representatives if they could speed the completion of these activities. Wapon's response was that several activities could be rushed, but that this would be at an additional cost to Ronald's.

The meeting came to a close after Ronald's management asked Wapon to submit in writing an estimate of the costs necessary to ensure completion by week 20.

Exhibit B

Memo

To: Ronald's

From: Wapon Company

Several of the activities comprised in the Wisconsin Dells project can be completed earlier than originally estimated. We are including a list of those activities and the amount by which they can be shortened. Also included are the additional costs incurred if these measures are undertaken.

Activity	No. of weeks reduction	Cost of reduction
B	1	500
G	1	400
	2	800
	3	1300
	4	2000
H	1	200
	2	350
J	1	400
M	1	500
	2	1200

Please note that the time to complete some activities cannot be shortened at all. Only those activities which are listed above can be shortened. Some can be shortened by just 1 week, but others can be shortened by 4 weeks. For example, activity M can be shortened by 1 or 2 weeks. The cost to shorten it by 2 weeks, however, is more than double the cost of shortening it by 1 week.

This morning Ronald's received the information included in Exhibit B.

Time is running short, and Ronald's management would like to decide which of those activities should be rushed to completion.

QUESTIONS

1 Draw the network for this project.
2 Identify the critical path.
3 What is the earliest the project can be completed if no speedup is incurred?
4 What activities should be rushed to completion? What will be the cost of this speedup?

APPENDIX A: PERT

INTRODUCTION

The network model developed in the beginning of this chapter assumed that each activity time was known with certainty. For some projects, this may be an unrealistic oversimplification. In research and development projects, for example, seldom are many activity times known with certainty. In these situations it would be more appropriate to use PERT, a technique which incorporates into the analysis the risk associated with each activity.

TIME ESTIMATES

In PERT it is assumed that those responsible for each activity can make three estimates. The first is an estimate of activity time if everything proceeds exceptionally well. It is called an optimistic time. The second is an estimate of the most likely time to complete the activity. The third is an estimate which allows for the occurrence of difficulties and delays in completing the activity. It is called a pessimistic time.

An example of a project with these PERT time estimates is given in Figure 15A-1. The three time estimates for each activity can be summarized as an expected value in the following way:

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

where t_e = expected time
t_o = optimistic time
t_m = most likely time
t_p = pessimistic time

For example, the expected time for activity A is 4 weeks.

Activity	Optimistic time	Most likely time	Pessimistic time
A	1 week	4 weeks	7 weeks
B	4	7	16
C	5	8	11
D	1	7	13

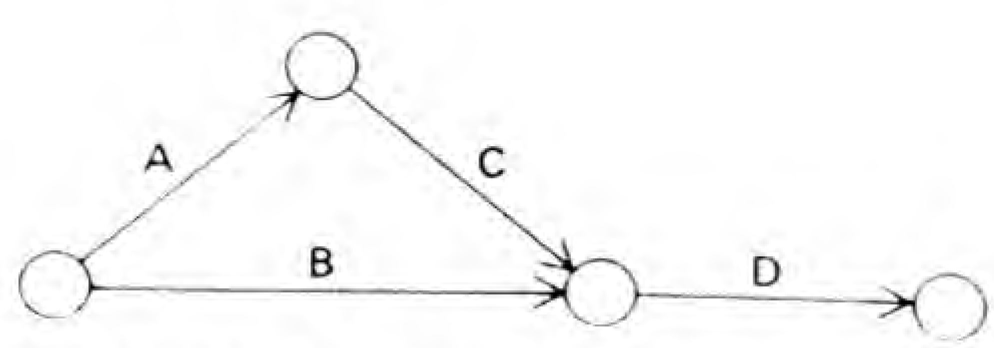


Figure 15A-1 PERT network and time estimates.

Activity	Expected time
A	4 weeks
B	8
C	8
D	7

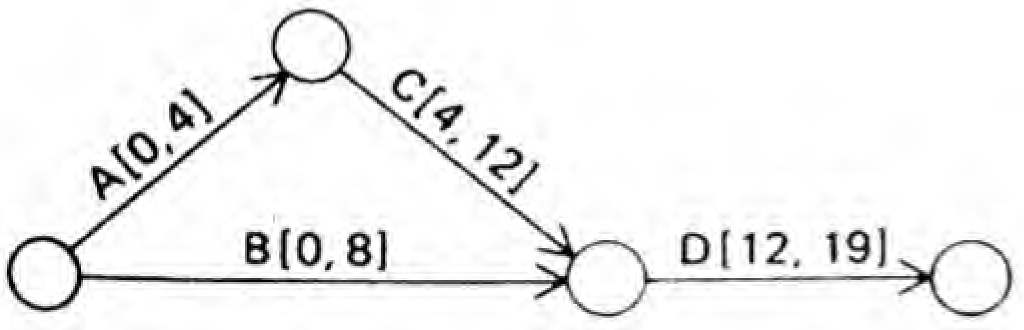


Figure 15A-2 PERT network and expected times.

$$t_e = \frac{1 + 4(4) + 7}{6} = \frac{24}{6} = 4$$

This and the remaining expected values are presented in Figure 15A-2.

A well-developed theory lies behind the computation of the expected value in this way. It has been found that a "beta probability distribution" best describes the distribution of time estimates. The formula given above is the method for computing the mean of a beta distribution. Not only does this method make sense theoretically, but it makes sense intuitively. It seems reasonable to weight the most likely estimate much more heavily than the optimistic and pessimistic estimates. Since a full development of the theory behind the beta distribution is beyond the scope of this chapter, the computation of these expected values will have to be accepted on this intuitive basis.

ANALYSIS OF THE NETWORK

Once the expected values have been computed, the analysis of the network can proceed using these values. This is done in Figure 15A-2. The critical path is ACD, and the earliest that the project can be completed is 19 weeks.

THE RISK ASSOCIATED WITH PROJECT COMPLETION

Let's take a closer look at this project completion date. Our results showed that the project would be completed in week 19. Clearly there was *no* certainty associated with the activity times: three estimates were given for each one. Then why should only one estimate be given for the completion of the project? By providing this single number, 19 weeks, the impression is given that this is a decision problem under certainty. It is not, and our next step is to express the risk associated with project completion in an explicit way.

The risk associated with project completion can be expressed by combining the risk of those activities which lie along the critical path ACD.

To turn to activity A, its risk can be described by the standard deviation of the activity completion time. Two extreme estimates are made for all activities: optimistic and pessimistic. Since plus and minus 3 standard deviations, or a total of 6 standard

deviations, should include nearly the full range of all possible completion times for an activity, the standard deviation can be estimated in the following way:

$$s = \frac{t_p - t_o}{6}$$

For activity A the standard deviation is therefore 1 week.

$$s = \frac{7 - 1}{6} = 1$$

The standard deviations for the remainder of the activities along the critical path are given in Table 15A-1.

In combining these measures of risk for each activity to determine the risk of the project, it is necessary first to convert the standard deviations to variances and then add. To convert a standard deviation to a variance simply requires squaring the standard deviation. This is done in Table 15A-1.

The last step is to add the variances along the critical path and then take the square root of this sum. The result will be the standard deviation of the critical path. Returning to our example, we have the following.

$$\text{Variance of critical path} = 1 + 1 + 4 = 6$$

$$\text{Standard deviation of critical path} = \sqrt{6} = 2.45$$

We can therefore conclude that the completion of the project depicted in Figure 15A-1 can be described by a distribution whose mean is 19 and whose standard deviation is 2.45. In fact a stronger statement can be made about this and other distributions which describe the risk associated with a critical path: They are normal.

Since the distribution is normal, certain conclusions can be drawn about the completion of the project with the help of Table 1 at the back of the book. For example, it can be said that there is a 68.26 percent chance that the project will be completed within plus or minus 1 standard deviation of the mean, or between

$19 \pm 1(2.45)$

16.55 and 21.45 weeks

There is a 95.44 percent chance that the project will be completed within plus or

Table 15A-1 Computation of Standard Deviation and Variance for Activities along the Critical Path

Activity	Optimistic time	Pessimistic time	Standard deviation	Variance
A	1	7	1	1
C	5	11	1	1
D	1	13	2	4

minus 2 standard deviations of the mean, or between

$19 \pm 2(2.45)$

13.10 and 24.90 weeks

These probability statements add a new dimension to the analysis and communicate, in an explicit manner, the risk surrounding the completion of the project. To many project managers, this extra dimension is well worth the extra effort.

PROBLEMS

15A-1 Find the critical path for the following project:

Activity	Precedent relationship	Time estimates, days: Optimistic	Most likely	Pessimistic
A	—	2	4	7
B	A	1	3	5
C	A	2	3	4
D	B	3	5	9
E	C	5	6	8
F	D	2	4	6
G	E	3	5	8
H	D	4	6	7
I	H	4	6	9
J	F	1	5	7
K	G	1	4	6
L	I, J, K	2	3	4

15A-2 Determine the likelihood that the project described in problem 15A-1 will be completed in less than 30 days.

15A-3 Find the critical path of the following project and determine the likelihood that it will be completed in more than 21 days.

Activity	Precedent relationship	Time estimates, days: Optimistic	Most likely	Pessimistic
A	—	2	5	8
B	—	4	5	7
C	—	6	8	12
D	A	1	3	7
E	D	2	4	5
F	B	8	9	11
G	C	4	5	7
H	G	6	8	10
I	E, F, H	1	2	3

15A-4 Find the critical path of the following project. What is the likelihood that the project will be completed between 21 and 23 days from the start?

Activity	Precedent relationship	Time estimates, days		
		Optimistic	Most likely	Pessimistic
A	——	2	4	6
B	——	1	3	5
C	A	4	5	6
D	C	4	7	10
E	B	5	6	7
F	E	1	2	3
G	E	2	2	2
H	F	3	5	7
I	G	3	6	9
J	D, H, I	2	4	12

15A-5 Find the critical path of the following project. What is the likelihood that the project will require more than 11 weeks?

Activity	Precedent relationship	Time estimates, weeks		
		Optimistic	Most likely	Pessimistic
A	——	2	3	4
B	A	4	5	6
C	——	2	6	10
D	B, C	1	2	3

15A-6 Is it likely for the noncritical path in problem 15A-5 to take longer than 10 weeks? Compute exactly how likely this is.

Since it is likely that a noncritical or near-critical path can become critical, it therefore seems reasonable for management to watch noncritical as well as critical paths. Should all noncritical paths be watched with equal emphasis?

What effect can these near-critical paths have on the kinds of probability statements that were made in problem 15A-5?

If you read Appendix B, you will find that these problems are resolved by the use of PERT simulation techniques.

APPENDIX B: PERT Simulation

INTRODUCTION

In Appendix A the probability estimates of project completion were based solely on the mean and standard deviation of the critical path. It is quite possible, however, that

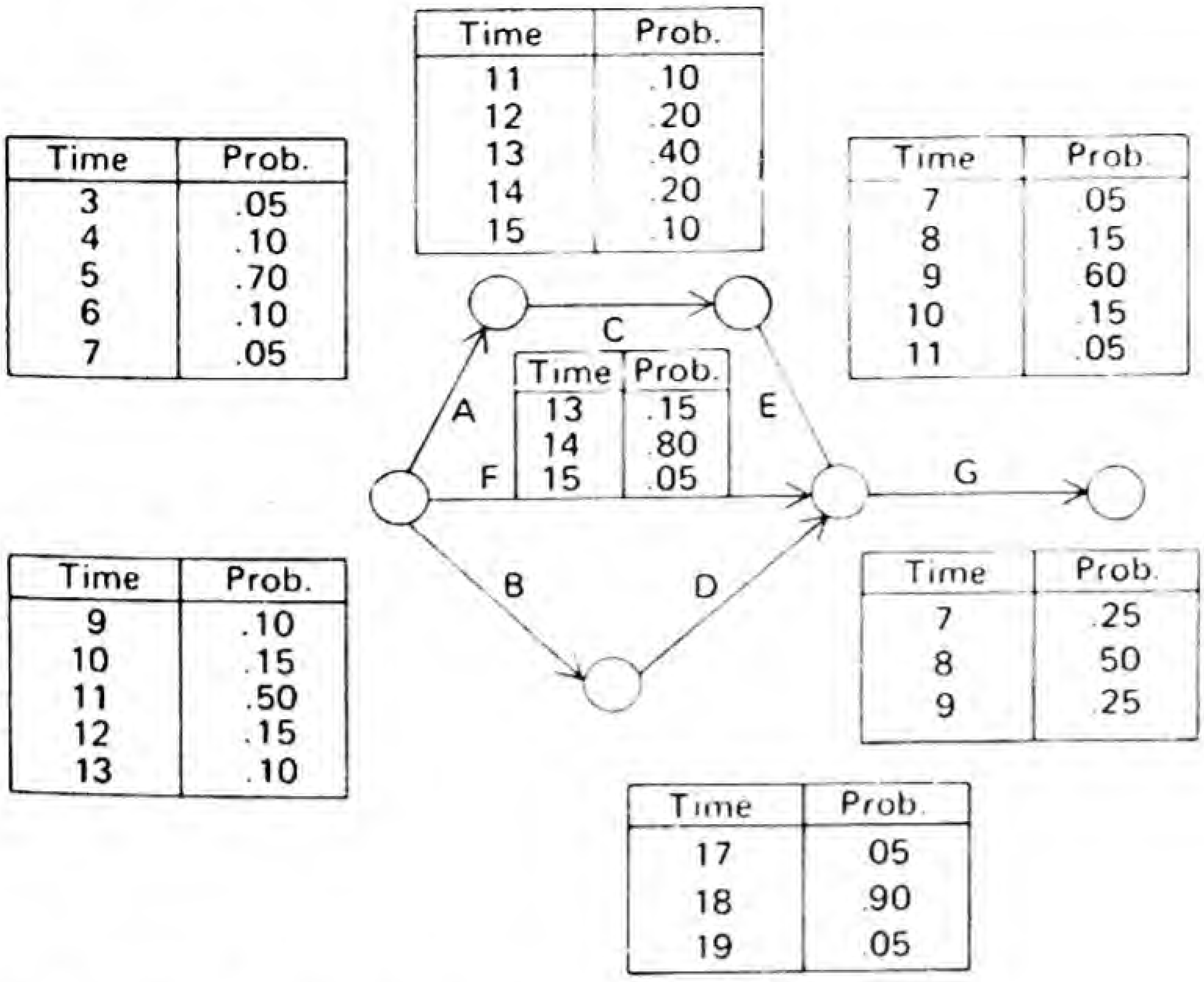


Figure 15B-1 PERT simulation network.

the completion of a project might also hinge on the mean and standard deviation of near-critical paths. If there is a noncritical path, for example, whose length is close to that of the critical path but whose standard deviation is larger, it is certainly possible for this path to become critical. Therefore an analysis which focuses only on the critical path—as PERT did—can lead to incorrect probability statements concerning project completion. A more appropriate analysis would require that attention be given not only to critical paths but also to those which are near-critical. To accomplish this, we now turn to PERT simulation.

THE NETWORK AND SUBJECTIVE PROBABILITY ESTIMATES

A PERT simulation network is shown in Figure 15B-1. Rather than an optimistic, most likely, and pessimistic estimate for each activity, a subjective probability distribution of activity times is determined in the following way. The person responsible for each activity is asked to list the various times that the activity could take to be completed. Then for each of these times, the likelihood of occurrence must be estimated. For example, it was determined that activity A could take 3, 4, 5, 6, or 7 weeks. The likelihood that it would take 3 weeks was determined to be 5 percent, the likelihood that it would take 4 weeks is 10 percent, the likelihood that it would take 5 weeks is 70 percent, the likelihood that it would take 6 weeks is 10 percent, and the likelihood that it would take 7 weeks is 5 percent. This and the remaining probability distributions for each activity are given in Figure 15B-1.

SIMULATION

Each replication in the simulation process will include the following steps:

1 Generate an activity time for each activity from its distribution
2 Determine the critical path and the project length
3 Repeat steps 1 and 2 until the desired number of replications are completed

First two-digit numbers are assigned to the times associated with each distribution, and then the random number table is used to generate activity times from each of these distributions. A sample replication is shown in Figure 15B-2. The result of this replication indicates that the project will be completed in 36 weeks and that the critical path includes activities B, D, and G. This result and the result of 19 additional replications are shown in Table 15B-1.

The average completion date for these 20 replications is 37.05 weeks. A histogram of completion dates is shown in Figure 15B-3. From it we can estimate the likelihood of different completion times. For example, the likelihood that the project will take more than 37 weeks is 6/20, or 30 percent. This estimate, however, is based on only 20 replications, and perhaps it would be better to complete more replications before an estimate is made.

Another useful piece of information which the simulation provides is a critical index for each activity. At the bottom of Table 15B-1, the percentage of the time that each activity was critical is shown. This is called a *critical index* and can be used as a measure of the likelihood that an activity will be critical. For example, activity A has a critical index of .25. This means that if the project were undertaken repeatedly, activity A would be found on the critical path 25 percent of the time. Said another way, activity A has a 25 percent chance of being on the critical path. With these indices, management knows which activities to watch closely, which to watch occasionally, and which are unlikely to be critical.

Quite noticeable, then, should be the fact that the identification of a critical path is missing. Only a critical index is given for each activity. This information, however, is

Activity	Time
A	5
B	11
C	12
D	18
E	10
F	13
G	7

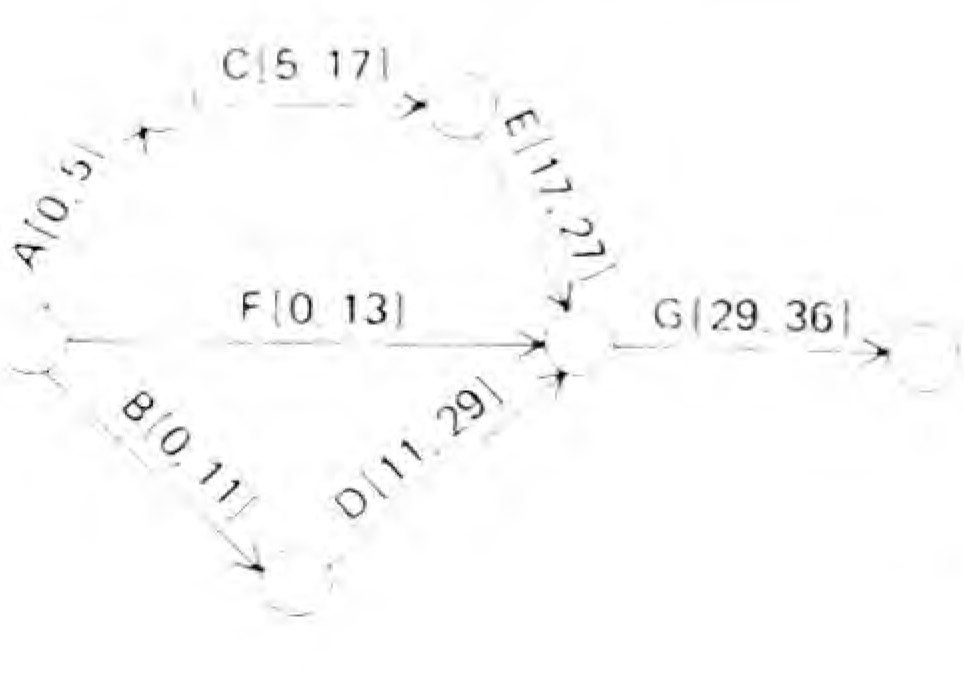


Figure 15B-2 Sample replication.

Table 15B-1 Simulation Results

	Critical path							
Replication	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**Completion date**
1	—	1	—	1	—	—	1	36
2	—	1	—	1	—	—	1	38
3	—	1	—	1	—	—	1	38
4	1	—	1	—	1	—	1	37
5	1	1	1	1	1	—	1	37
6	—	1	—	1	—	—	1	38
7	—	1	—	1	—	—	1	37
8	—	1	—	1	—	—	1	37
9	—	1	—	1	—	—	1	39
10	—	1	—	1	—	—	1	37
11	1	1	1	1	1	—	1	37
12	—	1	—	1	—	—	1	36
13	—	1	—	1	—	—	1	37
14	1	1	1	1	1	—	1	37
15	1	—	1	—	1	—	1	38
16	—	1	—	1	—	—	1	36
17	—	1	—	1	—	—	1	36
18	—	1	—	1	—	—	1	38
19	—	1	—	1	—	—	1	37
20	—	1	—	1	—	—	1	36
Critical index	.25	.90	.25	.90	.25	.00	1.0	

more useful than the information provided by the basic PERT method described in Appendix A, for it takes into account not only critical paths but near-critical paths as well.

COMPARISON OF THE RESULTS

It is very interesting to compare the results of the simulated PERT network with the results that would have been achieved if the network in Figure 15B-1 were analyzed using *only* expected values as if it were a decision problem under certainty. The expected values for each activity are computed and entered in Figure 15B-4. The earliest that the project can be completed, according to our new analysis, is the thirty-seventh week. The critical path is BDG.

In certainty analysis a single critical path is identified. In our problem it is implied that path BDG is the only one necessary for attention. The importance of the other paths is placed in the background. Under certainty analysis one never knows just how important these other paths are.

The major benefit associated with the incorporation of risk is that the relative importance of all activities is identified. There is no longer a critical path, only critical indices.

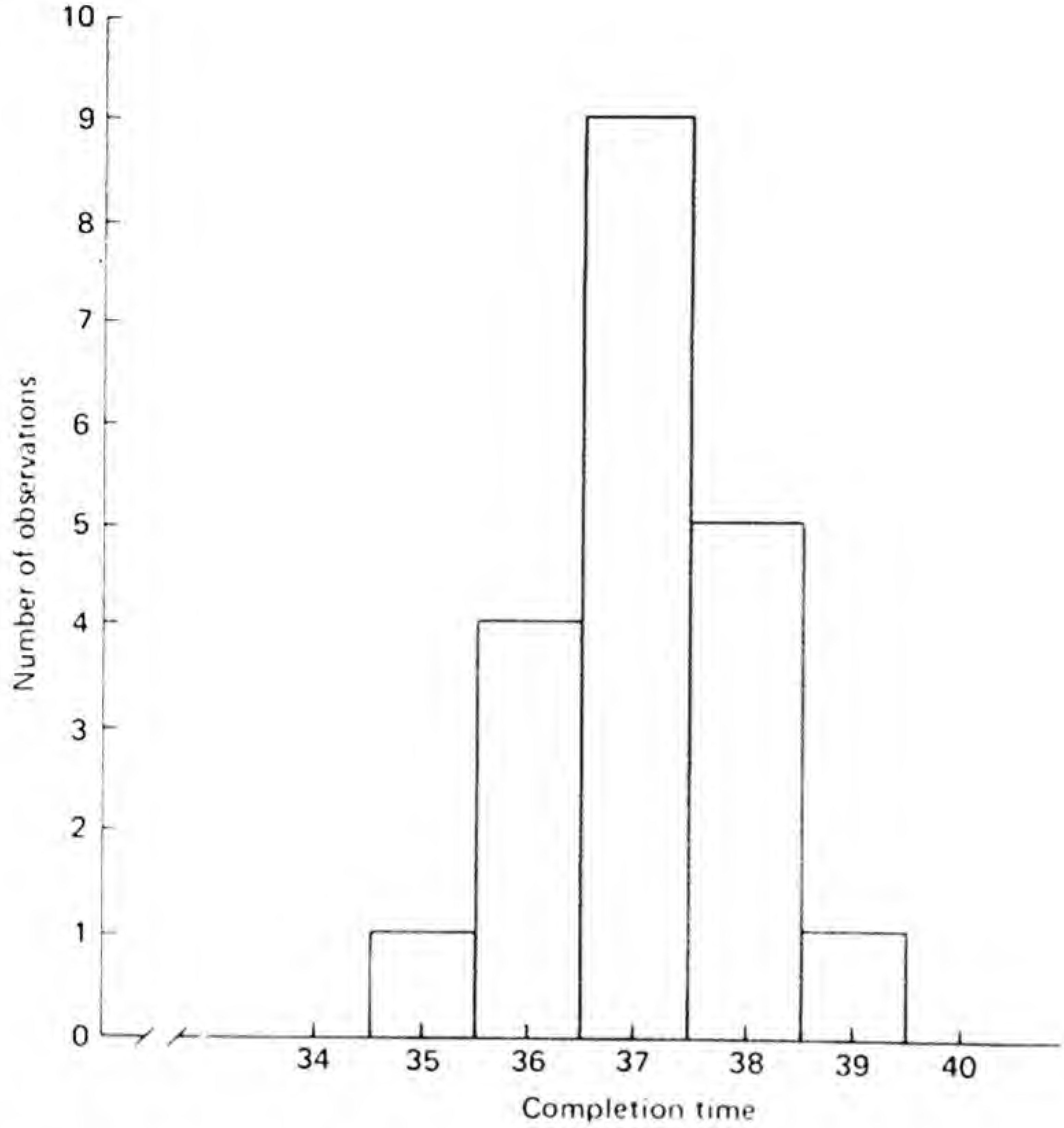


Figure 15B-3 Distribution of simulated completion dates.

Activity	Expected value
A	5
B	11
C	13
D	18
E	9
F	13.9
G	8

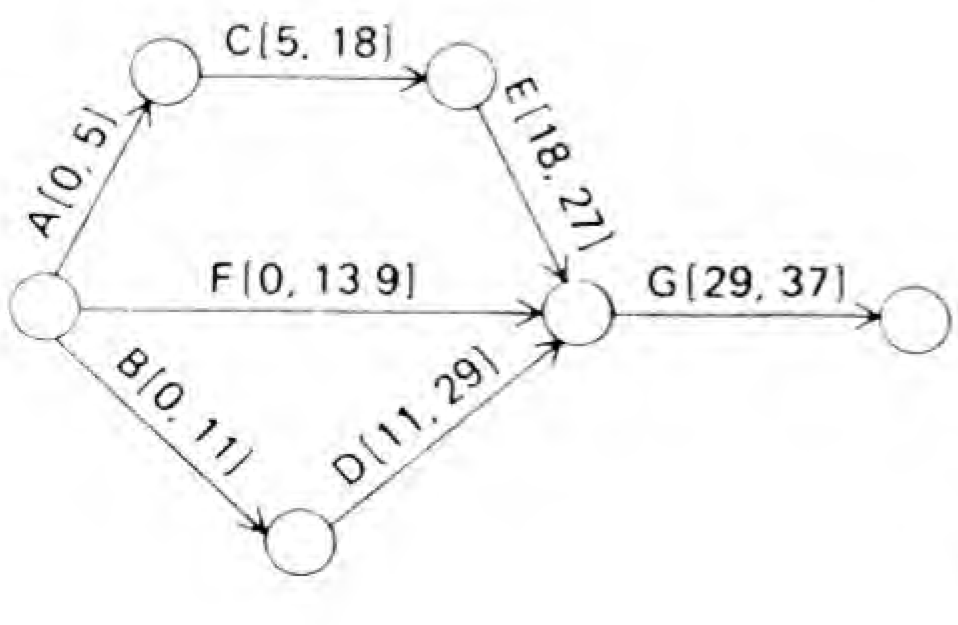
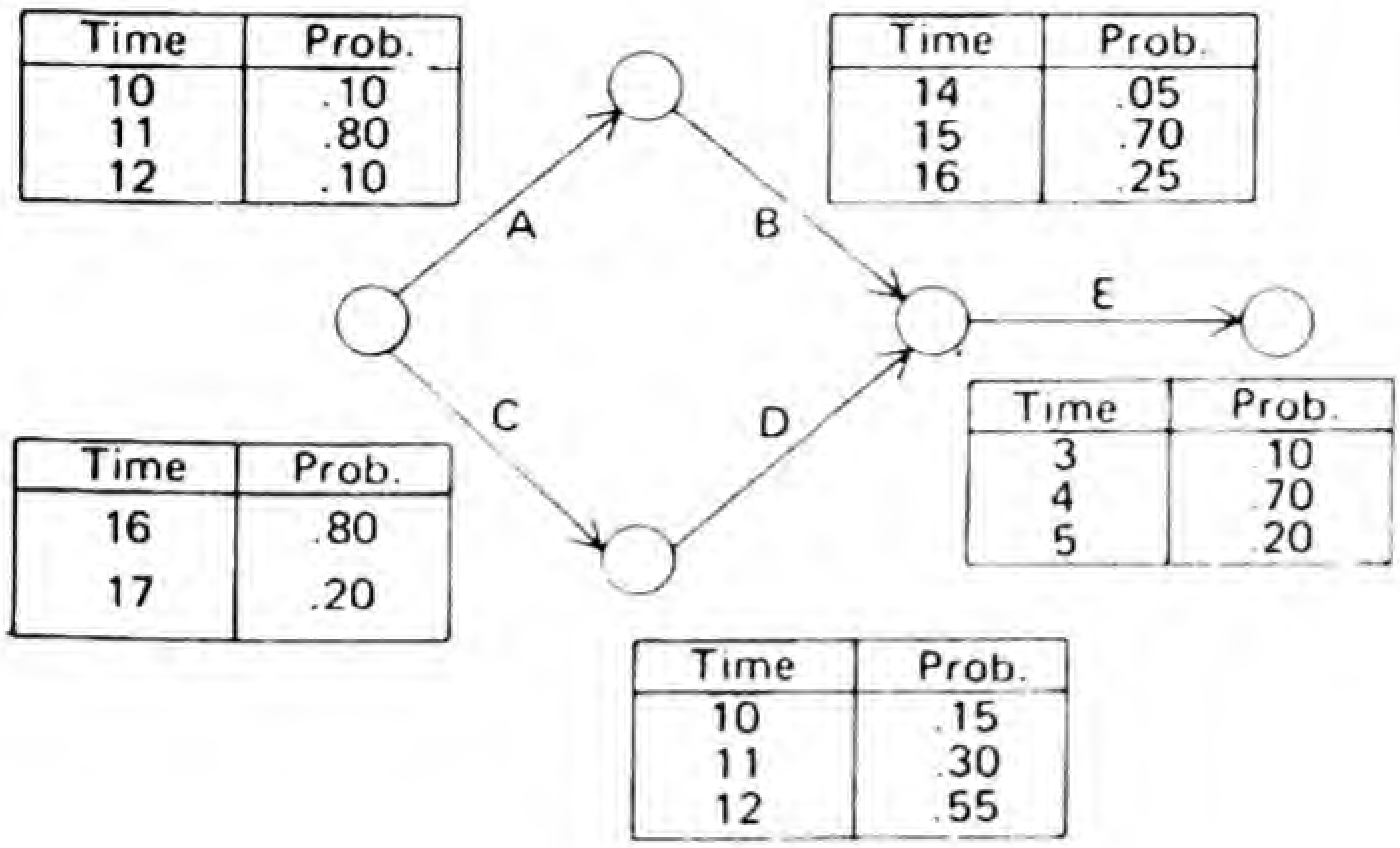


Figure 15B-4 The PERT simulation problem of Figure 15B-1 solved as a network problem under certainty.

PROBLEM

15B-1 Given the following network and subjective probability estimates of activity completion times, find the likelihood that the project will be completed in 29 days. Base your answer on 10 replications.



Time	Prob.
10	.10
11	.80
12	.10

Time	Prob.
14	.05
15	.70
16	.25

Time	Prob.
16	.80
17	.20

Time	Prob.
10	.15
11	.30
12	.55

Time	Prob.
3	.10
4	.70
5	.20

APPENDIX C: Critical Path Method

INTRODUCTION

The critical path method (CPM) was developed for situations where the project manager has the flexibility to channel resources to certain activities in the interest of improving the project's completion date. It does not, however, incorporate PERT'S ability to deal with risk. Therefore, CPM is a model under certainty.

ACTIVITY AND PROJECT COSTS

The central idea in CPM is that certain costs vary with time. These costs can be broken down into two categories: direct activity costs and indirect project costs. *Direct activity costs* are those costs which are incurred by the use of labor, machines, and money. There is an inverse relationship between these costs and the length of time it takes to complete an activity. In Figure 15C-1*a* it can be seen that as the expected completion date of an activity is shortened, the direct cost of performing that activity increases. This follows because additional resources are necessary to accomplish a decrease in activity time.

The second set of costs is related to the *indirect costs* associated with the entire project. These include supervision, some supplies, and a host of fixed expenses which are normally allocated to the project. Many of these increase with the duration of the project. Therefore if the total project length can be shortened, these costs will be saved. The direction of these project-related costs is shown in Figure 15C-1*b*.

We have identified two sets of costs, each working in an opposite direction. If an activity is shortened, certain costs increase; if the project is shortened other costs decrease. It can therefore be concluded that additional resources can be applied to an

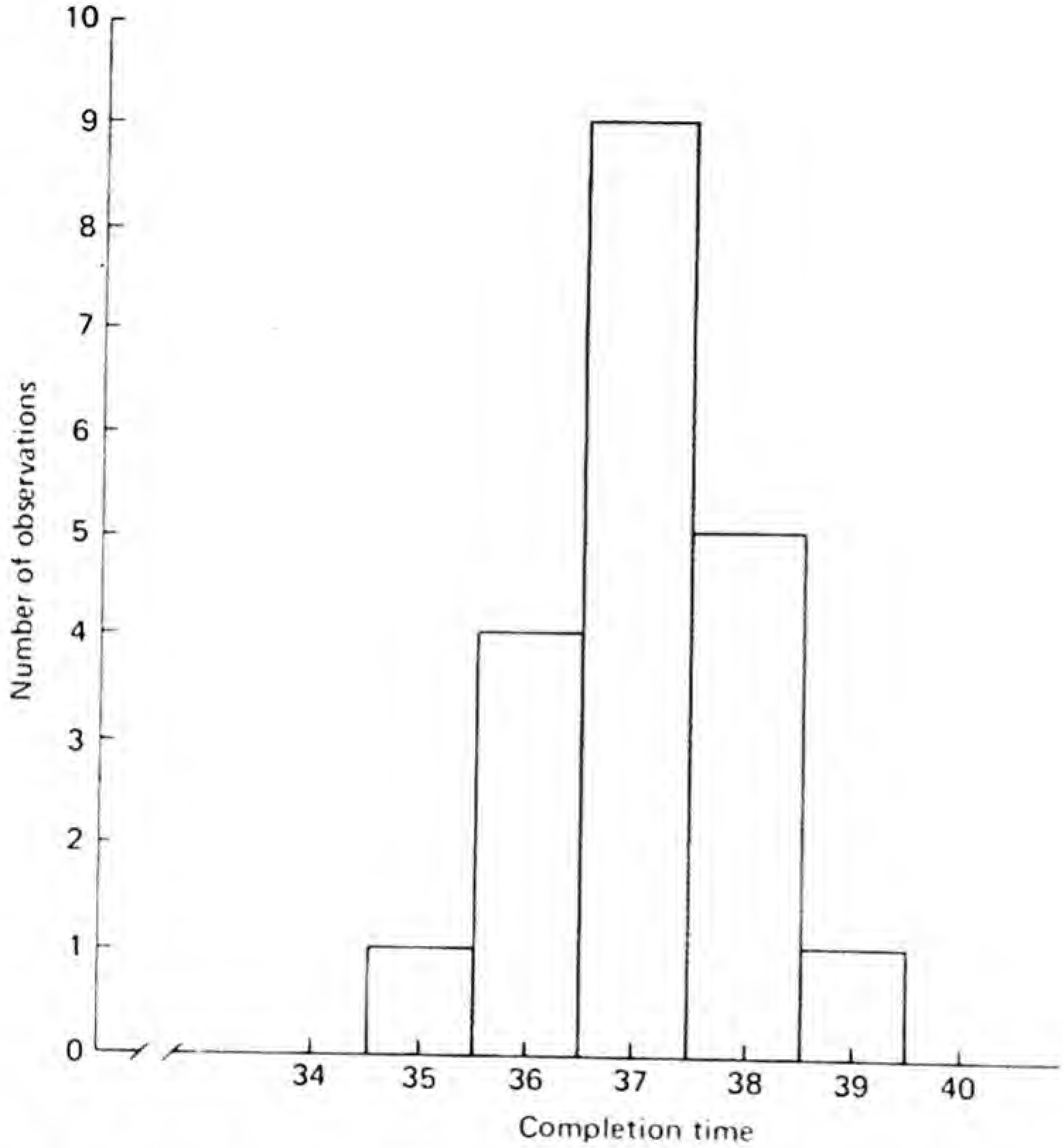


Figure 15B-3 Distribution of simulated completion dates.

Activity	Expected value
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B	11
C	13
D	18
E	9
F	13.9
G	8

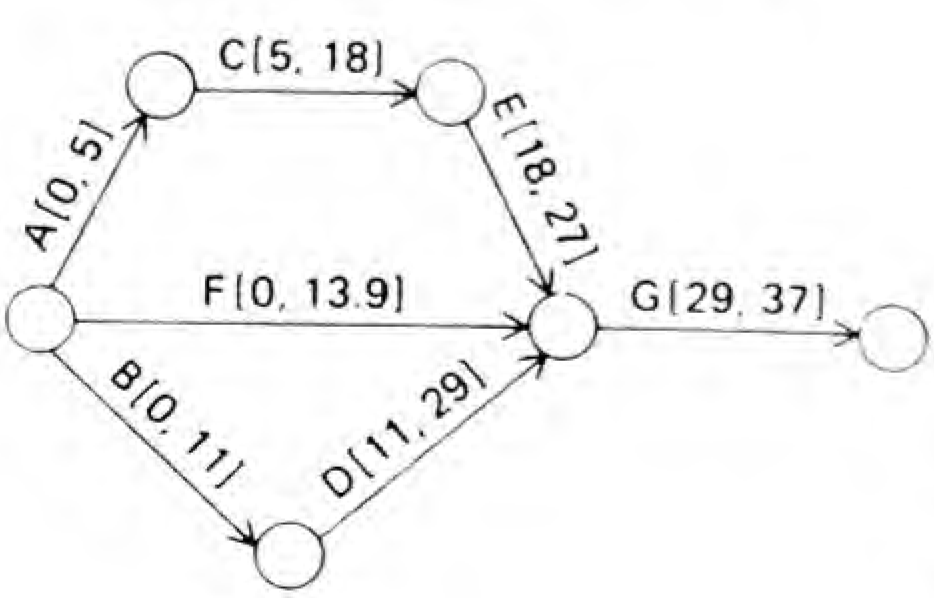
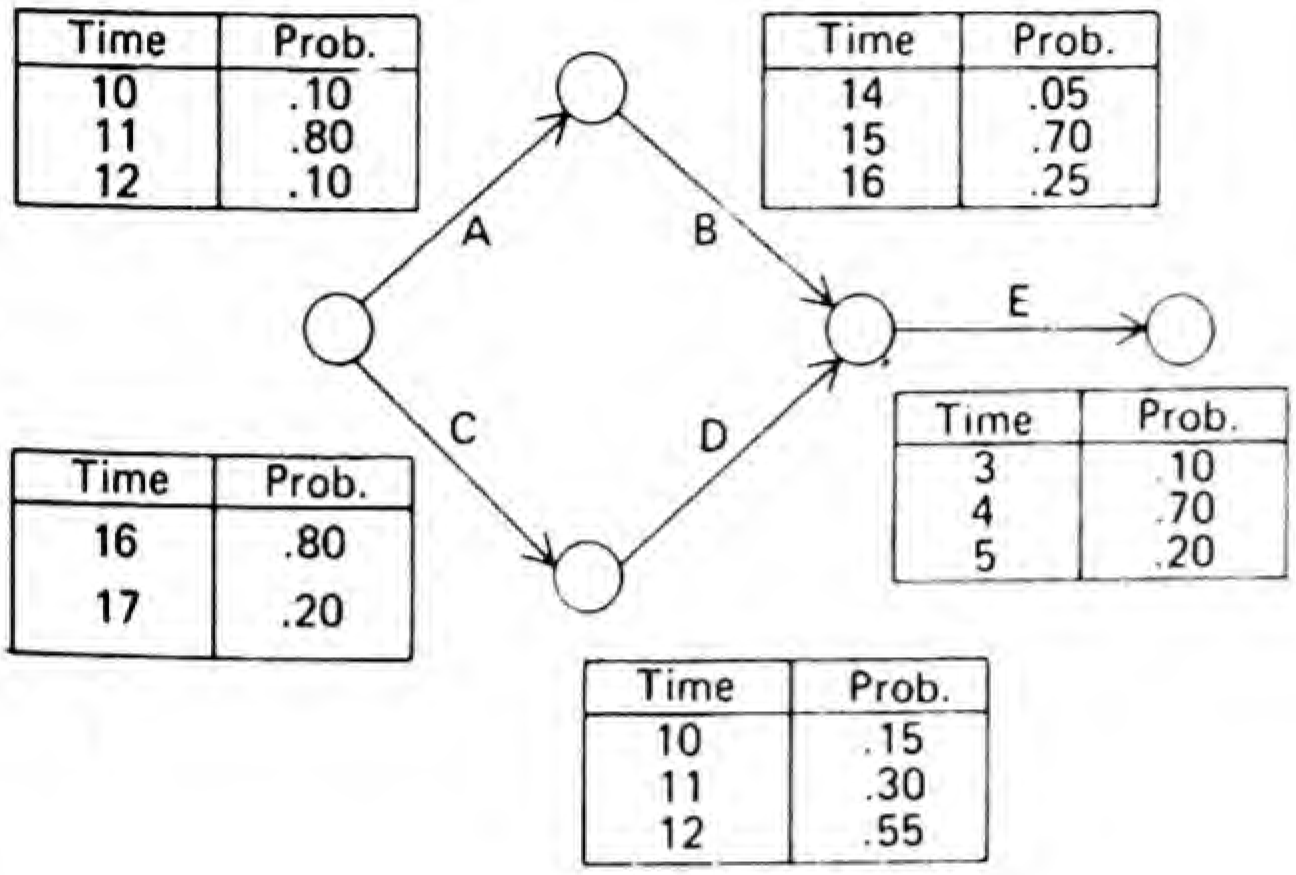


Figure 15B-4 The PERT simulation problem of Figure 15B-1 solved as a network problem under certainty.

PROBLEM

15B-1 Given the following network and subjective probability estimates of activity completion times, find the likelihood that the project will be completed in 29 days. Base your answer on 10 replications.



Time	Prob.
10	.10
11	.80
12	.10

Time	Prob.
14	.05
15	.70
16	.25

Time	Prob.
3	.10
4	.70
5	.20

Time	Prob.
16	.80
17	.20

Time	Prob.
10	.15
11	.30
12	.55

APPENDIX C: Critical Path Method

INTRODUCTION

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The second set of costs is related to the *indirect costs* associated with the entire project. These include supervision, some supplies, and a host of fixed expenses which are normally allocated to the project. Many of these increase with the duration of the project. Therefore if the total project length can be shortened, these costs will be saved. The direction of these project-related costs is shown in Figure 15C-1*b*.

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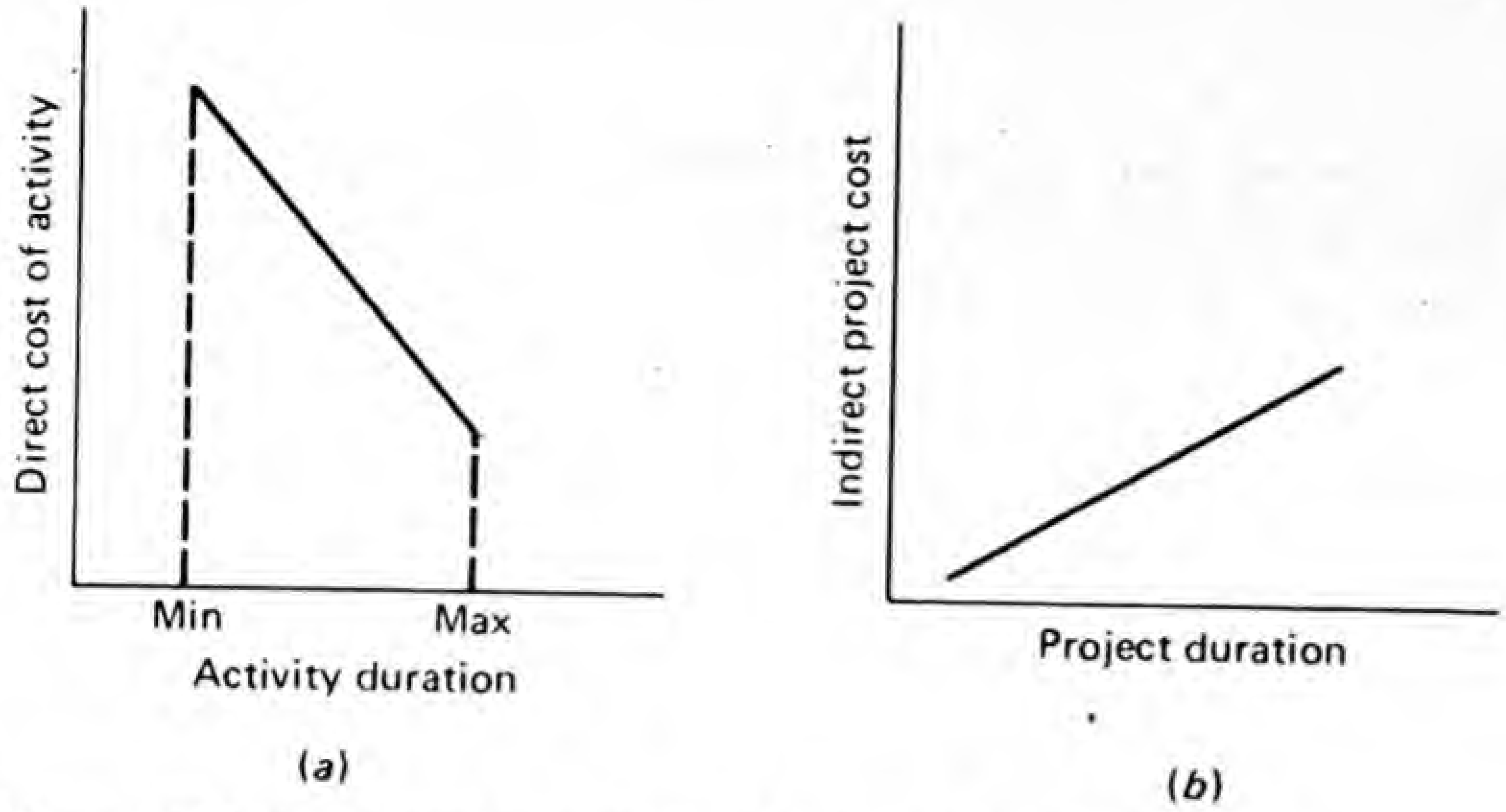


Figure 15C-1 Direct activity and indirect project costs.

activity provided that the cost of the additional effort is less than the saving which accrues from a shorter project cycle.

Consider the example shown in Figure 15C-2. The data above each arrow represent the average time or normal time for the completion of that activity, the minimum possible completion time if extra resources were employed, and finally, the cost of reducing the average time in dollars per day. Activity B, for example, has a normal completion time of 4 days and can be completed in a minimum of 2 days if sufficient resources are used. The cost of reducing the completion time is \$1 per day. Therefore if the activity time is reduced from 4 days to 3 days, an additional cost of \$1 is incurred. If the time is reduced to 2 days, the total cost of reduction is \$2. It is not possible to reduce the time further.

CPM TIME GRAPH

The problem is solved on a CPM time graph. First a time scale is drawn, after which the activity arrows are entered at their earliest start time. This is illustrated in Figure 15C-3. The only way that this schedule can be reduced is by the commitment of additional resources. For each day that the schedule is reduced, a saving of \$5 per day in indirect project costs is realized. Therefore if the project is shortened by 3 days, this indirect saving would total \$15.

Now we can compare these direct costs and indirect project savings to see whether any action should be taken. As it stands now, the project will be completed in

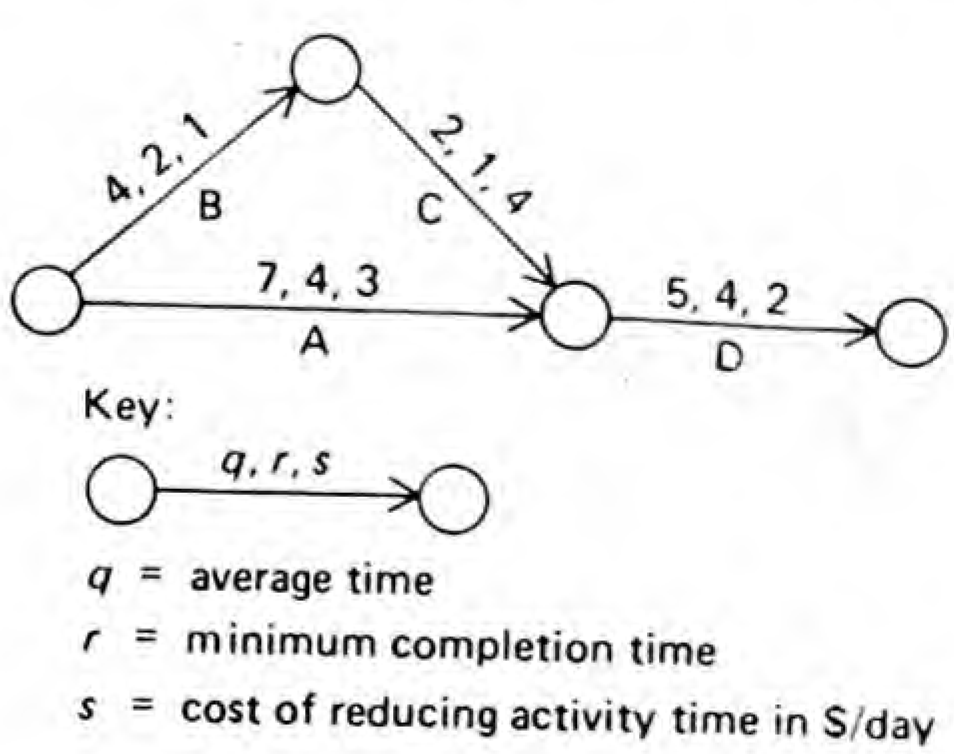


Figure 15C-2 Critical path method example.

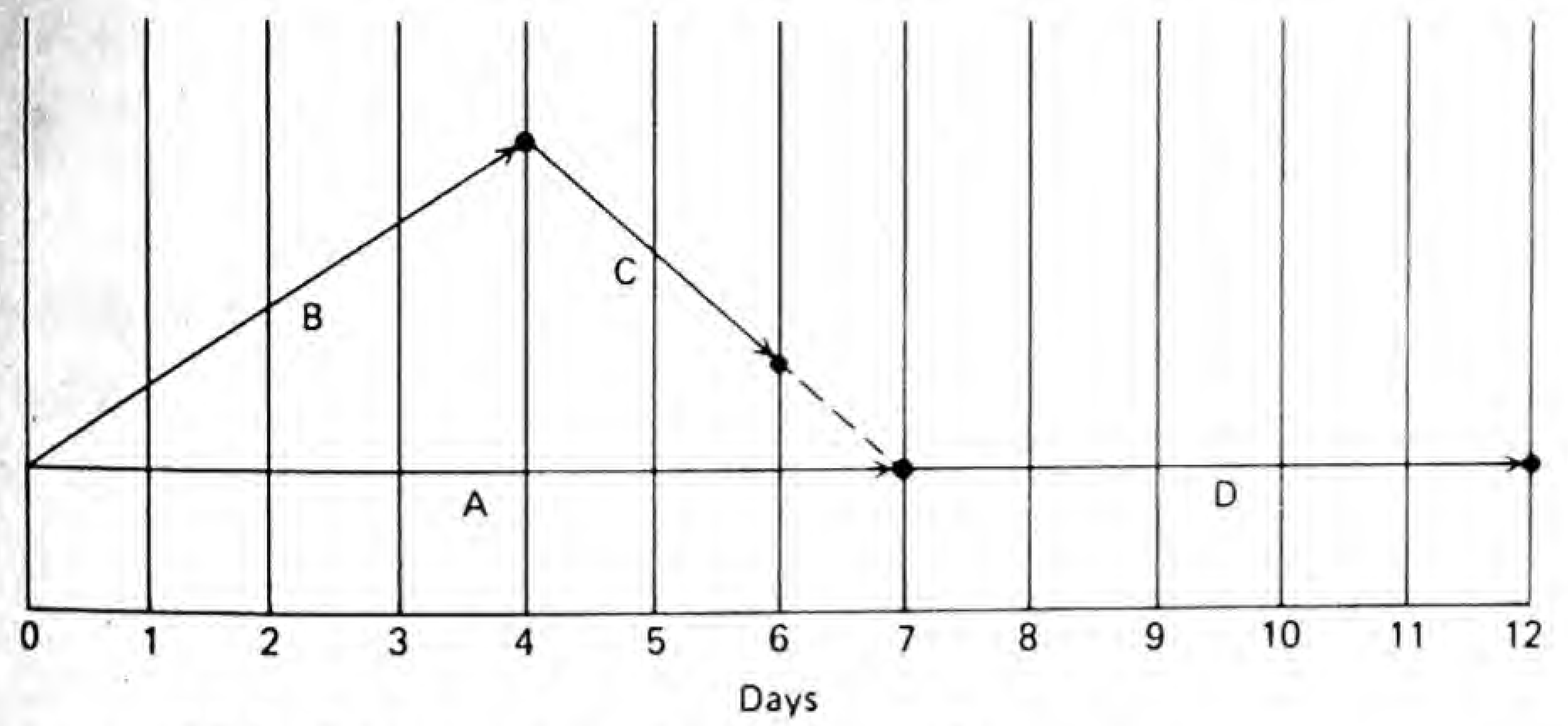


Figure 15C-3 Critical path method time graph.

12 days. The critical path of the project is AD, and to shorten the project length, it only makes sense to focus on activities which lie on the critical path. Therefore, either activity A or activity D should be shortened. Activity A can be reduced to a minimum of 4 days at a cost of \$3 per day, and D can be reduced to a minimum of 4 days at a cost of \$2 per day. First we will choose that activity which is least costly to shorten. Activity D is therefore shortened to its minimum of 4 days. Since the project now takes 1 day less, the indirect cost saving is \$5 and the net saving is \$5 − \$2 = \$3.

If we ignore those costs which are not under our control at this stage in the planning process, the controllable cost of the schedule *before* any changes were made was:

$$\begin{aligned}\text{Cost} &= \text{cost of reducing completion date} + \text{indirect costs}\\ &= 0 + (12 \text{ days})(\$5 \text{ per day})\\ &= \$60\end{aligned}$$

Now that we have reduced the schedule to 11 days, the cost becomes

$$\text{Cost} = \$2 + 11(\$5) = \$57$$

We have therefore effected a \$3 cost saving. The new CPM graph reflecting this change is found in Figure 15C-4.

The critical path of our new CPM graph is still AD. Activity D can no longer be shortened. The cost of shortening A is \$3 per day and it *could* be shortened to 4 days, but after it is shortened by 1 day, another critical path, BCD, emerges. Moving one step at a time, we first reduce A by 1 day. The controllable cost of this new 10-day schedule becomes

$$\text{Cost} = (\$2 + \$3) + 10(\$5) = 5 + 50 = \$55$$

The first cost in the first parentheses represents the \$2 cost that was incurred in the previous step, when we shortened the project from 12 to 11 days. The \$3 is the cost of shortening the project from 11 to 10 days. The new CPM graph is drawn in Figure 15C-5.

We now have two critical paths, AD and BCD. Since D cannot be shortened any

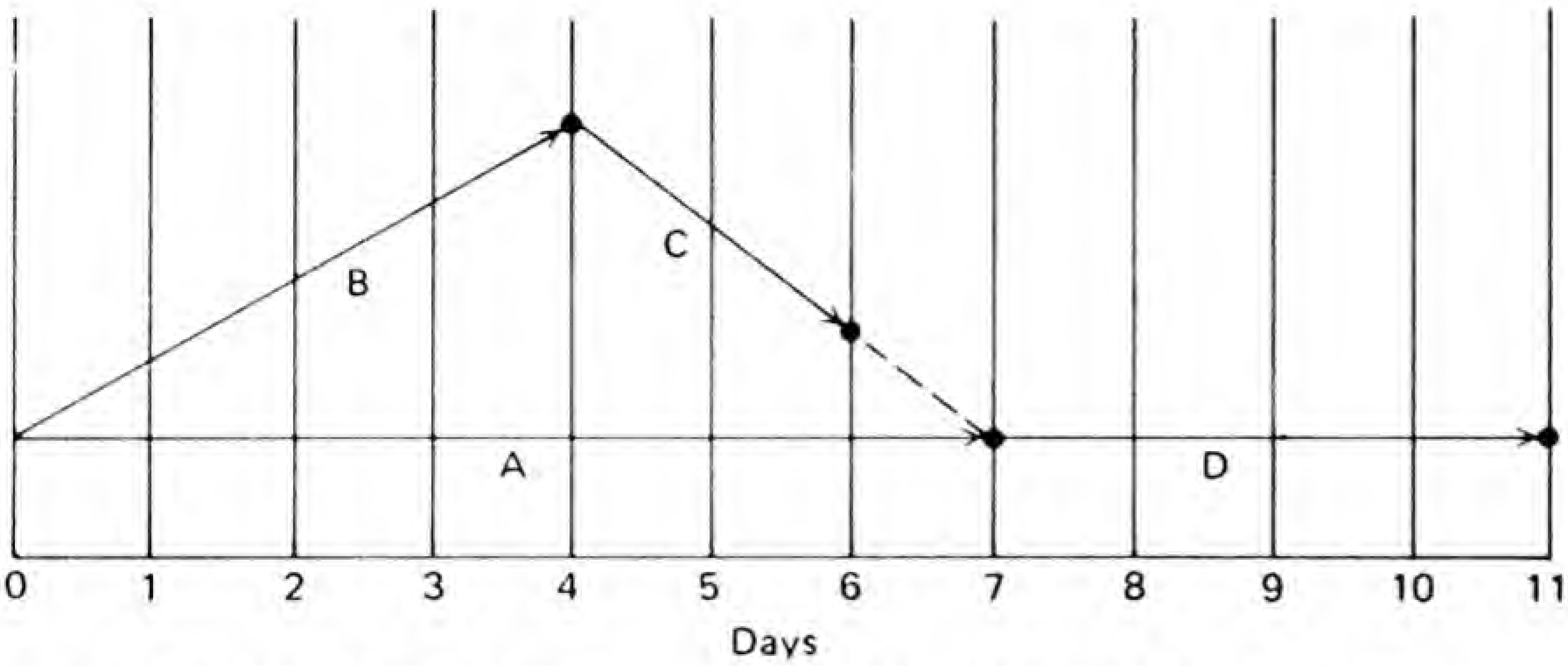


Figure 15C-4 Critical path method graph shortened to 11 days.

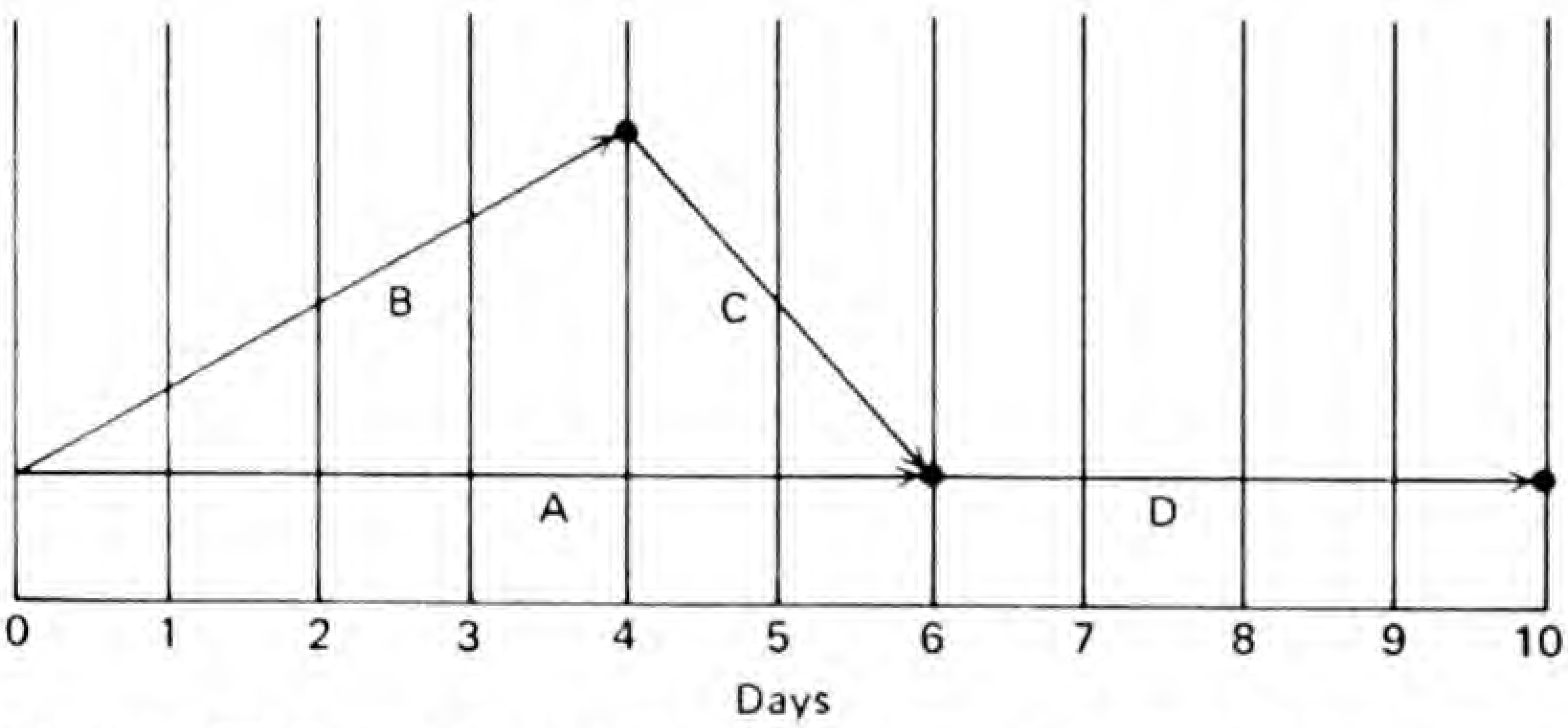


Figure 15C-5 Critical path method graph shortened to 10 days.

further, any reduction in the project must occur in activity B or C *and* in A. The choices we have, therefore, are to shorten B and A or C and A. The cost of shortening B and A is \$4 per day. The most that they can be shortened is by 2 days, since B cannot be completed in less than 2 days, and A cannot be completed in less than 4 days. If, on the other hand, we shorten C and A, the cost per day is \$7. We therefore choose to shorten B and A. The cost of the schedule is

$$
\begin{aligned}
\text{Cost} &= [\$2 + \$3 + 2(\$4)] + 8(\$5) \\
&= \$13 + \$40 \\
&= \$53
\end{aligned}
$$

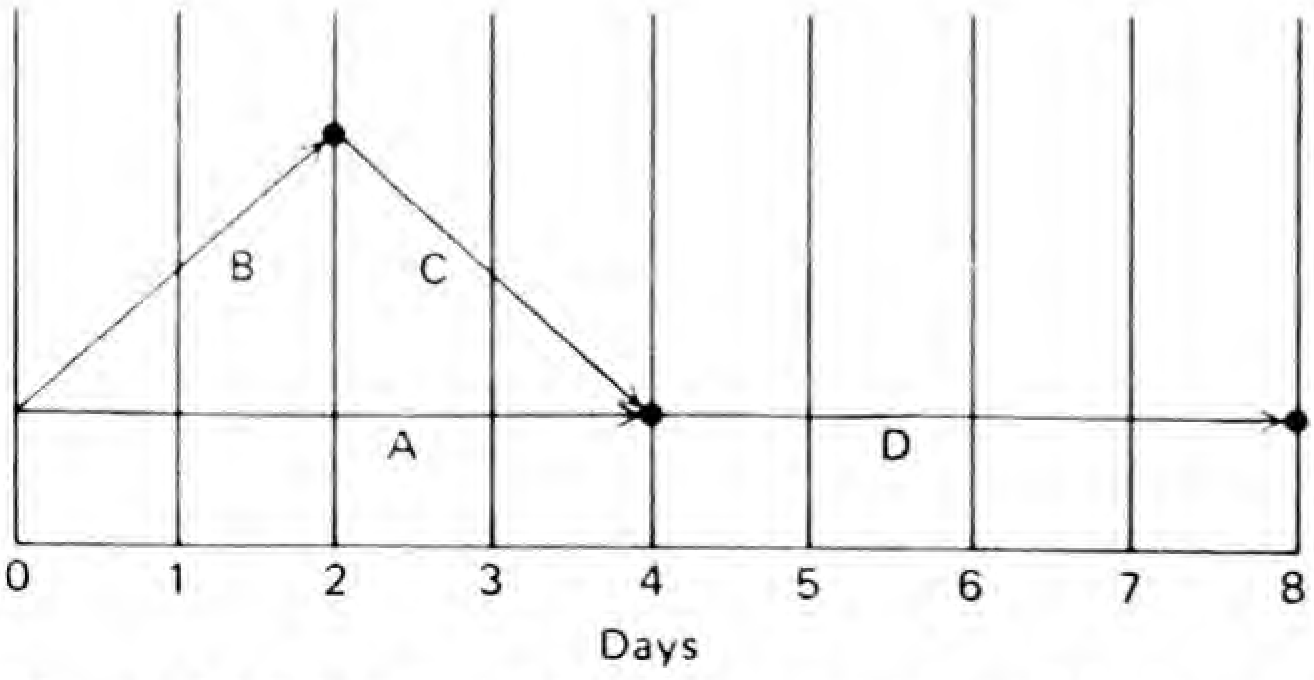
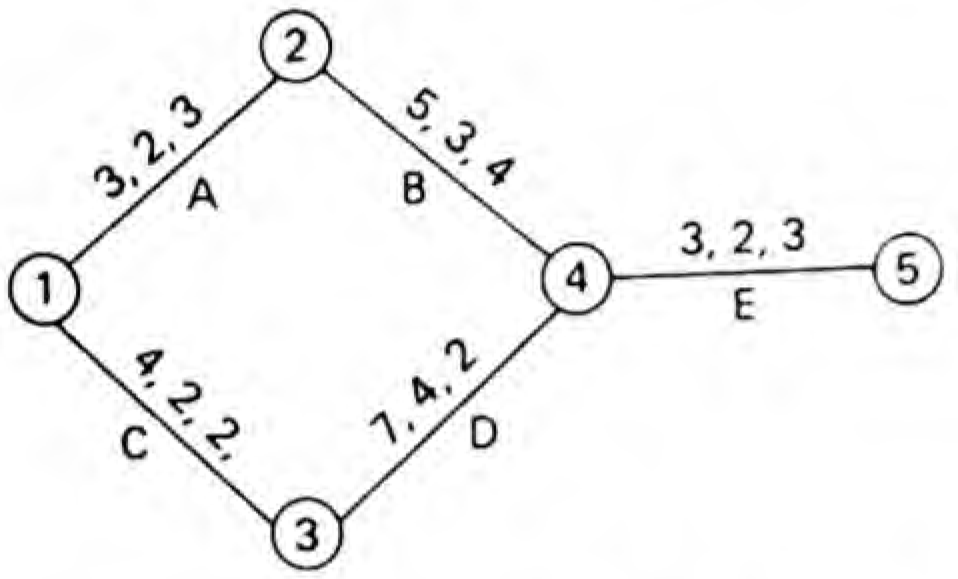


Figure 15C-6 Critical path method graph shortened to 8 days.

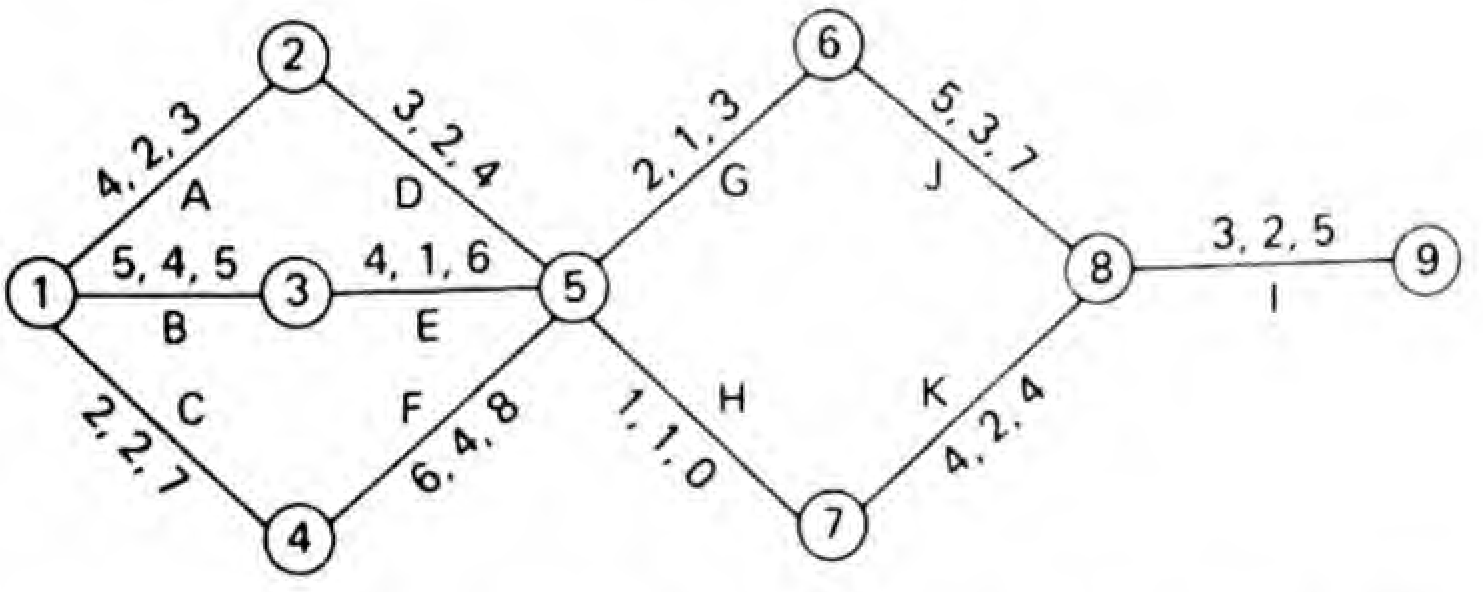
The new CPM graph is shown in Figure 15C-6. This is the schedule with the lowest cost, since any further reduction in project time must come as a result of decreasing activities C and A at a cost of $7 per day, which exceeds the saving that such a change would generate.

PROBLEMS

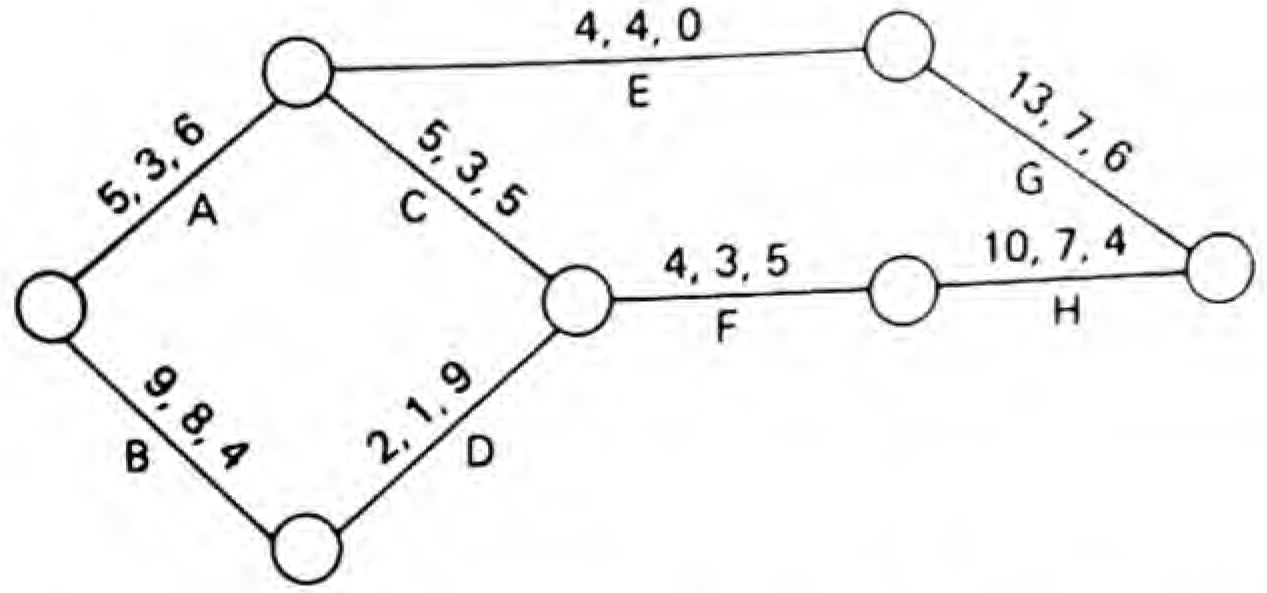
15C-1 Solve the following CPM problem: The figures above each activity represent the average time in which the activity can be completed, the minimum time, and the cost of reducing activity time in dollars per day. In addition, the saving which accrues if the project is shortened is $5 per day.



15C-2 Solve the following CPM problem: For each day that the project is shortened, a saving of $7 is realized.



15C-3 Above each activity in the following network is given the average time to completion, the minimum time, and the cost of reducing activity time in dollars per day. For each day the project is shortened a project cost of $12 per day is saved. What activities should be shortened, and how long will it then take to complete the project?



APPENDIX D: Dummy Activities

Consider the project described in Table 15D-1. Please take a pencil and paper and draw the network before reading further.

Many people will draw the network as it is shown in Figure 15D-1. Upon closer examination of this figure, however, it will be discovered that not only must activities A and F be completed before activity B can be started, but also activities A and F must be completed before activity C can be started. But this is the wrong representation because Table 15D-1 requires that *only* activity A is a predecessor of activity C. Figure 15D-1 is therefore an inappropriate representation of Table 15D-1.

This dilemma can be resolved by introducing a dummy variable. This is done in Figure 15D-2. A close examination of this figure reveals that all the precedent requirements of Table 15D-1 are indeed met.

A dummy variable is used whenever two or more activities have some *but not all* of their immediate predecessors in common. In our case activities B and C have some but not all of their immediate predecessors in common. On this occasion all the predecessors must be drawn leading to separate end events (circles), and dummy activities should be added from these events to comply with the precedent relationships. Since activities A and F were the immediate predecessors, they were drawn leading to separate events in Figure 15D-2. The dummy activity G was then added to comply with the precedent relationships in Table 15D-1.

Table 15D-1 Project

Activity	Precedent relationship	Activity time
A	—	5
B	F, A	6
C	A	5
D	B	12
E	D, C	4
F	—	3

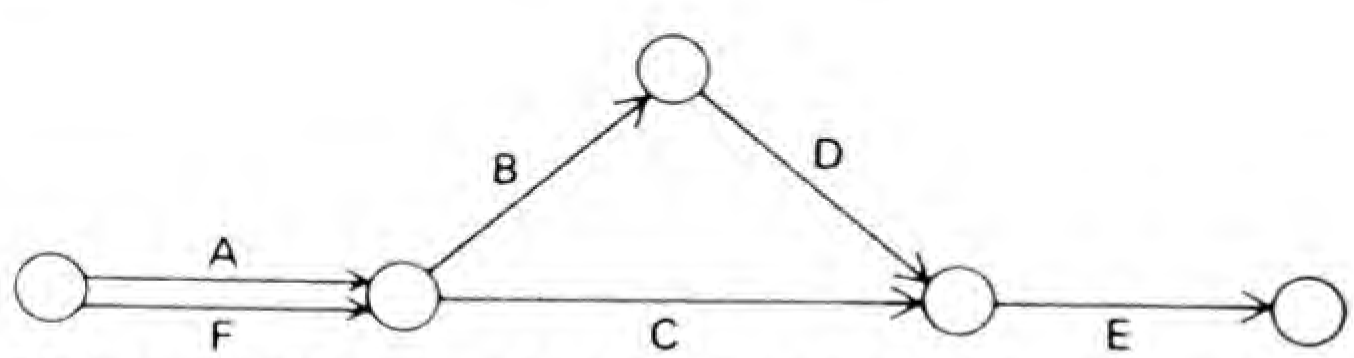


Figure 15D-1 Erroneous implication that activity F must precede activity C as well as activity B.

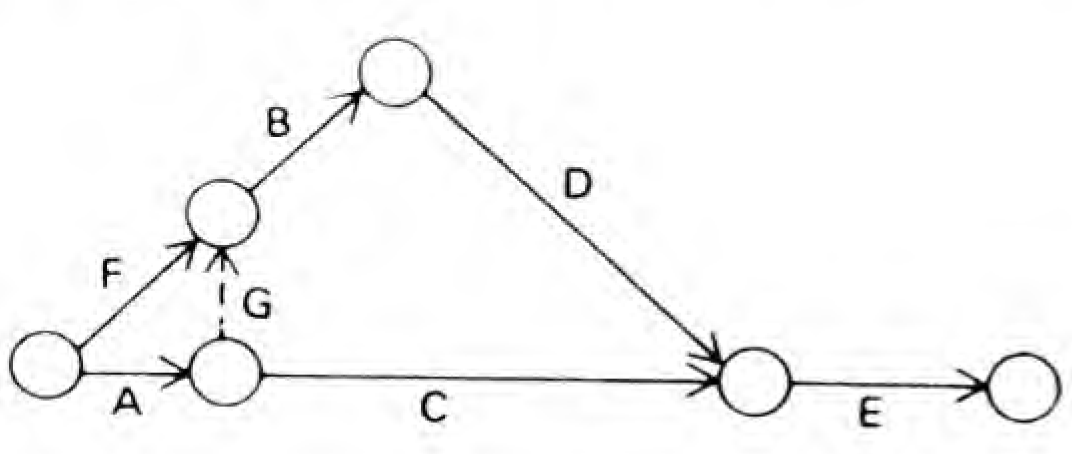


Figure 15D-2 Introduction of dummy variable.

PROBLEMS

15D-1 Draw the network for the following activities:

Activity	Precedent relationship
A	—
B	A
C	B
D	A
E	A
F	B, D, E
G	D, B
H	C, G
I	C, G, F
J	H, I

15D-2 Draw the network for the following activities:

Activity	Precedent relationship
A	—
B	—
C	—
D	—
E	A
F	B, A
G	C, B, A
H	D, C, B, A

15D-3 Draw the network for the following activities:

Activity	Precedent relationship
A	—
B	A
C	B
D	B
E	D
F	C
G	C, E
H	F
I	H, G

Chapter 16

Markov Models

INTRODUCTION

Markov analysis is used to study the behavior of a variable through time. Perhaps the most common application of this analysis is in marketing, where the variable studied is market share. The object of these studies is to determine the eventual share of the market which a product can obtain.

Rarely does the market share held by a company remain unchanged. A soft drink producer, for example, may currently sell 15 percent of all soft drinks sold. But this market share may change over time. The reason behind this is that some customers may switch to competitive brands on their next purchase, others may switch to this product from competitive brands, but some may remain loyal to the product.

This brand-switching process will lead to changes in the company's market share. Markov analysis is one means of studying this brand-switching behavior and predicting the behavior of the variable, market share, over time.

PROPERTIES OF MARKOV MODELS

Markov models are used to analyze a process which exhibits the following properties. First, there must be a finite set of possible outcomes. To continue

with the marketing example mentioned above, the set of outcomes would include all competitive soft drink products from which a consumer could choose.

The second property of the process is that the *probability* of the next outcome—also called a transition probability—is entirely dependent on the prior *outcome* only. Relating this to our example, the probability that a consumer will purchase this company's soft drink is dependent only upon the prior outcome or the drink purchased last time.[1]

The third property of the process is that the probabilities remain constant over time. For example, the probability that a consumer will purchase this company's soft drink having previously purchased a competitor's brand will remain unchanged over time.

We can therefore conclude that the process which is modeled by a Markov model should exhibit a finite set of outcomes, outcome probabilities dependent only on prior outcomes, and constant probabilities over time.

CASE STUDY: Consolidated Electric Power Company

The Consolidated Electric Power Company provides electric power to 200,000 subscribers in a large Midwestern city. Its power is generated by several large gas turbines which range in age from 1 to 15 years.

The only times at which all of these turbines are operating simultaneously is during the peak demand for electricity which occurs both in December and August. At other times only the most efficient generators are used while the rest are kept in reserve.

Turbines occasionally break down and must be repaired. Should it happen that an inadequate number of turbines are in operating condition to meet the power requirements of Consolidated's customers, the company must purchase power from other electric companies. The cost of purchasing this power is much higher than the cost of generating its own.

At the present time Consolidated Electric is preparing a long-range capital investment plan. As part of this plan it needs to know the percentage of time or likelihood that a turbine will be in operating condition. Given this information the company will be able to determine the number of turbines that will be needed during the next 5-year period.

Three types of turbines are currently in use, and Consolidated would like to proceed by analyzing the performance of each group separately. The first group includes the oldest turbines. The data for this group have been collected and are presented in Table 16-1.

[1] Markov models which exhibit this second property are said to be first-order Markov models. Higher-level models do take into consideration more than just the prior outcome, but these complex models are not covered in this chapter.

From \ To Initial state	Subsequent state: Operating (state 1)	Not operating (state 2)
Operating (state 1)	.80	.20
Not operating (state 2)	.40	.60

Table 16-1 Transition matrix, machine problem.

During any one day a turbine from this group may either be operating (state 1) or not operating (state 2). During the following day the machine may also be operating or not operating. If the turbine is initially operating, the probability that it will still be operating during the next day is 80 percent. The probability that it will not be operating is therefore 20 percent. If the turbine is initially not operating, the likelihood that it will be operating the next day is 40 percent and the probability that it will not be operating is 60 percent. This machine behavior is recorded in the form of a transition matrix; it is shown in Table 16-1.

The outside left-hand column of Table 16-1 represents the initial state of the turbine, and the outside top row represents the subsequent state or outcome of the turbine 1 day later. The values within the matrix represent the probabilities of moving from one state to the next.

It can be concluded that this case exhibits the properties discussed in the previous section. First there is a *finite* set of two possible outcomes: operating and not operating. Second, outcome probabilities depend only upon prior outcomes. Third, the probabilities can be assumed to be constant over time.

THE CHANGES OVER TIME

The process described in Table 16-1 can be represented by probability trees. The first probability tree, in Figure 16-1, describes the sequence of outcomes *if* the turbine is initially in state 1 (operating). The second tree, in Figure 16-2, describes the sequence of events *if* the turbine is initially in state 2 (not operating).

The Changes over Time Given Initial State 1

Probability of State 1 on Day 2 Suppose the machine is initially in state 1 (Figure 16-1). The likelihood that it will be in state 1 on day 2 is read directly from Table 16-1: it is .8. This can be expressed in the following way:

$$P\left(\begin{matrix}\text{state 1 on}\\ \text{day 2}\end{matrix}\,\middle|\,\begin{matrix}\text{state 1 on}\\ \text{day 1}\end{matrix}\right) = .8$$

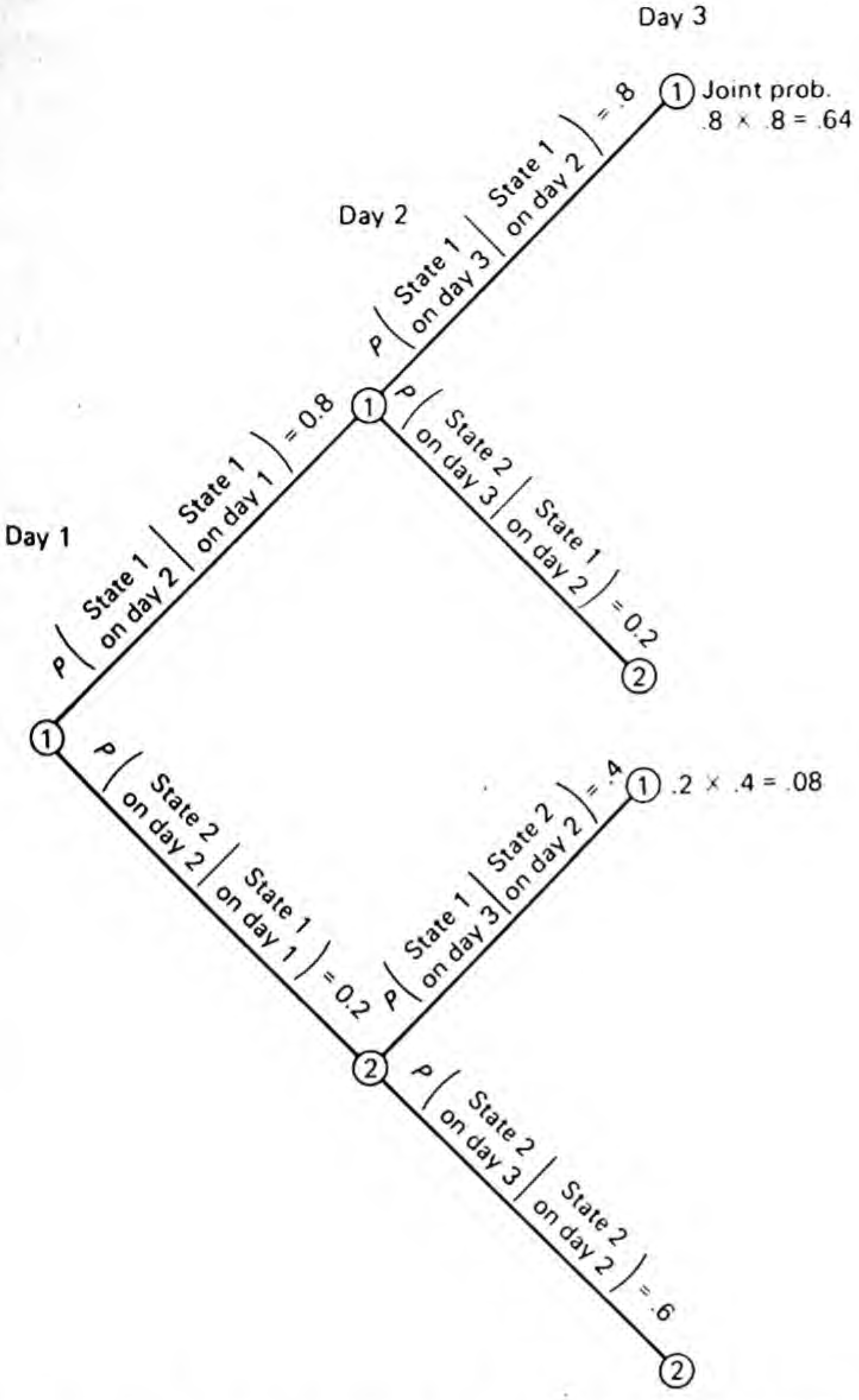


Figure 16-1 Probability tree for days 1, 2, and 3 given initial state 1.

Probability of State 1 on Day 3 The likelihood that the machine will be in state 1 on day 3, given that its initial state was 1, can be determined by identifying the two paths through the tree that terminate in state 1 on day 3 and summing the joint probabilities shown in Figure 16-1. This likelihood can be expressed also in the following way:

$$P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 3} & \text{on day 1}\end{array}\right) = P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 2} & \text{on day 1}\end{array}\right) P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 3} & \text{on day 2}\end{array}\right)$$

$$+ P\left(\begin{array}{c|c}\text{state 2} & \text{state 1}\\ \text{on day 2} & \text{on day 1}\end{array}\right) P\left(\begin{array}{c|c}\text{state 1} & \text{state 2}\\ \text{on day 3} & \text{on day 2}\end{array}\right)$$

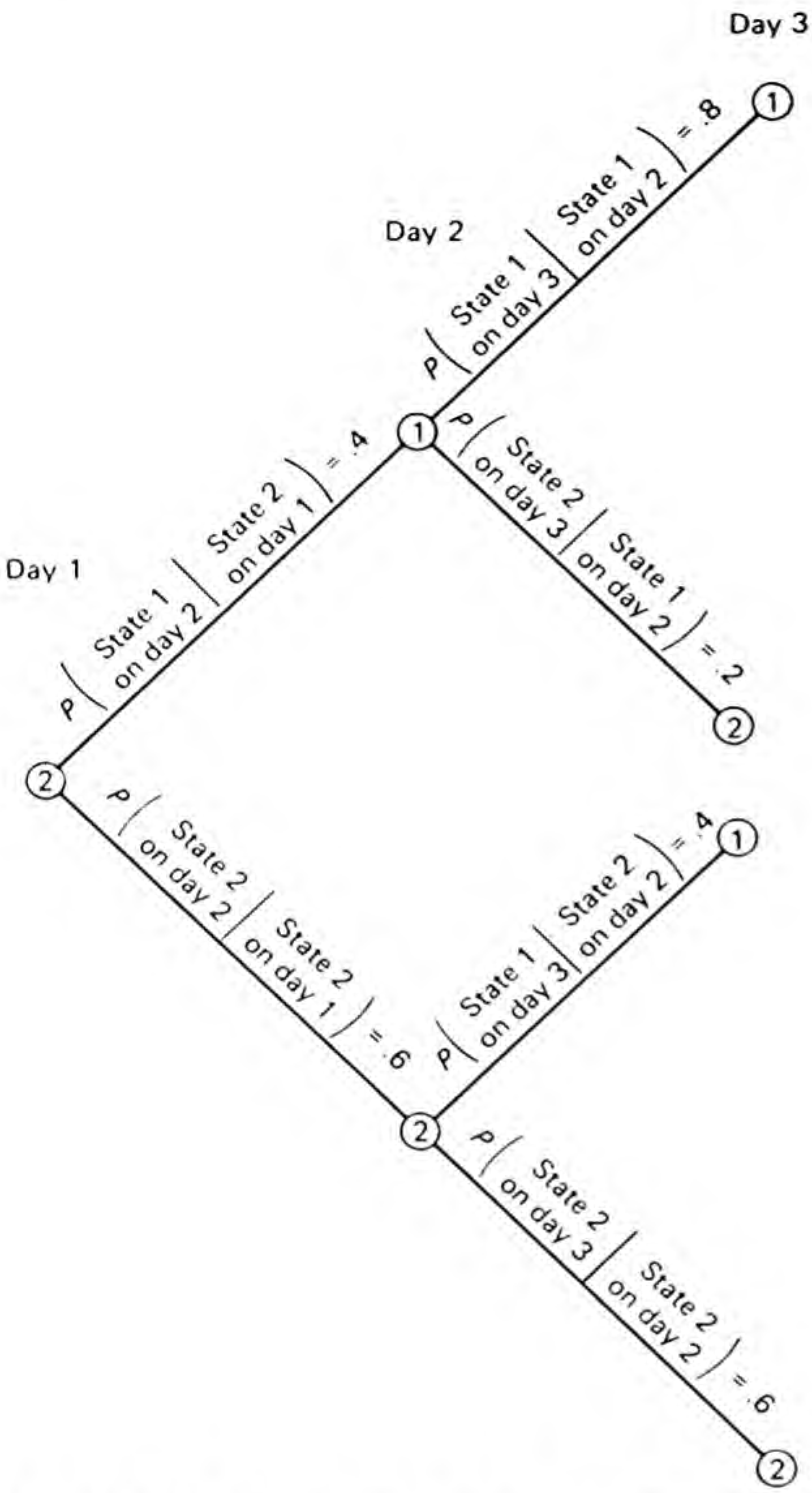


Figure 16-2 Probability tree for days 1, 2, and 3 given initial state 2.

which we will simplify to:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day 3} & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day 2} & \text{on day 1}\end{pmatrix}(.8)$$

$$+P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day 2} & \text{on day 1}\end{pmatrix}(.4)$$

and finally we have

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day 3} & \text{on day 1}\end{pmatrix} = .8(.8) + .2(.4) = .64 + .08 = .72 \tag{16-1}$$

We can therefore conclude that if the machine starts off in state 1, the likelihood that it will be in state 1 on day 3 is .72. Since it must be in either state 1 or state 2, the likelihood that it will be in state 2 on day 3 is $1.00 - .72 = .28$.

Thus far we have determined that if the machine starts in state 1, the likelihood that it will be in state 1 on day 2 is .8 and the likelihood that it will be in state 1 on day 3 is .72. Next we will compute the likelihood of the machine being in state 1 on day 4 given that it started in state 1.

Probability of State 1 on Day 4 Computing the probability that the machine will be in state 1 on day 4, given that it started in state 1, can be simplified by referring to Figure 16-3.

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\right)$$
$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\right)$$

or we can write this as

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right)(.8)$$
$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right)(.4) \tag{16-2}$$

and finally we have

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) = .72(.8) + .28(.4) = .576 + .112 = .688$$

Correspondingly we have

$$P\left(\begin{matrix}\text{state 2}\\ \text{on day 4}\end{matrix}\middle|\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) = 1.000 - .688 = .312$$

We can now conclude that if the machine starts off in state 1, the likelihood that it will be in state 1 on day 2 is .8, the likelihood that it will be in state 1 on day 3 is .72, and the likelihood that it will be in state 1 on day 4 is .688.

General Expression If Equations 16-1 and 16-2 are reexamined, a pattern can be seen. That is, we can write the general expression for the probability

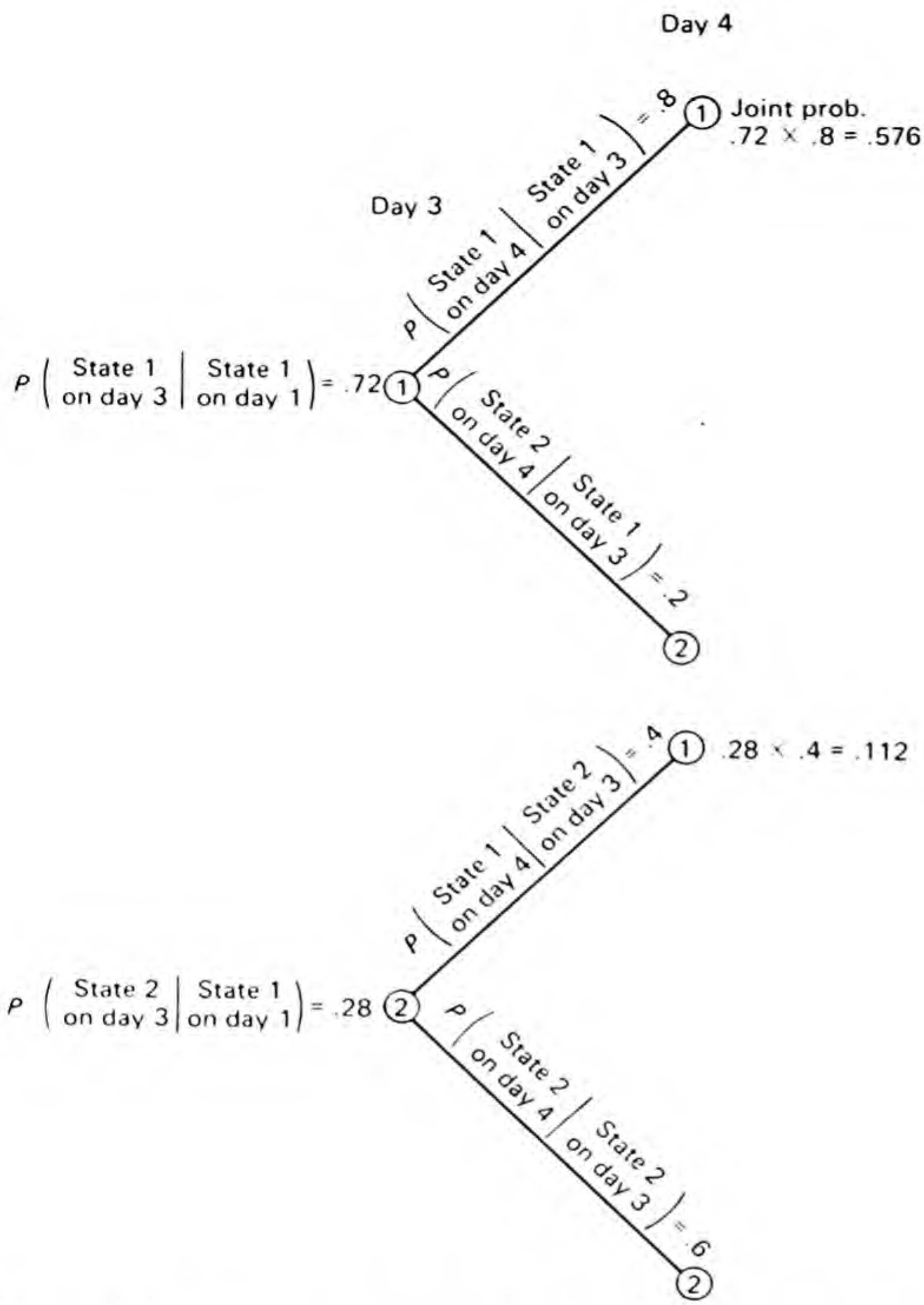


Figure 16-3 Probability tree for days 3 and 4 given that the system started in state 1 on day 1.

of being in state 1 on any day n, given that the machine started in state 1 on day 1, in the following way:

$$P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{array}\right) = P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day } n-1 & \text{on day 1}\end{array}\right)(.8)$$
$$+ P\left(\begin{array}{c|c}\text{state 2} & \text{state 1}\\ \text{on day } n-1 & \text{on day 1}\end{array}\right)(.4) \qquad (16\text{-}3)$$

For example, applying this to day 5, we have

$$P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 5} & \text{on day 1}\end{array}\right) = P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 4} & \text{on day 1}\end{array}\right)(.8)$$

$$+P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day 4} & \text{on day 1}\end{pmatrix}(.4) = .688(.8) + .312(.4) = .6752$$

Correspondingly

$$P\left(\begin{array}{c|c}\text{state 2} & \text{state 1}\\ \text{on day 5} & \text{on day 1}\end{array}\right) = 1.0000 - .6752 = .3248$$

we could continue and compute these probabilities for days 6, 7, 8, etc. Since the computations should be familiar to you by now, only the results are reordered in Table 16-2.

Steady State It can be seen from Table 16-2 that the probability of being in state 1 on some future day tends toward ⅔ *given* that state 1 was observed on day 1. To prove an important point, we will now examine the probabilities of being in states 1 and 2 given that we *started* in *state 2* on *day 1*.

The Changes over Time Given Initial State 2

Probability of State 1 on Days 2 and 3 Figure 16-2 represents the possible sequences on day 2 and day 3 if the machine is not operating (state 2) on day 1. The likelihood that it will be operating on day 2 can be read directly as .4. The likelihood that it will be in state 1 on day 3 can be expressed as follows:

$$P\left(\begin{array}{c|c}\text{state 1} & \text{state 2}\\ \text{on day 3} & \text{on day 1}\end{array}\right) = P\left(\begin{array}{c|c}\text{state 1} & \text{state 2}\\ \text{on day 2} & \text{on day 1}\end{array}\right) P\left(\begin{array}{c|c}\text{state 1} & \text{state 1}\\ \text{on day 3} & \text{on day 2}\end{array}\right)$$

$$+P\left(\begin{array}{c|c}\text{state 2} & \text{state 2}\\ \text{on day 2} & \text{on day 1}\end{array}\right) P\left(\begin{array}{c|c}\text{state 1} & \text{state 2}\\ \text{on day 3} & \text{on day 2}\end{array}\right)$$

Table 16-2 Probability of Machine Being in State 1 or 2 on Day n Given That It Started in State 1

Day n	$P\left(\begin{array}{c\|c}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{array}\right)$	$P\left(\begin{array}{c\|c}\text{state 2} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{array}\right)$
1	1.0	.0
2	.8	.2
3	.72	.28
4	.688	.312
5	.6752	.3248
6	.67008	.32992
7	.668032	.331968
8	.6672128	.3327872
9	.6668850	.3331150
10	.6667540	.3332460
11	.6667016	.3332984
12	.6666805	.3333195

or

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day 2}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.8)$$

$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day 2}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.4) \tag{16-4}$$

or

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = .4(.8) + .6(.4) = .56$$

Correspondingly

$$P\left(\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = 1.000 - .56 = .440$$

These are entered in Table 16-3.

Probability of State 1 on Day 4 Next we turn to period 4. (See Figure 16-4.) The probability of being in state 1 on day 4, given that the system started in state 2 on day 1, can be expressed in the following way:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\,\middle|\,\begin{matrix}\text{state 1}\\ \text{on day 3}\end{matrix}\right)$$
$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) P\left(\begin{matrix}\text{state 1}\\ \text{on day 4}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 3}\end{matrix}\right)$$

Table 16-3 Probability of Machine Being in State 1 or 2 on Day n Given That It Started in State 2

Day n	$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)$	$P\left(\begin{matrix}\text{state 2}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)$
1	.0	1.00
2	.40	.60
3	.56	.44
4	.624	.376
5	.6496	.3504
6	.65984	.34016
7	.663936	.336064
8	.6655744	.3344256
9	.6662297	.3337703
10	.6664918	.3335082
11	.6665966	.3334034
12	.6666385	.3333615

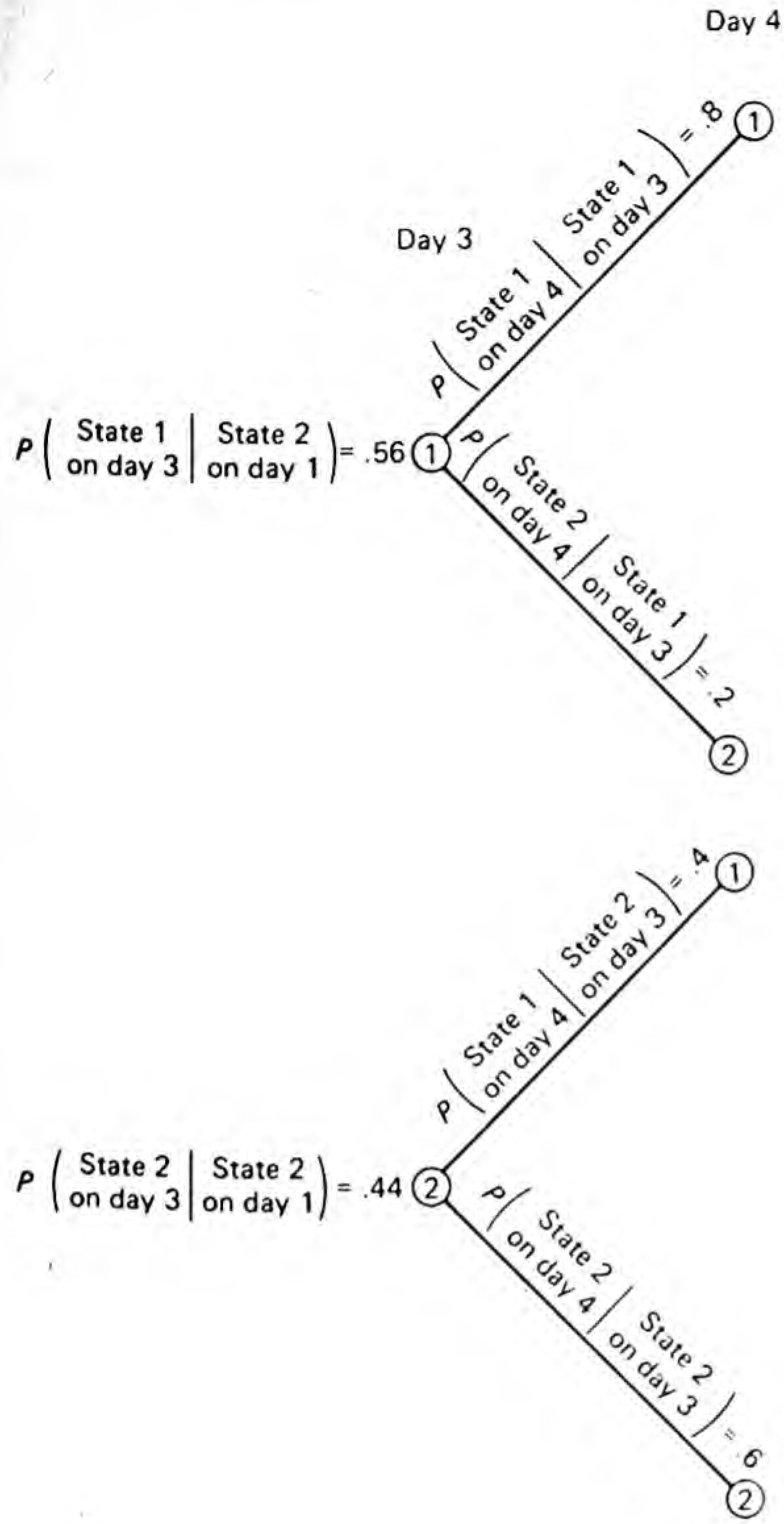


Figure 16-4 Probability tree for days 3 and 4 given that the system started in state 2 on day 1.

or

$$P\begin{pmatrix}\text{state 1} \\ \text{on day 4}\end{pmatrix|\begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}) = P\begin{pmatrix}\text{state 1} \\ \text{on day 3}\end{pmatrix|\begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix})(.8)$$

$$+ P\left(\begin{matrix}\text{state 2} \\ \text{on day 3}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right)(.4) \tag{16-5}$$

and

$$P\left(\begin{matrix}\text{state 1} \\ \text{on day 4}\end{matrix}\,\middle|\,\begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right) = .56(.8) + .44(.4) = .624$$

Correspondingly

$$P\begin{pmatrix}\text{state 2} \\ \text{on day 4}\end{pmatrix} \left| \begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right) = 1.000 - .624 = .376$$

These are entered in Table 16-3.

General Expression Once again, if we reexamine Equations 16-4 and 16-5, a pattern can be seen. We can write the general expression for the probability of being in state 1 on any day n given that the machine started in state 2 on day 1. This is expressed in the following way:

$$P\left(\begin{matrix}\text{state 1} \\ \text{on day } n\end{matrix} \middle| \begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1} \\ \text{on day } n-1\end{matrix} \middle| \begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right)(.8)$$

$$+ P\left(\begin{matrix}\text{state 2} \\ \text{on day } n-1\end{matrix} \middle| \begin{matrix}\text{state 2} \\ \text{on day 1}\end{matrix}\right)(.4) \qquad (16\text{-}6)$$

The general expression can be applied to days 5, 6, 7, etc. The results are recorded in Table 16-3.

Steady State By referring to Table 16-3 it can be seen that if the machine starts in *state 2* (out of order), the likelihood that it will be in state 1 during some future period tends toward ⅔.

Referring to Table 16-2, you will recall that we discovered that if the machine started in *state 1*, the likelihood that it would be in state 1 during some future period would *also* be ⅔. We can therefore conclude that *regardless of the initial state* the likelihood of the machine being in state 1 during some future period is ⅔. This is called the steady-state probability of being in state 1.

In the next section we turn to a much more efficient way of computing the steady-state probability.

Computing Steady-State Probabilities

Given That the Initial State Is State 1 In the previous section we learned that the probability of being in state 1 after several time periods n tends toward ⅔. Therefore we can write:

$$P\left(\begin{matrix}\text{state 1} \\ \text{on day } n+1\end{matrix} \middle| \begin{matrix}\text{state 1} \\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1} \\ \text{on day } n\end{matrix} \middle| \begin{matrix}\text{state 1} \\ \text{on day 1}\end{matrix}\right) \qquad \text{as } n \to \infty$$

$$(16\text{-}7)$$

since both the left- and right-hand sides will tend toward ⅔ as n gets very large.

Equation 16-3 was stated in the following way:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n-1 & \text{on day 1}\end{pmatrix}(.8)$$
$$+P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day } n-1 & \text{on day 1}\end{pmatrix}(.4)$$

This can be written in the following way:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n+1 & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.8)$$
$$+P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.4) \qquad (16\text{-}8)$$

Combining this with Equation 16-7, we have:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.8)$$
$$+P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.4)$$

since

$$P\begin{pmatrix}\text{state 2} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix} = 1 - P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}$$

We can now write:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.8)$$
$$+\left[1-P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}\right](.4)$$

Multiplying, we have:

$$P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix} = P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.8)$$
$$+.4-P\begin{pmatrix}\text{state 1} & \text{state 1}\\ \text{on day } n & \text{on day 1}\end{pmatrix}(.4)$$

Collecting like terms and solving, we have:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right)(1 - .8 + .4) = .4$$

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 1}\\ \text{on day 1}\end{matrix}\right) = \frac{.4}{.6} = \tfrac{2}{3}$$

It is therefore in this way that we can solve for steady-state probabilities. The answer we have just derived does indeed correspond with the one determined in the previous section: The likelihood that the machine will be in state 1 on some future day n (given that it started in state 1) is $\tfrac{2}{3}$.

Given That the Initial State Is State 2 We have determined previously that the initial state of the machine has no bearing on its steady-state probabilities. It will now be shown that the probability of being in state 1, given that the machine started in state 2, is also $\tfrac{2}{3}$.

Following the same method used in the last section, we have the following:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n+1\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) \qquad \text{as } n \to \infty \tag{16-9}$$

From Equation 16-6 we can write:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n+1\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.8)$$
$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.4)$$

Combining with Equation 16-9, we have:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.8)$$
$$+ P\left(\begin{matrix}\text{state 2}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.4)$$

and

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(.8)$$
$$+ \left[1 - P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)\right](.4)$$

Collecting like terms and solving:

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix} \middle| \begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right)(1 - .8 + .4) = .4$$

$$P\left(\begin{matrix}\text{state 1}\\ \text{on day } n\end{matrix} \middle| \begin{matrix}\text{state 2}\\ \text{on day 1}\end{matrix}\right) = \frac{.4}{.6} = 2/3$$

This result shows that the probability of being in state 1 on some future day n is 2/3—given that state 2 was the initial state. In the previous section it was shown that the probability of being in state 1 on some future day n is 2/3—given that state 1 was the initial state. Once again we conclude that the steady-state probability of being in state 1 is 2/3 regardless of the initial state.

EXAMPLE OF A MARKOV MODEL: MARKET SHARE

Suppose that only Ford and Chevrolet compete in the marketplace for lower-priced full-size cars. Of primary interest to both companies is the share of the market which they can win. The data presented in Table 16-4 represent the brand-switching behavior of Ford and Chevrolet customers.

The outside left-hand column of this transition matrix represents the car which is currently owned; the outside top row represents the next purchase. The values within the matrix represent the probability that a customer will move from one brand on the present purchase to the same or another brand on the next purchase. For example, the likelihood that a present Ford owner purchases another Ford is 70 percent. The likelihood that a Ford owner switches to a Chevrolet on the next purchase is 30 percent.

We can also conclude that the example exhibits the properties discussed in the beginning of the chapter. First, there is a finite set of outcomes: Ford and Chevrolet. Second, outcome probabilities depend only upon prior outcomes. Third, the probabilities will be assumed to be constant over time.

This model and its solution can be valuable to the decision maker in many ways. First, just the formulation of the model in terms of a transition matrix focuses attention on the fact that changes in market share are dependent on winning new customers, losing customers, and retaining some out of brand loyalty. The matrix also identifies the degree to which new

From present purchase \ To next purchase	(state 1) Ford	(state 2) Chevrolet
Ford (state 1)	.70	.30
Chevrolet (state 2)	.40	.60

Table 16-4 Transition matrix for Ford and Chevrolet brand-switching behavior.

customers are won from competitors and the degree to which customers are lost to competitors. This insight may be valuable in developing marketing strategy.

From this model the decision maker will also be able to predict market share at some future point in time. For example, if Ford's current market share is 30 percent, the model can be used to determine its future market share. These calculations are presented in the next section, where the eventual market share for Ford is computed.

Computing Steady-State Probabilities

Suppose that Ford would like to know its steady-state market share. In other words, it would like to determine the steady-state probability of being in state 1 in some future period n.

Following the same pattern developed in Equation 16-8, we have:

$$P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n+1\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right) = P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(.7)$$
$$+ P\left(\begin{array}{c}\text{state 2}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(.4)$$

and since

$$P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n+1\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right) = P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right) \qquad \text{as } n \to \infty$$

we can write

$$P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right) = P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(.7)$$
$$+ P\left(\begin{array}{c}\text{state 2}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(.4)$$

and

$$P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right) = P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(.7)$$
$$+ \left[1 - P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)\right](.4)$$

Collecting terms and solving:

$$P\left(\begin{array}{c}\text{state 1}\\ \text{on day } n\end{array}\middle|\begin{array}{c}\text{state 1}\\ \text{on day 1}\end{array}\right)(1 - .7 + .4) = .4$$

$$P\left(\begin{matrix}\text{state 1}\\\text{on day } n\end{matrix}\,\middle|\,\begin{matrix}\text{state 1}\\\text{on day 1}\end{matrix}\right) = \frac{.4}{.7} = 4/7$$

We can therefore conclude that Ford's eventual share of the market will be 4/7, or 57 percent.

HOW USEFUL IS THE MODEL?

Most Markov model applications have been in marketing, and the purpose of these models has been to determine the eventual market share of a product or service. Perhaps the most useful aspect of this modeling process has been the insight into complex systems which these models provide.

In the beginning of the chapter it was stated that in simple Markov models the probability of going into any state depended only on the current state and not on any previous states. Therefore if consumers indeed have "no memory" and if their next purchase is not influenced by previous purchases, the Markov model may be a reasonable representation of reality. If consumers do have a memory, however, the use of this Markov model may prove misleading.

In the beginning of the chapter it was also stated that the probabilities found in the transition matrix were assumed to be constant over time. Again we have an assumption which must be compared with the system being modeled. Is it reasonable to assume that the probability of switching from a Ford to a Chevrolet will not change over time? If so, we are implying that advertising, promotion, pricing strategy, and competitors' actions will not change the likelihood of a customer switching out of a Ford and into a Chevrolet. The model assumes that these probabilities will not change, and once again if this does not seem like a reasonable representation of reality, use of the model may be misleading.

In conclusion the usefulness of the model to the decision maker depends upon how close the assumptions of the model fit reality. It may be, however, that even though the correspondence may not be close, the model can still provide valuable insight into a problem.

SUMMARY

Markov models are used to model a process which exhibits a finite set of outcomes, outcome probabilities dependent only on prior outcomes, and constant probabilities over time. Analysis of these models showed that regardless of the initial state, the probability that the process would be in a certain state on some future date would be a constant. This was called its steady-state probability.

QUESTIONS

1 What properties must be observed before a Markov analysis can be used to model a system?

2 Suppose that one of the turbines in the Consolidated Electric Power case is currently inoperative. What is the likelihood that it will be operating at some date in the future?

3 Is a transition matrix likely to remain unchanged over time? Explain.

4 What can cause a transition matrix to change over time?

PROBLEMS

16-1 Using the method employed in determining Table 16-2, show that the probability of being in state 1 on days 1 through 8 tends toward 40 percent. The transition matrix is given below. Assume that the process begins in state 1.

From \ To	State 1	State 2
State 1	.7	.3
State 2	.2	.8

16-2 Using the method employed in determining Table 16-2, show that the probability of being in state 1 on days 1 through 8 tends toward .18. The transition matrix is given below. Assume that the process begins in state 1.

From \ To	State 1	State 2
State 1	.1	.9
State 2	.2	.8

16-3 Return to the example of a Markov model for market share earlier in the chapter and compute Chevrolet's steady-state market share. (See Table 16-4.)

Suppose that Chevrolet is currently enjoying a 60 percent market share. What will its market share eventually be?

If the assumptions of this model are valid, what can be said about the relationship of the current market share to the eventual market share?

16-4 The likelihood that a machine which is currently operating will remain operating in the next period is .7. The likelihood that it will be out of order in the next period is .3. If the machine is out of order in this period, the likelihood that it will be out of order also in the next period is .4 and the likelihood that it will be in order is .6. What is the steady-state probability for the machine being out of order given that it starts off out of order?

16-5 Given the following transition matrix, compute:

a The steady-state market share for brand X (assume brand X as the initial state).

b The steady-state market share for brand Y (assume brand X as the initial state).

From brand \ To brand	X	Y
X	.3	.7
Y	.4	.6

16-6 Given the following transition matrix, compute:

a The steady-state market share for brand Alpha.

b The steady-state market share for brand Beta.

From brand \ To brand	Alpha	Beta
Alpha	.2	.8
Beta	.6	.4

16-7 Given the following transition matrix, compute:

a The steady-state market share for brand X (assume brand X is the initial state).

b The steady-state market share for brand Y. This is called an absorbing Markov chain. Can you explain why?

From brand \ To brand	X	Y
X	1	0
Y	.2	.8

16-8 The three major competitors for low-priced full-size automobiles are Ford, Chevrolet, and Plymouth. A transition matrix describing purchasing behavior is given below. Find the steady-state market share for Plymouth (assume the initial state is 1), Ford, and Chevrolet.

From \ To	Ford (state 1)	Chevrolet (state 2)	Plymouth (state 3)
Ford (state 1)	.5	.3	.2
Chevrolet (state 2)	.2	.6	.2
Plymouth (state 3)	.2	.4	.4

CASE STUDY: Golden Snacks

Mr. Jack Golden owns and operates 25 snack bars in discount stores located within Greater Cleveland. His snack bars sell soft ice cream, hamburgers, hot dogs, soft drinks, and popcorn.

The space for the snack bars is leased from several discount chains. The leases are written to cover a 5-year period, and over half of them include options to renew.

The danger in leased space is that if the tenant operates a very profitable concession, the discount chain may decide not to renew the lease and thereafter operate the concession themselves. In the last 10 years this has happened three times to Mr. Golden.

At the present time Mr. Golden is considering an addition to his chain of snack bars. This addition is located in a new discount store scheduled for opening in 6 months.

The lease for this space will be in effect for 5 years with no option to renew. Management of the discount chain, however, has assured Mr. Golden that if they are both happy, there is no reason why another lease cannot be negotiated.

The terms of the lease call for a rental payment of 6 percent of gross receipts. However, this rental payment must not be less than $300. Therefore Mr. Golden must guarantee a minimum rental payment of $300 per month.

From a demographic study of the area surrounding the store, the discount chain has compiled some interesting figures which have been made available to Mr. Golden. The store is expected to draw an average of 15,000 shoppers per week. Records show that opening week will bring 30,000 shoppers. The estimate is that 15 percent of them will spend an average of 50 cents at the snack bar. The discount chain therefore forecasts a first-week gross of $2250 for the snack bar.

Mr. Golden has had enough experience to know that 50 percent of those who make a purchase at his snack bar will return next week for another purchase. Of those who do not make a purchase this week, only 10 percent will make a purchase next week. Mr. Golden's records also show that the average expenditure in his snack bars is 50 cents.

There will be several expenses incurred each month in addition to the lease payments. Wage costs are $400 per week, food costs are one-third of gross receipts, repairs average $20 per week, administrative costs average $50 per week, while $25 is allocated to miscellaneous and $75 to utilities.

To finance the fixtures, counters, and refrigeration equipment needed, Mr. Golden will borrow $30,000 from a local commercial bank. The rate of interest on the loan will be 10 percent. The equipment will have a $5000 salvage value at the end of 5 years and little or no salvage value at the end of 10 years.

QUESTIONS

1 Determine Mr. Golden's steady-state market share.
2 Estimate the profitability of this investment.
3 Suppose that special in-store promotions could increase the percentage of people who were not customers of the snack bar in the previous period but will be customers in this period from 10 to 15 percent. What effect will this have on the profitability of the snack bar? If this in-store promotion costs $100 per week, is it worth it?
4 Should Mr. Golden sign the lease?

Chapter 17

Quantitative Methods in the Organization

CASE STUDY: Criminal Court Scheduling

The criminal courts in Middlesex County were overloaded for many years. Recently the situation had become even worse. In response to pressure from all sides, the mayor's office hired a consulting firm to analyze the problem.

The consulting firm found that few things could be done to alleviate this overload. Crime rates had increased, and court facilities had failed to keep pace with this increased load. It was impossible to change either of these two factors in the short run.

The consultants did find, however, that poor scheduling of criminal cases often resulted in too many cases scheduled on one day and not enough on the next. Better scheduling, they concluded, could result in a greater number of cases being processed through the same facilities.

With the full cooperation of the mayor, the consulting firm developed a scheduling model. The model took into consideration the expected length of time necessary to hear a particular case, the availability of judges, courtrooms, lawyers, police, and witnesses. It also considered the priority of the case by scheduling cases involving serious crimes such as murder ahead of cases involving less serious crimes. Two months ago the mayor presented this

model to a group of county officials including court administrators, lawyers, and judges. It was the first time that the court administrators had heard of the model. They listened to the mayor make his presentation, and when he was through they voiced their opinions. First they felt that the model could not possibly take into consideration all the factors that they did when a schedule was made. Several examples were given. Second, they pointed out that the model required tremendous amounts of data which were difficult to obtain. Even if the data were accessible it was unlikely that their computers could handle this extra burden. Third, they felt that scheduling problems should not take priority over the efforts to expand physical facilities. Fourth, they felt that the consulting firm lacked an understanding of their problems and that if anyone was to make changes in the scheduling system it should be the administrators themselves.

The atmosphere during the meeting continued to be negative. It was apparent that the time and money which had gone into the model were wasted. Only the mayor was convinced of its potential.

In the months that have passed since this meeting, the model has collected dust while the court system has remained overloaded.

SUCCESSFUL QUANTITATIVE METHODS

The use of quantitative methods in the court scheduling case was completely unsuccessful. The blame, however, cannot be placed on the model itself. It was excellent. Where then can this blame be placed? Perhaps it can be placed on the management role played by the mayor's office. The mayor ignored some of the steps that should have been taken if one is to expect the successful use of these methods in an organization. A complete list of these steps includes the following:

1 Recognizing and defining the problem
2 Determining the criterion by which alternative strategies will be compared
3 Estimating payoffs
4 Deciding who in the organization will solve the problem
5 Developing a model of the system
6 Developing a decision support system which includes the model
7 Testing the decision system before it is implemented to ensure that it does what it is expected to do
8 Implementing the decision system
9 Controlling the decision system

The purpose of this chapter is to explore several of these steps in greater detail.

RECOGNITION AND DEFINITION OF THE PROBLEM

It is not easy to recognize when a problem exists. There is the story of the management scientist who was taking a plant tour of a large printing company. He stopped in front of a printing press to see 10 jobs waiting for their turn on the machine. He thought that here indeed was a complex problem. If the 10 jobs could be taken in any order, there were over 3 million possible sequences! His curiosity got the best of him and he asked the operator how this "complex" sequencing problem was solved. The operator replied, "I don't have any problem. I just take whichever one I want to." The management scientist was astounded. He realized that the operator was unaware that a problem even existed.

In general, problems are first recognized from symptoms. For example, problems in inventory control may be suggested by high inventories or an unusually large number of stockouts. Production scheduling problems may be suggested by the late shipment of finished goods or high per unit production costs. Teller scheduling problems at a bank may be suggested by long waiting lines. In the court scheduling case we saw that a high backlog of criminal cases suggested problems in the scheduling system.

Management's job is to be constantly on the lookout for symptoms and to dig beneath the surface in an effort to find out just what the real problems are.

Some practitioners might say that management already has too many problems and that it should not look for more. But better managers disagree. They are always on the lookout for symptoms. They continually screen problems which arise and select for further study only those which have a potentially high payoff. Problems with low payoffs, which might include the printing press sequencing problem, might go unsolved indefinitely. Indeed, managers must know when to say no!

Once it has become clear that a problem exists, the problem must be carefully defined. If it is defined in an ambiguous way, the solution which follows is unlikely to be useful. This step, then, must *never* be ignored.

DETERMINING THE CRITERION BY WHICH ALTERNATIVE STRATEGIES WILL BE COMPARED

Alternative strategies are compared on the basis of a criterion. In some situations it is clear which single criterion is to be used. For example, most inventory control problems are solved by comparing alternative ordering strategies on the basis of cost alone. In other situations, however, there may be several criteria which must be considered. Marketing decisions, for example, are often made on the basis of revenue, profit, and market share. Health care decisions are based on such criteria as availability of service, cost, and quality of service. Economic decisions made by the federal government generally consider such criteria as costs, unemployment, rate of growth, inflation, and balance of payments.

When there are several possible criteria or when it is unclear what the criteria should be, no further progress on the problem should be made until this issue is resolved. Without agreement on the criteria, there will be no agreement on the strategy to solve the problem. If one decision maker desires to minimize per unit costs while the other wishes to maximize revenues, there may be little chance that any agreement on market share strategy will be reached.

Whether or not a quantitative model is used, it is still very difficult to resolve a multiple-criteria decision problem. In practice these problems are often reduced to their most dominant single criterion. This was certainly true of all the quantitative models presented in this book. All of the systems which were modeled were reduced to a single criterion. In the court scheduling case, for example, the simulation model which was developed by the consultants had as its criterion the maximization of the number of cases heard in the court system.

But limiting a model to a single criterion has its shortcomings. First, there must be general agreement on the criterion chosen. Second, care must be exercised in using the results of this model in the decision-making process. In spite of these shortcomings, the use of such a model is often very productive.

ESTIMATING PAYOFFS

Before any major commitment is made to study a problem, there must be some assurance that the payoff or benefits from the study will exceed the cost. It would make no sense to spend \$50,000 on a project to determine better sequencing rules for a printing shop if the expected savings are \$2,000 per year.

How can estimates be made? There are two ways. The first is subjective and the second more scientific. Subjective estimates can be based on experience with similar problems in the same field or with similar problems in other divisions of the company. For example, a new inventory system which saved one company 3 percent might be expected to save at least that amount for another company whose problems are even worse.

The second method for making an estimate is based on the concepts of statistical sampling. In this method, the problem is sampled, the sample is studied for potential savings, and then it is assumed that the percentage savings in the sample will be similar to the percentage savings in the population. For example, a large firm was experiencing several problems in inventory control: costs were high and stockouts were frequent. Management felt that a major change in the inventory system was necessary. It estimated that a complete redesign would cost \$200,000. But before it approved this project, a preliminary study was undertaken. The study was based on a random sample of 100 items from a total of 10,000 items kept in stock. The results showed that an improved system could reduce total inventory costs by 5 percent. This meant that \$80,000 per year could be saved. Since the cost of

the redesign was $200,000, and since the benefits would occur for at least 5 years, it was decided to proceed with the project.

As we discovered in the court scheduling case, the likelihood of a successful payoff does not depend solely on the technical aspects of the model. It also depends upon some management and organizational factors. First, it depends upon whether or not management and operating personnel feel that there is a real need for the new information. If they do not feel there is, the results will never be used and the expected payoff will never be realized. Second, the ultimate user must be willing to cooperate in the study. Third, the study must receive the support of top management. These factors will be considered in the next section.

DECIDING WHO WILL SOLVE THE PROBLEM

Once it has been decided that a problem should be studied, it must be determined who should be responsible for the study. Should it be the sole responsibility of management scientists, or should they work together as a team with others who also share an interest in the problem? There are three basic methods of team organization: the pure management science team, the management science team with periodic middle-management interaction, and the management science–management team.

Pure Management Science Team

The pure management science team is generally composed of management scientists, computer scientists, and statisticians. Most often they receive assignments from management and have very little formal obligation to interact with other groups or levels within the organization.

The group can define the problem as they see it. Consequently, conflict and compromise with others in the organization can be kept at a minimum during this stage of the project.

Because of this isolation there may be a tendency to overmodel the system. Overmodeling occurs when more detail or more elements than necessary are included in the model. The result is that the development of the model may take too long, that the end user may receive more information or be required to supply more data to the problem than necessary, and that the decision system may be too complex to use.

To prevent overmodeling, the management science group must ask themselves the following questions.[1] First, will the added detail delay the project? Second, will the added detail improve the payoff of the model? Third, will the added detail help management to avoid major mistakes that a more general model would have ignored? Fourth, will the greater detail make the model more difficult to use and consequently limit its use?

[1] Harvey Shycon, "Perspectives on Management Science Application," *Interfaces* (The Bulletin), vol. 1, no. 2, February 1971, pp. 62–64.

Overmodeling is only one reason why the model may not be very useful to the end user. Another may be attributed to the fact that the group had the freedom to define the problem as they saw it. Unfortunately, this may not be the same way that the end user saw it, and it therefore may be quite unlikely that it will ever be used as part of the decision-making system.

Management Science Team with Formal Communication to Management

The next form of organization is the management science team with formal communication ties to middle management. In general, this management science team is still a *pure* management science team, but a difference is that periodic sessions are scheduled with middle management. The purpose of the sessions is to keep management continually informed about the progress of the project. Management, on the other hand, can use the sessions to criticize, suggest, provide direction, and give support to the study.

Under this form of organization the management science team will undoubtedly face many more problems and conflicts than under the first form of organization. But because management is not actually part of the daily team, it is often unclear how this input will affect the final model and decision system. At the very least, this form of organization will ensure that more problems are encountered in the system design stage and fewer problems at the implementation stage, with hope of a higher likelihood of project success.

Management Science–Management Team

The third form of organization is the integrated management science and management team approach. This group includes management scientists, representatives from middle management, and the end user. In addition, formal lines of communication may be established through periodic meetings with the top administrative level in the organization.

Each member of the group is considered to be an equal working partner who shares the common goal of the group, which is to maximize the payoff from the project. In spite of several problems in dealing with a heterogeneous group such as this, more often than not the advantages far outweigh the disadvantages.

A common source of conflict is that the members of the group usually have different intuitive models of the real-world system and its decision problems. The elements that one person sees as dominant will not necessarily be the ones that others see. For example, one person may see a problem as an inventory control problem with the order quantity as the major issue. Another person may see it as an integrated production and inventory control problem with both order quantity and the impact of these orders on production schedules as the major issue. These different views, and possibly different objectives, will often introduce a high level of conflict in the early stages of the project.

There are substantial benefits, however, from taking this approach. First,

the diversity of interests, the wide range of experience, and the different perspectives on the problem will ensure that a true systems approach will be taken. In other words, it is likely that the problem will be viewed from its widest consequences.

Although more conflicts will arise during the initial stages of the project than under any of the other form of organization, few problems will be encountered when the decision system is actually implemented. Projects approached in this way will generally have a high probability of success.

The management scientists may be only a fraction of the team, but their role on the team remains unchanged. They have the primary responsibility for structuring the problem and designing the eventual decision system. The role of management is to provide judgments, criticism, direction, and support, while the role of the end user is to criticize, provide judgments, and ensure that the system will ultimately be usable.

Two examples of the management science–management team approach will be given. The first occurred in a major insurance company, and the second in a large paper products company. Both examples are real; only the names have been changed.

The top executives at the Adams Insurance Company were convinced that a large-scale model which simulated many of the interacting departments of their company would be helpful for decision-making purposes. They expected the model to be used, for example, in examining how an increase in the different types of insurance the company sells would affect the portfolio mix of its investments and the long-term profitability of the company.

The senior vice president determined the composition of the project team. He chose the vice president of economic research, the vice president of marketing, representatives from the investment departments, and two from management science. The team met weekly and reported on a monthly basis to the senior vice president.

Although the project has yet to be completed, the comments of the team members are quite interesting. They believe that they have all gained a more thorough understanding of both the functional areas of the business and the interrelationships between them. Even if the project were terminated now, they feel that the insight already gained has been worth the effort and expenditure. Also unanimous is the conviction that the finished project will have a high payoff and that it will definitely be used in the decision-making process.

Another example of a management science–management team approach was reported by Harvey Shycon in *Interfaces*.[1] The Hall United Paper Products Corporation is an integrated producer of paper goods. It operates a nationwide complex of paper mills, corrugating facilities, and box-manufacturing plants. These facilities are not perfectly balanced with one another. That is, the capacities of the paper mills are not balanced with the corrugating

[1] Ibid.

facilities, nor are the corrugating facilities balanced with the box-manufacturing facilities. They do not need to be, since there are ready markets for their semifinished products, and ready supplies of semifinished products can easily be purchased in the industrial marketplace. When semifinished products are purchased by Hall, however, the price is always higher than the production costs for the same item. Costs and capacities also vary among the plants; consequently, it is not clear at any given time what the best combination of plants might be for producing a given set of products. Nor is it clear which plants should ship to which customers.

In order to solve this complex purchasing, scheduling, and shipping problem, the president of the company, Joe Hall, supported the development of a large-scale linear-programming model. He appointed a team consisting of the head of management science, the vice president of finance, the vice president of marketing, and the manager of inventory planning and scheduling.

The president met on a weekly basis with the team and participated in the discussions and decisions, which focused on such issues as the level of detail in the model, the degree to which the model should reflect the actual conditions at each manufacturing location, the extent to which financial considerations were to be included, and the flexibility which the model should have concerning the varying products and volumes.

With this careful integration of middle management, management scientists, and top management, it should not be surprising that the project was a complete success. The model is used weekly as an invaluable aid in the decision-making process.

Additional Evidence

A total of 16 projects were studied by Shycon.[1] The projects were separated into three categories, the first including those undertaken by pure management science teams, the second by management science teams with formal communication to management, and the third by management science–management teams.

Of the five projects in the pure management science category, only one was completely successful; of the four projects in the second category, three were completely successful; and of the seven projects in the management science–management team category, six were completely successful.

It can be concluded that a pure management science approach may not be the best strategy for ensuring project success.

The Composition of the Team

The fully integrated management science–management team should include representatives from at least four areas in the organization: management

[1] Ibid.

science, computer science, general management, and the problem area to be studied.

It is important that the individuals of the management science group themselves be applications-oriented. They must, of course, have a strong background in their discipline, but they must also be willing to make the compromises necessary to ensure the successful application of their techniques. Therefore, a management science–management team must have one or more management scientists who are sensitive to the needs of management and are willing to work with management toward common goals.

Since many decision systems use a computer for collecting, processing, and presenting large quantities of data, computer scientists are often a necessary addition to the team. The selection of this member must be deliberate and cautious. The person must understand the process of collecting and manipulating data for the purpose of developing relevant and useful *management* information. In short, this person needs to be much more than just a computer programmer.

The third area from which team members must be chosen is middle management. They should have a broad perspective of the company's operations and be capable of comprehending the complete system under study and its problems as they currently exist. Especially useful will be their contributions during the formulation stages. This input should ensure the relevancy of the model.

The fourth area from which a team member must be chosen is the problem area itself. This person will likely possess a thorough knowledge of all the details. In fact, so much detail may be known that those at this level rarely admit to the possibility of developing a general decision-making system. They are often quick to point out some specific event such as "the November 2, 1973, shortage of raw materials that no decision system could possibly have forecast." They conclude that because of such variations from normal activities it would be absurd to rely on sophisticated techniques. The management scientist should reply that no decision system is designed to operate independently of human intervention. It is but an aid in the decision-making process and, as such, is not the final word. In short, the decision system is designed to function side by side with the ultimate users. For this reason it requires their inputs and above all their cooperation during the design stage.

Where in the Organization Does the Management Science Group Belong?

There are several ways in which the management science group can be integrated into the organizational structure of the firm. The first strategy is one in which all the management science skills are located within a group at corporate headquarters. This corporate group may undertake studies requested at either the corporate or divisional level and may also lend their personnel for divisional studies. This approach has several advantages and disadvantages. It is an advantage that centralized groups of this type in-

variably develop a strong sense of professionalism. The group members share many areas of common interest, enjoy the cross-fertilization of ideas, and are constantly studying and learning. The result is a highly competent and up-to-date management science team: their expertise is often unquestionable.

One major disadvantage in this form of organization, however, is that the very level of professionalism which the group enjoys might be its biggest liability. Because of the professional focus, the group members may look toward outside sources for recognition of their work. These sources include research publications, the presentation of papers at professional meetings, and peer approval. The danger, then, is that the group's informal goals and objectives may be so far removed from the urgency of actual divisional problems that their orientation is not toward the client but toward the profession. Certainly this is not always true, but experience has shown that under this form of organization there will always be a tendency to drift in the direction of professionalism rather than toward client-centered activities.

The second strategy still maintains a corporate management science capability, but it also includes a management science staff in each of the divisions. The purpose of the corporate staff is to provide assistance to the groups at the division level. This might even include a loan of personnel for large studies. In general, the corporate staff is more specialized than the divisional staff, but the dangers of professional drift still apply to both groups.

The next two strategies represent the direction in which many people feel that the management science field is headed. The first of these is one in which a strong management science capability at the corporate level is maintained, but in addition management scientists are dispersed throughout the line organization of the divisions. For example, some management scientists are assigned permanently to production, inventory control, marketing, finance, and perhaps engineering. In this way the expertise of a professional group is maintained at corporate headquarters, and the orientation toward client-centered activities is maintained at the divisional level. It is in the best interest of the company to keep the channels of communication wide open between the management scientists at both of these levels.

There are an increasing number of people who believe that the only way to ensure that management scientists will be client-centered or applications-oriented is to disperse them completely in the line organization of the firm. The last strategy, then, has no corporate skills in this area, only division level skills. Moreover, there is no management science group at all, just management science skills.

Is one strategy better than the others? No. The strategy finally chosen depends upon the firm. What works for one might not work for another.

DEVELOPING A MODEL OF THE SYSTEM

It is at this point in the process of using quantitative methods that attention is directed toward the details of the model itself. First, the system within which

the problem exists must be carefully studied. What are the elements which compose this system? Which ones are to be included and which excluded from the model? How are they interrelated? Only after these questions are answered and there is some agreement among the team members can the model be formulated.

During this stage the data requirements of the model must also be considered. What data will be needed? Are the data available within the organization, or must they be obtained outside? How often must those data be updated? Are computer facilities needed to collect, process; and store these data?

THE DEVELOPMENT OF A DECISION SUPPORT SYSTEM WHICH INCLUDES THE MODEL

A model generally includes only the dominant elements of a system. Countless other elements of secondary importance are often omitted. Consequently, the model represents a somewhat limited view of this real-world system. Those who use the model must be aware of this shortcoming and must obtain additional information when necessary. This information may be obtained from experience, intuition, others in the organization, trade associations, or the competitive environment of the firm.

A model cannot stand alone. It is not the only component in a decision support system. It is only *one* component. It provides one of the *many* sources of information that are required to support the decision-making process.

Those responsible for developing the model should therefore maintain a perspective which is broader than the model itself. They must take into consideration the fact that the model will be used in conjunction with other sources of information and that the success of the model depends upon the effectiveness of the entire decision support system.

This perspective should also be expressed when training the ultimate users of the decision system. It should be made clear that the model does not have all the answers. Perhaps if the court scheduling model had been acknowledged to be but a part of a court scheduling system, the end users would have been more receptive. It should have been pointed out to them that the best the model could do would be to *recommend* a schedule. It would then be the responsibility of court personnel to take this and other relevant information into consideration before the final schedule would be made.

TESTING THE DECISION SYSTEM

It is not enough just to test the model to ensure that it does exactly what it was designed to do. The entire decision system must be tested to make sure that the model and the system are effective in dealing with the decision

problem. This means that the ultimate user must use the system on a trial basis and that the behavior of the decision maker must be observed to determine if any changes need to be made in the system. Seldom will a decision system meet objectives at the beginning. Changes are inevitable.

IMPLEMENTATION OF THE DECISION SYSTEM

The project team including the management scientists must remain involved with the project through the actual implementation and initial operation of the decision system. If interest in and control over the project is given up too early, changes may be made by those using it, and this could jeopardize the effectiveness of the system. Since many of the users may not have been involved in the design phase of the system, they may view these changes from a very narrow point of view, neglecting the broader system-wide implications.

It is essential that the team keep the implementation phase in mind at all times during the progress of the project. The decision system which is finally developed must be kept simple and easy to use. If computer output is used, it must be presented logically, be easy to read, and easy to use. Every detail of the system should be worked out: who is responsible for inputs, who will receive the output, what kinds of inputs will be required, are they easy to obtain, when will these inputs be required, and last but definitely not least, what training programs will be necessary to familiarize operating personnel with the new system?

Only when these implementation problems are carefully considered in advance can the chances of damaging changes and incorrect use of the system be minimized.

CONTROLLING THE DECISION SYSTEM

Many of the problems that management faces are recurrent. Inventory, media selection, portfolio analysis, and production scheduling are all examples of problems which arise over and over again. Consequently, the decision systems which are developed to deal with these problems are also used over and over again. But real-world systems change. Prices, rates of return, efficiencies, capacities, and fixed charges change. The elements in the system and the relationship between them change. As these changes occur, the decision system that was designed at some point in the past becomes less and less relevant to the real-world system and its real-world problems that exist today. Somehow the decision system must therefore be controlled. When it becomes clear that the model is no longer a useful representation of reality, the system must be modified. Whether this is every year, or every 2 years or every 5 years is impossible to say. It is imperative, however, to establish formal review or control procedures to ensure the continued success of the decision system.

SUMMARY

The purpose of this chapter has been to point out that the successful use of management science methods requires much more than the choice and analysis of the "right" model. It also requires that meaningful problems with potentially high payoffs be studied, that the studies receive the support of top management, that a team approach probably be employed, and that the decision system be carefully implemented and controlled. Neglect of these important aspects of a study could doom it to failure.

QUESTIONS

1 Why do you think the criminal court scheduling model met with such resistance?
2 If indeed court scheduling is a problem, how would you proceed to have the problem studied and solved?
3 A large commercial bank was considering the development of a simulation model to help the bank's investment group in making bond portfolio investment decisions. What steps would you suggest be taken in order to ensure the success of this project?
4 Why must decision systems be periodically reexamined and modified?

CASE STUDY: Howard Company

The Howard Company is a large machine shop which manufactures parts used by the defense and aircraft industries. Most of its orders are for small quantities, and seldom are two orders alike. Each order is scheduled through a unique set of manufacturing steps according to the specifications detailed in the customer's engineering drawings.

To accommodate this variety in jobs Howard operates 1000 machines grouped into 120 functional work centers. One work center is completely devoted to drilling, another grinding, another to painting, and so on. The company employs 400 hourly workers, and at any one time there are between 1000 and 3000 orders in progress.

When a contract is awarded to Howard, the job is placed on a master schedule. It is scheduled early enough to reasonably ensure that it will be completed by the date promised to the customer. In many contracts it is stipulated that Howard must pay a penalty if this due date is violated.

The schedule not only specifies the date on which the job will be started, but it also identifies the sequence of operations and work centers necessary to manufacture the part. But when the part actually arrives at a work center, it often must wait its turn in line for an available machine. These waiting lines can cause significant delays and can expose the job to late penalties. In the past it has been up to the discretion of the machine operator as to which job is taken next on the machine.

Because of these waiting lines, jobs often fall behind schedule. This situation occurs so frequently that Howard employs 20 expeditors whose responsibility it is to identify jobs which are behind schedule and put pressure on the appropriate work center to give these jobs top priority. Often, however, the expeditors become aware of the problem too late, and the job is still not delivered on schedule.

One year ago the president of the company, Allan Ross, decided to take steps to end three persistent problems: high per unit manufacturing costs, late penalties, and inefficient use of workers and machines. He contacted a nationally known consulting firm in New York City and asked for its help.

During the next 9 months the consulting firm designed a scheduling system directed at solving these problems. The focal point of this system was a simulation model which tested thousands of possible schedules each day and recommended the schedule which would minimize the lateness of jobs. It was designed in such a way that each evening the location of each job was entered into the company's computer through remote terminals located at the work centers. The simulation model was then run and the results made available to work center foremen at the start of the next shift. The results not only specified the order in which the jobs were to be scheduled on the machines but also identified critical jobs that were to arrive during the day and the work center from which these jobs were coming.

Three months ago the consulting firm presented this system to Howard's

top management. The officials received it very enthusiastically and recommended that it be implemented on a trial basis as soon as possible. It took the consulting firm two months to install the system and during this time three briefing sessions of 1 hour each were held for the foremen and managers of the shop. For most, it was the first they had heard of the new system.

Last month the trial began, and although it is still too early to reach a final conclusion, it appears that the system is a failure. The computer output almost never corresponds to the jobs at the work centers. Consequently, the schedule makes little or no sense. The foremen are still required to schedule according to their best judgment and the expeditors are busier than ever.

Allan Ross finally decided to have the consultants confront his managers and foremen. At the meeting the consultants were quick to place the blame on the foremen.

"You people," said one of the consultants, "make no effort to enter the location of the jobs on the remote terminals. Without that information it is impossible for our system to work."

The foremen admitted that on occasion they were too busy to enter the information but contended that most of the time it was done.

Steve Romez, foreman of grinding operations, was next to speak. He said that the system didn't give the foremen the information they needed. "Scheduling," said Steve, "depends upon more than the due date of a job. Equally important to me is the need to keep my machines and workers busy. Last week your computer schedule told me to keep three of my machines and workers idle for the first 2 hours of the day so that they could be available for a priority job when it arrived from Nancy Cook's area. That's crazy. Instead I loaded those machines with a 5-hour job. The priority job had to wait only 2 hours, and I kept my people busy all day. I don't think your system will ever work because it doesn't take into consideration the problems that we have on the shop floor."

When the meeting finally concluded, the feeling of the consultants was that the managers and foremen were unwilling to use the system properly. The managers and foremen, on the other hand, were convinced that the consultants did not understand their problems.

QUESTIONS

1 What were the symptoms that suggested the existence of scheduling problems?
2 What criterion did the consulting company use in solving the scheduling problem? Are other criteria relevant? Is it difficult to use more than one criterion in the development of a model?
3 What factors would you take into consideration when estimating the payoffs and costs of this simulation model?
4 Should the simulation model be expected to solve all the scheduling problems?
5 What shortcomings do you think the use of outside consultants contributed to the eventual failure of the new scheduling system? How could they have been used more efficiently?

6 How did the consulting firm fail in the implementation of the decision system?

7 Once an effective simulation model is designed for the shop, it will be unnecessary to modify the model for several years. Do you agree with this statement or disagree? Explain why.

8 Do you think the information available to the decision maker from the output of this simulation model is timely and relevant to the decisions that must be made? Explain.

9 Write a brief paragraph describing your final position on the case. One possible strategy is to terminate the services of the consulting firm, while another strategy is to enforce the use of its system in the plant. Other strategies are also possible. Support your position!

Chapter 18

Computers, Information Systems, and Quantitative Methods

INTRODUCTION

Most applications of quantitative methods require the use of a computer. The purpose of this chapter is to present the fundamentals of computers and describe the way they are used to support these methods.

THE DEVELOPMENT OF COMPUTER TECHNOLOGY

The phenomenal success of computers in administration can be attributed to the fact that there has *always* been a need for the collection, storage, processing, and presentation of large volumes of data. Before organizations began using computers these tasks were either undertaken by large clerical staffs or left undone altogether. For example, the preparation of company payrolls in the past required the maintenance of a large staff. However, as the cost of computer processing dropped, and as clerical wages increased, few, if any, large companies could still afford these old methods. They turned to computers which promised them a dramatic increase in efficiency and a reduction in costs. The need had been there all the time. As the cost of computer processing came down, its use, not surprisingly, went up.

The First Generation

The first commercial computer, Univac 1, became available in 1951. It was designed around the vacuum tube, a component which characterized all first-generation computers from 1951 to 1959. The cost of these machines was high, as was the per unit cost of processing data. Only the largest companies could afford the luxury of a computer during this period.

Second Generation

Second-generation computers, which were available between 1959 and 1964, utilized the new technology of transistors. The speed of the computer was greatly increased and the cost per unit of information processed dropped dramatically.

Third Generation

The third generation of computers utilized a new technology again: it was the technology of integrated circuits. Once more, the cost per unit of information processed dropped.

Now computers are available in all price ranges and in sizes from hand-held calculators up. Technology has succeeded in bringing computers within every administrator's reach.

THE COMPONENTS OF A MANAGEMENT INFORMATION SYSTEM

Computers are part of larger systems whose objective it is to present administrators with relevant, timely, and useful management information. Consequently, systems of this kind have often been referred to as *management information systems*. But management information systems should be separated, more appropriately, into data processing and decision support systems. First, we will look at data processing systems.

Data Processing Systems

Computers were first used as part of data processing systems. Even today most computer installations are used in this way. The most common example can be found in billing and payroll. Every month, for example, you receive a bill from the telephone company: it is prepared by a computer. Stored in the computer is your telephone number, the class of service (private line, business, etc.), your base rate, and all toll calls. At the end of each billing period these items are summarized and totaled on your bill. All this takes a fraction of a second and costs very little when compared with accomplishing the same number of steps by hand.

Characteristics of Data In general, data processing systems handle a high volume of data. Thousands of checks, bills, invoices, and inventory records

all require that the system be carefully designed to collect, store, process, and present this information in an efficient way.

The level of detail required in such data processing systems is high. Every employee must be included in the payroll file, and every customer must be included in the billing file. In addition, the level of accuracy must be high. Most people would object to being paid for the approximate number of hours worked.

Characteristics of the System The complexity of these data processing systems is high. The set of instructions for the computer, which is called a computer program, is generally long and complex. Its intellectual complexity, however, is low since the program merely carries out the steps which were previously accomplished by hand. No complex model building phase is necessary.

It is also quite important that the entire data processing system including the computer program be very efficient. With large volumes of data, each step must be performed in as little time as possible. Since the program is used on a daily or weekly basis, any small savings that can be made will accumulate over time.

Finally, a word should be said about the consequence of a failure or breakdown in a data processing system. It can be disastrous. An entire segment of the administrative process can come to a halt.

Decision Support Systems

The concept of a decision support system is more recent than that of a data processing system. Its focus is on the tactical and strategic decisions that administrators must make and on the development of information systems that can prove to be useful in this decision-making process. Quite often, a quantitative model, of the kind covered in this book, is used as part of this system.

For example, consider a decision problem which is faced on a weekly basis by a nationally known producer of baked goods whose product line includes 50 kinds of crackers and cookies. The company has six plants throughout the country, and each plant is able to produce any of the products. The output from these plants is shipped to one of four distribution centers, and from there the product is shipped to wholesalers and large retail chains. The problem is to determine which products should be made on which machine at which factory and shipped to which distribution center. To help the company make this decision, it uses a decision support system, including a large-scale linear-programming model. Extremely important is the fact that the company, and most others who use quantitative models, does not necessarily implement the precise outcome of the model. The model may be weighted heavily in reaching the final decision, but it is generally not the last word. That is why the model is said to be a part of the decision *support* system and not the decision system itself.

In the next two sections the characteristics of the data requirements and of the decision support system itself will be presented and contrasted with those of a data processing system.

Characteristics of Data In general, the volume of data which must be handled in a decision support system is much lower than that handled by a data processing system. The requirements for detail may also be much lower and there may be less emphasis on accuracy.

The major problem, however, is that the data which are needed are not always found in the traditional categories of accounting records. It is often necessary to collect data which were previously unavailable. The data needed in a data processing system, on the other hand, are usually quite available.

Characteristics of the System The complexity of the computer system including its programs is generally not high, but the intellectual complexity of the model may be high. Often, indeed, the model may be the most complex aspect of the system.

It is not important that the system be efficient because decision support systems do not get the intense use common to some data processing systems. The payoff does not come from repeated use of a system which has a series of small savings but rather the intermittent use of a system with large potential payoffs associated with each use.

Finally, there is another difference between these systems that is worth mentioning. Data processing systems generally focus on the past and the present. Decision support systems, on the other hand, focus on the present and future, since this is the relevant time frame for decision purposes.

Management information systems, then, comprise data processing systems and decision support systems. The focus in this book has been on quantitative models which have been found to be useful in decision support systems. Since the computer may be an integral part of these systems, familiarity with how computers work and how they can be programmed is essential. These topics are covered in the next sections.

COMPUTER FUNDAMENTALS

A computer system is composed of the three basic units shown in Figure 18-1. These include the input, output, and central processor units. In addition, secondary storage devices may be used.

Input

When a program is to be run on a computer, it must be entered, along with the data, in an input device. Depending upon the computer system, several methods may be used. One method is to punch the program and data onto a card using a key punch. The cards are then read by an input device called a high-speed card reader. The purpose of this device is to transfer the program

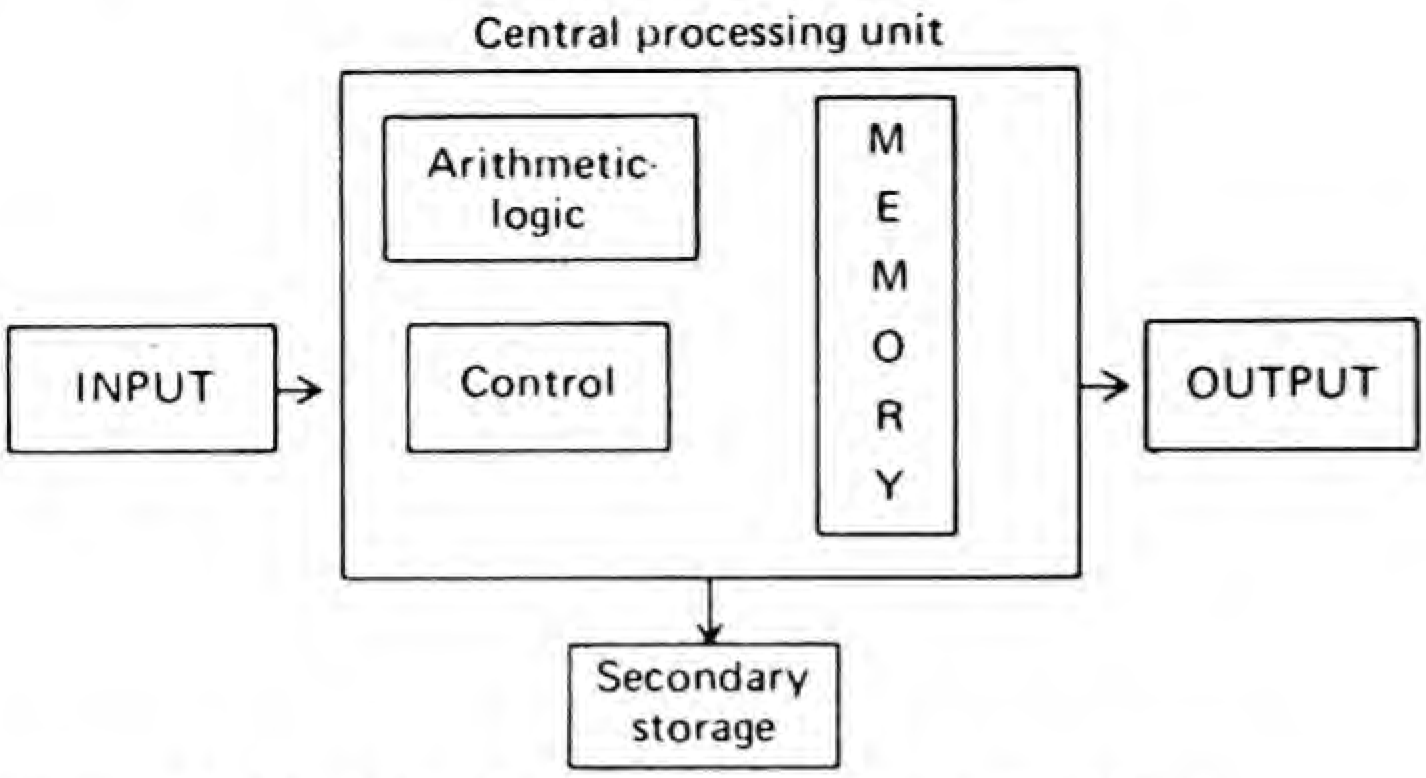


Figure 18-1 The basic units of a computer system.

and data into the central processor. In some computer systems the input can be entered directly through remote terminals, thereby eliminating the need to keypunch cards.

Central Processor

The central processor includes the control, arithmetic, and memory units. When programs and data are first entered through input devices, they are stored in the memory or storage unit.

Memory The purpose of the memory unit is to hold programs and data, to hold intermediate information during the execution of the program, and to hold the result just before it is released as output.

The memory or primary storage unit inside the central processor can be expanded by the addition of secondary storage devices located outside the central processor. The purpose of this is to increase the capacity of the computer.

Arithmetic-Logic Whenever the program requires the processing of data, the relevant data are transferred to the arithmetic-logic unit. The purpose of this unit is to make calculations such as additions, multiplications, subtractions, divisions, and comparisons.

Control Every unit of the computer is under the direction of the control unit. It receives instructions from the primary storage unit. After the interpretation of these instructions, the control unit may direct the memory unit to retrieve data, direct the arithmetic-logic unit to perform the necessary computations or comparisons, or direct the appropriate output devices to process the results.

Output The purpose of the output devices is to convert the results of the program, which are in the form of electronic impulses, into readable form.

This may be done on a high-speed printer, a video display unit, a teletype, or punched cards.

COMPUTER SYSTEMS

There are many kinds of computer systems including batch, remote batch, multiprogramming, inquiry, and general-purpose time-share systems. It is impossible to say which one is better, because the application determines the system that should be used. Consequently, it is necessary to know something about each one.

Batch Processing

All first-generation computers and many second and third-generation computers can be classified as batch processing systems. The major difference between this and other systems is that the computer is totally dedicated to the job being run. That is, other jobs must wait in a queue until the one being processeu is completed. Then the next job in line is run while the others wait, and so on. This approach has two drawbacks, including the inefficient use of the central processor and the time it takes to complete a job, called the turnaround time.

In most batch facilities the following sequence of steps occurs. First, the data and program are entered, the program is executed, and the output becomes available. During most of these steps the central processor is entirely dedicated to this one job and as a result a considerable amount of idle time may be incurred. For example, when the program requires that data which are stored in memory be processed, the other units in the system sit by idly as the search for the data is undertaken. In strict batch processing, little can be done to overcome this problem.

The turnaround problem is the second drawback of batch processing. Usually a job is brought to the central computing facility, and output is picked up some minutes or hours later. The wait depends upon the number of jobs in the queue and on their length. It is not unusual for this process to take 24 hours. This delay, together with the inconvenience of dropping off and picking up the job, casts an unfavorable shadow over the exclusive use of batch processing for most decision support systems.

Many decision support systems require real-time capabilities. This means that there must be an insignificant period of time between entering information in the computer and receiving a response. In general, if the response time is less than 2 or 3 minutes, the system is said to have real-time capabilities. Batch processing certainly does not have this capability.

Many decision support systems also require that the user interact with the computer in a conversational way. If a system is fully conversational, it must be capable of a fast response *and* must also provide the user with the opportunity of using the computer as a problem-solving tool. By definition

conversational systems are also real-time. Therefore batch processing systems do not fit in this category, either.

Remote Batch

One method for reducing the turnaround time is to use remote batch processing. In this method terminals are placed in convenient locations. Jobs are entered from these terminals and transmitted over a communications system to the central computer, where they take their place in queue waiting for their turn to be executed. Once completed, the output is transmitted back over the communications system to the input-output terminal. Except for the convenience of remote input-output, the system operates basically as a batch process. That is, it is not conversational or real-time.

Multiprogramming

Multiprogramming is a technique which allows the central processor to process more than one program at a time. This overcomes one of the basic inefficiencies of the batch process. It works in the following way. If the central processor is idle at any time during the execution of a program, the next program in the queue is temporarily processed. When the first program once again requires the use of the central processor, it switches from the second to the first. For example, if the primary program directed the computer to search for data in a storage file, during the time it would take to seek and retrieve this information the central processor would switch to the next job. A major advantage of multiprogramming, therefore, is that the central processor is kept busy most of the time, and the computer becomes more efficient. The turnaround time, however, may still be substantial. Again this system is neither real-time nor conversational.

Inquiry Systems

An inquiry system operates in real time. Input-output devices in this system are located at remote points and connected to the central computer by means of a communications system. The system is designed in such a way that the remote terminal operator can originate an inquiry and receive an answer from the central computer in a few seconds. Typical uses of such systems are stock quotation services and airline reservation systems. In these applications a trivial amount of computation and a considerable amount of search for particular data held in storage are required. Many of these systems are multiprogrammed, since a substantial portion of the system time is spent searching for data. Although the system is real-time, it is not conversational because computers in this category are quite specialized and cannot be used as a general problem-solving tool.

General-Purpose Time-Share Systems

In a general-purpose time-share system, the user has the impression of being the only one using the machine: the response to inputs is nearly instan-

taneous. This is accomplished by allocating a slice of the computer's time to each of several users located at remote terminals. In this time slice a user has complete use of the central processor, after which the next user has exclusive use, and so on, the central processor eventually returning to the first user. The cycle is so fast that the user gets the impression of continuous, exclusive use. The system is therefore real-time.

The system is general-purpose in that users can retrieve information, use standard computer programs, or write their own programs in Basic, Fortran, and so on. Because this computer system can be used as a general problem-solving tool and because it is real-time, it is classified as conversational.

In some decision support systems it is essential that the computer have this conversational capability. Consider, for example, a decision support system used by a major oil refinery which was designed to help its production managers formulate weekly schedules. At the center of this system is a linear-programming model. From a remote terminal, which is connected to a general-purpose time-share system, the model is run using available data and the results are observed. Then production managers make several changes in the data for the purpose of observing the effect that other levels of workforce, equipment, raw materials, demand, and production efficiencies might have on the schedule. These changes are entered in the remote terminal, the linear-programming model is run again, and the results are observed. Quite often several more changes are made before the schedule is finalized. By interacting with the computer in this conversational way, decision makers can gain valuable insight into the response of the production system to changes in the data. The benefit to the oil refinery has been in the development of more efficient production schedules.

COMPUTER SOFTWARE

How do we communicate to the computer the logical sequence of steps to be executed? Unfortunately the computer has a very limited vocabulary. Therefore the program that describes these steps must be written in a language the computer understands. Several are currently used, such as Fortran, Basic, Cobol, and PL/1. Each has its own language rules, and if you are interested in programming, these rules must be learned.

Both Fortran and PL/1 are general-purpose languages which can be used in business and engineering applications. Cobol is a special-purpose language directed at simplifying the preparation of programs which focus on data processing problems in business. Basic is another special-purpose language which was developed to make it quite easy for beginners to be able to program the remote terminals found in most general-purpose time-share systems. A few elementary rules of Basic as well as an example are given in Appendix B.

It would seem inefficient if every time a popular technique such as linear programming was to be run on the computer, a program had to be written.

Fortunately, most computer systems permanently store this and other commonly used programs in their secondary storage units. When a user wants to analyze a linear-programming problem, it is necessary only to supply the data and call for the stored program. The user therefore needs to know very little about computer programming.

Permanently stored programs are called "canned programs" or "software packages." Besides linear programming, they are generally available for the transportation method, for network analysis (PERT), for decision-tree analysis, and to a less extent for forecasting and inventory control.

Unfortunately, the logical sequence of steps followed in one simulation model may be different in the next. Therefore, no single canned simulation model has ever been developed. There are, however, many common *sets* of steps in the simulation process, and two languages, GPSS and Simscript, are used to exploit these common aspects. They make the programming job easier.

When a canned routine cannot be found to meet the needs of the user, then, of course, a program must be written. If, on the other hand, a canned routine will meet these needs, all the user needs to know is how to call the routine and the format that the data should take when entered in the computer. An example of a canned linear-programming routine used on a general-purpose time-share system is given in Appendix A.

QUANTITATIVE METHODS AND COMPUTERS

The models developed in this book can often be found as part of computer-based management information systems. We will now return to several of these models and examine how computers are used.

Decision-Tree Analysis

Small decision-tree problems are solved by hand, but large ones are often solved by using canned programs.

In the canned programs the user must supply, as input information, the size of the decision tree, the probabilities, and the payoffs. The computer then calculates the expected payoff for each action alternative, and often the program has the additional capability of performing sensitivity analysis on the probability and payoff estimates.

Both batch and general-purpose time-share systems have been used to solve decision-tree problems. For very large problems with hundreds of branches the time-share system proves to be too inefficient and time-consuming; for these situations the batch system is used.

If the time-share system can be used, its conversational benefits allow the decision maker to explore the consequences of a series of changes in payoffs and probabilities on the expected payoffs of the action alternatives. The benefits derived from this interactive process are highly regarded by those

who have used such systems. For it is through this interactive process that they gain additional insight into the nature of their decision problem.

Linear Programming

Most computer systems have canned programs for solving linear-programming problems. All that is necessary is to enter the data associated with the objective function and constraints in the appropriate format, and the canned program does the rest.

Nearly all linear-programming problems encountered in practice are quite large. Some have thousands of variables and hundreds of constraints. Consequently data are often prepared on punched cards by skilled keypunch operators and read into batch processing systems.

Simulation

Computers are essential in running simulation models. Not only can the simulation model itself be long and complex, but the desired number of replications can be in the thousands. There is no way that simulations of this size can be executed by hand.

Because one simulation problem may be quite different in form from another, no single general-purpose simulation program can be written. Each simulation model must be programmed separately. There are special-purpose languages, however, such as GPSS and Simscript, which make the programming of these models easier. Nonetheless running a simulation model on the computer requires computer programming skills.

Both batch and time-share systems are used to run simulations, but as the model becomes more complex, the preference shifts to the use of batch systems.

Inventory Control

Nearly all modern inventory control systems place heavy demands upon the computer. The records kept and the decisions made all require the collection, storage, and analysis of tremendous volumes of data.

Inventory systems have characteristics of both data processing and decision support systems. They are data processing systems because they must collect, store, and present data, and they are decision support systems because they are used to determine the routine decisions of when and how many units to order.

Once again no general-purpose canned program can be written for an inventory system. Each one has its own special characteristics. But economic order quantity formulas are frequently an integral part of these programs.

Quite often inquiry-type systems are used in inventory control. Remote terminals may be found in many warehouse locations, and in these terminals are entered additions and depletions from stock. The terminals also supply status reports on any item. Since these terminals are not used in a problem-solving mode, only real-time capability is necessary.

Networks

Large-scale network models such as PERT often require the use of a computer. Canned programs that are available require the user to specify each activity, its predecessor, and the associated time estimates. The computer then develops the network and determines the critical path.

Since time estimates may change during the life of the project, the network is periodically updated by the introduction of revised estimates.

Both batch and inquiry systems have been used for network analysis. The benefit of an inquiry system is that changes in estimates can be entered from remote locations. In addition the progress of the project can be monitored from those locations. For example, managers thousands of miles from the site of a project can receive continuous updates on critical path status.

SUMMARY

The purpose of this chapter has been to explore the use of computers in the support of quantitative methods. Early use was directed at data processing problems, but more recently there has been a significant increase in the use of computers as part of decision support systems. Therefore, the modern manager must be familiar not only with the use of quantitative methods but with the computer as well.

QUESTIONS

1 Describe the differences between the use of computers in data processing and decision support systems.
2 Do you think that computer-based management information systems emphasize quantitative or nonquantitative data? What is the consequence of this for the decision-making process?
3 To what extent must managers be familiar with computer programming?
4 Why are batch processing systems generally unsuitable for use in decision support systems?
5 What role will the computer play in an organization by the year 2000?

APPENDIX A: Canned Routines

To illustrate the use of canned routines, a linear-programming problem will be presented; the computer inputs that are required will be identified; and the solution generated by a general-purpose time-share system will be explained. Although the format of the input data may differ slightly between computer systems, the example shown here should make it quite easy for you to use any canned linear-programming routine.

Consider the Jerry Company case presented in Chapters 7, 8, and 9. It is summarized below:

$$\begin{aligned} \text{Max } P = 20C + 30M & \\ 2C + 1M &\leq 8000 \\ 2C + 4M &\leq 16{,}000 \\ C, M &\geq 0 \end{aligned}$$

The first step in using a canned routine is to summarize the problem in inequality form.

Turning now to the printout illustrated in Figure 18A-1, we see that the next step is to load the simplex routine. This transfers the program from secondary storage facilities to the central processor. The name of the program used was **SIMPLEX. Therefore the first entry was LOAD**SIMPLEX. For your convenience, each entry required by the user has been underlined.

After the command has been given to load the program, the computer responds that it is READY.

Next, the data must be entered. This *must* be done in the following way. First, the left-hand-side constraint coefficients are entered. Each constraint is entered on its own data line, and each coefficient is separated from the next one by a comma. After the constraints are entered, the right-hand-side values are entered, and finally the objective function coefficients.

Once the data are entered, the letters RUN are typed and the program will be started.

The program asks several questions, and the user must provide the answers before it will continue. It asks whether this is a maximization or a minimization problem, the number of activities (or variables) in the problem, and the number of "less than or equal to" constraints, the number of "equal to" constraints, and the number of "greater than or equal to" constraints. The remaining questions refer to the level of output detail you would like.

In regard to the output, some explanations are necessary. First, the term "B-VECTOR" is nothing more than the right-hand-side values. The dual evaluators are the shadow prices. The variable Z represents the value of the objective function.

Comparing this output with the final basis in Table 9-3, we can see that they are identical.

Reading the computer output, the solution is:

Activity number	Variable name	Value
1	C	2666.67
2	M	2666.67

The profit is $Z = 133{,}333$.

```
LOAD **SIMPLEX
READY

1000  DATA 2,1
1001  DATA 2,4
1002  DATA 8000,16000
1003  DATA 20,30
RUN

SIMPLEX     19:18  MAY 13, 1975

TYPE  DES FOR A DESCRIPTION AND INSTRUCTIONS,
TYPE  INS  FOR INSTRUCTIONS ONLY,
TYPE  OMIT  FOR OMITTING BOTH.
? OMIT

IS THIS A 'MAX' PROBLEM OR A 'MIN' PROBLEM?MAX
HOW MANY ACTIVITIES DOES YOUR PROBLEM HAVE?2
HOW MANY CONSTRAINTS IN YOUR PROBLEM (<=, =, >=) ?2,0,0
DO YOU WNAT THE INITIAL TABLEAU PRINTED?YES
DO YOU WANT THE INTERMEDIATE BASIC SOLUTIONS PRINTED?YES
DO YOU WANT THE FINAL TABLEAU PRINTED?YES
STRUCTURAL ACTIVITIES        1 - 2
SLACK ACTIVITIES             3 - 4
PHASE II INITIATED AFTER 0      ITERATIONS

              INITIAL TABLEAU
```

Figure 18A-1 Computer output of linear-programming example.

X (1)	X (2)	X (3)	X (4)
2	1	1	0
2	4	0	1
OBJ FN ROW:			
−20	−30	0	0

B VECTOR

8000
16000
OBJ FN ROW:
0

INITIAL BASIC SOLUTION

ACTIVITY	VALUE
3	8000
4	16000

Z= 0

BASIC FEASIBLE SOLUTION AFTER ITERATION 1

ACTIVITY	VALUE
1	2666.67
2	2666.67

Z= 133333

THE SOLUTION IS MAXIMAL.

OPTIMAL DUAL SOLUTION

CONSTRAINT	DUAL EVALUATOR
1	3.33333
2	6.66667

OPTIMAL TABLEAU

X (1)	X (2)	X (3)	X (4)
1	0	.666667	−.166667
0	1	−.333333	.333333
OBJ FN ROW:			
0	0	3.33333	6.66667

B-VECTOR

2666.67
2666.67
OBJ FN ROW:
133333.

TIME 3 SECS.

APPENDIX B: BASIC Programming: Some Rules and an Example

INTRODUCTION

In this appendix a few rules of Basic programming will be presented, along with an example, in order to provide you with just enough of the language to be able to write simple programs but not enough to write more complex ones.

STATEMENT CATEGORIES

There are three major categories in which the vocabulary of the Basic language can be placed. These include arithmetic, input-output, and control categories. These categories and examples of the statements which fall into them are given in Table 18B-1.

In Table 18B-2 a sample program is presented which is capable of computing and printing the sum of two numbers. The category into which these statements fall is identified next to each one. Notice that the steps in the sample program are written in a logical order. First the data are supplied (A and B), then the sum is computed, then the results are printed, and then the computer is informed that the end of the program has been reached. In addition, every step is numbered in sequence.

STATEMENT RULES

It is essential that certain rules be followed when writing statements. These rules are organized according to the three categories of Table 18B-1.

Table 18B-1 Statement Categories

Categories	Statements
Arithmetic	LET
Input-output	READ, DATA, INPUT, PRINT
Control	GO TO, END

Table 18B-2 Sample Program and the Statement Categories

Program	Statement category
100 INPUT A, B	Input
110 LET C = A + B	Arithmetic
120 PRINT C	Output
130 END	Control

Arithmetic Statements

An arithmetic operation is performed by using the word LET, followed by an equal sign and an arithmetic expression. Examples are given below:

```
LET  C = A + B
LET  Z = X/Y
```

In the first example C is set equal to A plus B, and in the second example Z is set equal to X divided by Y.

The symbols which can be used in formulating these mathematical expressions are limited to the ones shown below:

Symbol	Operation
+	Addition
−	Subtraction
*	Multiplication
/	Division
↑	Exponentiation

For example, if A is to be multiplied by E, it must be written as A * E; and if S is to be raised to the fifth power, it must be written as S ↑ 5.

In Table 18B-2 the letters A, B, and C are called variables. A variable is a quantity that can assume (or be assigned) any of a given set of values. In Basic, variables can be denoted by single letters.

To ensure that the arithmetic statement is executed in the right order, parentheses are always used. For example, consider the following two statements:

```
LET  A = B + C/2
LET  A = (B + C)/2
```

To the computer they are quite different. In the first statement, C will be divided by 2 and the result will be added to B. In the second statement, however, B will be added to C and then this *sum* will be divided by 2. Therefore, operations inside parentheses are always performed first.

Input-Output Statements

The purpose of the input-output statements is to direct the computer to transfer data between the central processor and input-output terminals. The words used to accomplish this are READ, DATA, INPUT, and PRINT.

1 READ statements direct the computer to read data which are to be found in a following DATA statement.

2 DATA statements provide the data requested by a READ statement.

3 INPUT statements direct the computer to ask the user to input the necessary data through the terminal.

4 PRINT statements direct the computer to type the output on the user's terminal.

READ and DATA Statements The READ statement includes the list of the variables for which the data are to be read. Consecutive variables must be separated by a comma.

```
200  READ  A, C, X
```

This statement will direct the computer to find the next DATA statement, and read the first three as the values of *A*, *C*, and *X*.

The DATA statement is written in the following way:

```
300  DATA  125, 10, 1.5
```

The computer will therefore assign the following values:

A = 125
C = 10
X = 1.5

It can be concluded that whenever a READ statement is used, a DATA statement must be included before the END of the program.

INPUT Statements The INPUT statement directs the computer to ask the user for input data. Consider the following statement:

```
500  INPUT  A, B
```

When this program is executed, the computer will print a question mark on the terminal and wait for the user to type the values of the two variables. Once these values are received, the program will be continued.

PRINT Statements The PRINT statement is used to print output on the terminal. It can be used in the following way:

```
1000  PRINT  A, X
```

The computer will print the values which it has stored for variables A and X.

The PRINT statement can be used also to print headings and labels in the following way:

```
100  PRINT  "THE ANSWER IS"
```

Whatever is within quotation marks will be printed out on the terminal.

Control Statements Control statements are used to transfer the control to various parts of the program. One mechanism for accomplishing this is the GO TO statement.

```
125  GO TO  100
```

Table 18B-3 Introduction of GO TO Statement

```
100 INPUT A, B
110 LET C = A + B
120 PRINT C
125 GO TO 100
130 END
```

Table 18B-4 Introduction of READ and DATA Statements

```
100 READ A, B
110 LET C = A + B
120 PRINT C
125 DATA 10, 15, 25, 60
126 GO TO 100
130 END
```

When the computer arrives at this statement, it will be directed to return to statement 100.

SAMPLE PROGRAMS

The program illustrated in Table 18B-2 can be analyzed in the following way. The computer will start executing the program at statement 100. A question mark will automatically be typed on the terminal, and the execution of the program will temporarily be halted until the data for A and B are entered. Once entered, their sum will be computed and then the results printed on the terminal.

Suppose that the user wanted to use this program several times. It would therefore be convenient to have the computer return to statement 100 and start again. This can be accomplished by introducing a GO TO statement after statement 120. This is done in Table 18B-3.

If instead of using an INPUT statement, READ and DATA statements were used, the actual data would have to be provided in the program. The revised program is shown in Table 18B-4. The first time through the summation sequence the program would read the values of 10 and 15 for A and B. When statement 126 is reached, the computer returns to statement 100 and proceeds to read the *next* two values in the DATA statement. When it returns once again, you will be informed that the computer has run out of data and your program will be terminated.

PROBLEMS

18-1 Write a program for computing the average of five numbers which are supplied as input by the user.

18-2 Write a program for the following expression:

$$Y = \frac{AB - C/2}{4}$$

The values of the variables will be supplied by the user.

18-3 Write a program for computing the standard deviation of the following five numbers: 10, 12, 16, 8, and 5.

18-4 Write a program for computing the economic order quantity. The user will supply the demand estimate and per piece cost. The inventory carrying cost is 10 percent per unit per year, and the fixed order cost is $100 per order.

18-5 Write a program for computing the total yearly cost of an inventory system. Use the data supplied in problem 18-4.

18-6 Assume that you are involved in a large project and would like to use the techniques of PERT analysis. Write a program which asks the user for their most likely, optimistic, and pessimistic estimates, and then computes the expected time from these estimates. The program should be designed in such a way that upon completion of one expected value the process is repeated.

18-7 The operations department of a large bank is trying to determine the customer waiting time (average time spent in the system) if four tellers are used. Their problem, however, is that they are not sure of the arrival rates and would like to examine the consequence of several different ones. The average service time is 2 minutes. Both the arrival and service patterns can be represented by Poisson distributions. The bank channels customers into one line so that the system can be represented as a single-channel arrival–multiple-channel service type. Write a program which can be used for this analysis.

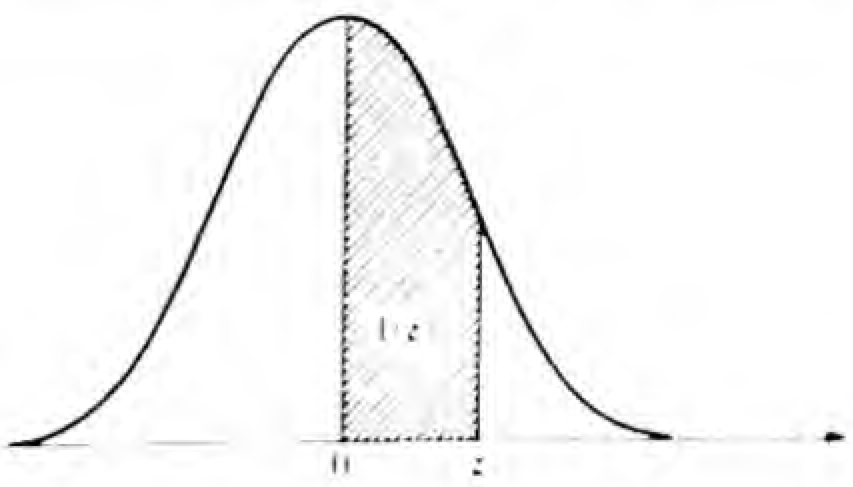
APPENDIX OF TABLES

Table 1 Normal Curve Areas

Area under the standard normal curve from 0 to z, shown shaded, is $A(z)$.

Examples. If Z is the standard normal random variable and $z = 1.54$, then

$$A(z) = P(0 < Z < z) = .4382$$
$$P(Z > z) = .0618$$
$$P(Z < z) = .9382$$
$$P(|Z| < z) = .8764$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

From Frederick Mosteller and Robert E. K. Rourke, *Sturdy Statistics*, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1973, p. 313.

Table 2 Random Numbers

53076	98356	71012	72913	57081	50378	24782	59604	68503	87115
67675	66328	31868	81477	44108	30976	97286	11185	85146	80501
11682	77634	35669	48952	11988	76536	47230	5101	56004	447
80779	52122	53345	65387	98605	43750	96415	48056	23627	15127
18002	40596	31530	47181	56785	91219	98542	55551	52010	42465
74217	45876	28094	41249	78516	63703	49785	37480	43103	15747
10332	39877	18937	66983	80203	5878	25213	86229	53043	65949
61641	6372	15397	58928	98026	56147	41803	27564	16801	49977
43916	21062	73364	23630	15617	27252	47003	80081	60913	65541
76522	33375	75162	85546	76477	51924	56899	87600	11637	32607
89274	71515	72226	76519	30405	13914	88805	6010	56571	62411
32280	90189	77336	40196	34756	49649	3071	44055	58885	55252
43878	83500	48961	20485	1404	55651	50495	41115	45496	87123
92040	11812	5970	16516	68018	38973	22272	71589	45584	73886
6359	2591	68104	91446	22993	21631	36380	53439	58588	61660
86313	36575	46153	37869	28847	5297	49395	51115	55731	29636
87662	73748	84288	29618	52395	70084	53295	15741	99974	74435
24285	66547	48913	73717	53551	65409	21248	56997	85000	35454
69058	76804	13315	96064	99615	35505	85816	44036	39764	6913
19215	42140	60830	83484	32894	51061	1645	31020	41339	67819
98487	1399	87220	35500	81485	9378	93659	14486	55827	24842
52206	36683	53151	73242	83242	19358	84254	96160	94802	49007
93275	36216	90302	89493	87642	66725	25401	72950	94381	32159
53993	7594	26512	74082	15915	70083	52465	93485	44128	31367
72734	80239	16993	40338	75399	85135	69427	2434	12818	3234
20823	36035	11101	1463	50068	5666	15313	75499	19621	44791
18203	39387	32534	41814	38175	88661	63836	62019	61531	77848
32030	42255	75125	48994	53716	47878	47411	84658	96763	92145
32626	32699	4949	5213	66642	15814	72257	7122	58807	78031
42589	5796	44046	50050	99642	28893	33291	92302	72065	16855
45818	6379	22194	95072	16463	4221	83866	11458	55554	54878
15188	51600	35924	1054	45465	84325	66904	2428	6880	18112
56103	99032	40717	51157	97265	49433	88443	47471	44185	87971
65959	49977	27971	19762	84439	79377	62707	42935	49401	56964
79082	70637	1453	40502	38065	52271	787	80762	35583	63603
22815	10297	4490	50398	44966	31144	64458	78696	88324	29508
62014	56652	42726	41713	37607	42636	54659	67205	259	57143
68594	76822	31362	80684	58100	28842	83207	58900	70070	40103
77866	66073	79150	37985	43637	42600	17149	95556	96339	72246
45063	57832	12466	54501	11314	96443	75026	50885	27642	93890
12734	19579	3176	47576	48602	45393	85097	31695	9856	67375
64837	54064	78104	1776	60400	68663	45096	90193	81828	92087
56683	73431	70175	43790	96677	58275	50851	93630	88019	27138
7263	98095	12841	26402	65134	54661	9645	58195	71850	4143
48058	25821	89441	36557	28807	6601	42032	54674	82623	80138
48051	19019	48678	40788	20914	17274	18822	53466	84952	87683
25867	34598	86636	57225	10061	94130	83161	13024	7262	97539
70560	25520	90985	66630	30384	61363	11016	17455	98874	84514
11130	30614	38973	22382	80797	53708	25215	88232	38211	67251
70042	12020	12248	37901	60797	46160	44691	89594	88327	32170

Table 2 Random Numbers (*Continued*)

81220	89905	96270	3623	90920	91528	5124	78102	99305	12198
2014	96843	72247	97374	98257	89117	15199	63100	32723	29380
73567	4964	20262	80530	5760	16541	92486	53998	12614	1070
8847	68279	65016	31205	24808	61308	56989	76232	46886	64975
64935	70822	84952	87846	56112	91074	54363	74277	8944	64363
7043	80474	49962	12441	29898	63879	24339	20601	16021	76895
29407	42824	38890	40494	29968	3065	38387	41567	93671	28325
59049	58423	97500	22786	81410	70815	77952	50940	82326	85601
77766	66495	97116	42433	51279	30837	60086	45643	32648	54973
17725	66003	9927	38545	98164	78503	96499	30562	87855	64955
81120	90401	87452	65591	844	70565	30688	12251	40816	49486
37314	78605	98343	58618	90702	41387	15051	15600	60412	68847
85817	44878	74780	7073	10142	27440	94016	70152	20730	44038
50924	66059	64853	69569	43822	42330	49230	87444	57565	47094
27781	31945	57841	20968	99685	70551	16834	82722	78239	35710
68139	46107	92977	40399	35679	69097	96061	96834	62644	80971
58373	47705	76396	8519	42943	42827	42016	38177	33948	43368
57099	85151	85064	99127	35661	78037	34949	35350	32095	6267
40550	81835	99200	8138	65345	10632	36604	75115	39045	93913
57156	81887	50162	10946	48234	68378	62644	80983	54443	53328
346	45443	34641	29396	31907	48056	63436	65724	32923	27460
20551	66428	30398	24441	21103	13002	85275	7796	26772	31961
13237	18786	17842	81901	64066	73767	4053	17150	96371	4372
85865	56546	37401	64846	62861	33533	31696	11715	9585	99254
95711	50585	29939	70075	44808	61069	19937	57696	77096	2724
5360	11764	58574	46894	72658	290	87552	64119	42070	92144
4992	47278	52559	86678	98174	14857	63431	60280	38274	29566
90597	81910	73621	59393	58848	615	9830	42081	3249	20599
19083	11456	53840	56265	58926	14049	22596	92848	12921	5625
56383	15958	15256	19494	19480	74910	36517	89190	87469	82434
5254	7036	73613	50525	70951	52644	10981	82883	37911	70583
13242	23236	26973	31103	23110	48336	1108	98940	50512	58286
2078	59372	38074	32130	41471	61442	89835	27125	81561	27098
98302	17526	69175	52521	49220	54538	47889	58361	36290	63880
77196	1375	63116	48593	56607	5183	37144	9873	84918	53938
58134	51069	10277	84679	17500	53137	59070	38531	84397	38345
43426	36102	77651	52946	69670	101	116	15814	72050	1696
43107	19419	44704	2231	11132	60806	78995	84566	5640	89349
32612	18515	49308	64825	42509	45466	57749	29870	1691	76404
26610	70580	45769	57831	10834	17156	41694	18903	33120	22320
37474	37018	85106	41007	38652	19310	36791	60364	21221	30898
4539	99115	23110	2108	89143	20156	15174	38180	36639	10213
41476	2827	2481	59144	11743	21120	30667	91562	38376	31092
38167	24378	58748	20093	12822	12344	33373	72754	134	33442
7121	57739	20073	93123	85809	41746	70687	51690	25597	66914
37304	68971	50934	76143	57872	12168	58969	38353	8263	89050
51342	79992	72212	62334	73850	48866	26419	81749	13486	64870
85490	21180	89743	36291	65229	87035	51856	90132	21086	96808
42013	35035	19893	14783	50447	37474	37616	77744	44506	5727
53459	18355	94140	92910	74259	76318	31448	65133	47542	14312

Table 3 Present Value of $1.00

Years Hence	1%	2%	4%	6%	8%	10%	12%	14%	15%	16%	18%	20%	22%	24%	25%	26%	28%	30%	35%	40%	45%	50%
1	0.990	0.980	0.962	0.943	0.926	0.909	0.893	0.877	0.870	0.862	0.847	0.833	0.820	0.806	0.800	0.794	0.781	0.769	0.741	0.714	0.690	0.667
2	0.980	0.961	0.925	0.890	0.857	0.826	0.797	0.769	0.756	0.743	0.718	0.694	0.672	0.650	0.640	0.630	0.610	0.592	0.549	0.510	0.476	0.444
3	0.971	0.942	0.889	0.840	0.794	0.751	0.712	0.675	0.658	0.641	0.609	0.579	0.551	0.524	0.512	0.500	0.477	0.455	0.406	0.364	0.328	0.296
4	0.961	0.924	0.855	0.792	0.735	0.683	0.636	0.592	0.572	0.552	0.516	0.482	0.451	0.423	0.410	0.397	0.373	0.350	0.301	0.260	0.226	0.198
5	0.951	0.906	0.822	0.747	0.681	0.621	0.567	0.519	0.497	0.476	0.437	0.402	0.370	0.341	0.328	0.315	0.291	0.269	0.223	0.186	0.156	0.132
6	0.942	0.888	0.790	0.705	0.630	0.564	0.507	0.456	0.432	0.410	0.370	0.335	0.303	0.275	0.262	0.250	0.227	0.207	0.165	0.133	0.108	0.088
7	0.933	0.871	0.760	0.665	0.583	0.513	0.452	0.400	0.376	0.354	0.314	0.279	0.249	0.222	0.210	0.198	0.178	0.159	0.122	0.095	0.074	0.059
8	0.923	0.853	0.731	0.627	0.540	0.467	0.404	0.351	0.327	0.305	0.266	0.233	0.204	0.179	0.168	0.157	0.139	0.123	0.091	0.068	0.051	0.039
9	0.914	0.837	0.703	0.592	0.500	0.424	0.361	0.308	0.284	0.263	0.225	0.194	0.167	0.144	0.134	0.125	0.108	0.094	0.067	0.048	0.035	0.026
10	0.905	0.820	0.676	0.558	0.463	0.386	0.322	0.270	0.247	0.227	0.191	0.162	0.137	0.116	0.107	0.099	0.085	0.073	0.050	0.035	0.024	0.017
11	0.896	0.804	0.650	0.527	0.429	0.350	0.287	0.237	0.215	0.195	0.162	0.135	0.112	0.094	0.086	0.079	0.066	0.056	0.037	0.025	0.017	0.012
12	0.887	0.788	0.625	0.497	0.397	0.319	0.257	0.208	0.187	0.168	0.137	0.112	0.092	0.076	0.069	0.062	0.052	0.043	0.027	0.018	0.012	0.008
13	0.879	0.773	0.601	0.469	0.368	0.290	0.229	0.182	0.163	0.145	0.116	0.093	0.075	0.061	0.055	0.050	0.040	0.033	0.020	0.013	0.008	0.005
14	0.870	0.758	0.577	0.442	0.340	0.263	0.205	0.160	0.141	0.125	0.099	0.078	0.062	0.049	0.044	0.039	0.032	0.025	0.015	0.009	0.006	0.003
15	0.861	0.743	0.555	0.417	0.315	0.239	0.183	0.140	0.123	0.108	0.084	0.065	0.051	0.040	0.035	0.031	0.025	0.020	0.011	0.006	0.004	0.002
16	0.853	0.728	0.534	0.394	0.292	0.218	0.163	0.123	0.107	0.093	0.071	0.054	0.042	0.032	0.028	0.025	0.019	0.015	0.008	0.005	0.003	0.002
17	0.844	0.714	0.513	0.371	0.270	0.198	0.146	0.108	0.093	0.080	0.060	0.045	0.034	0.026	0.023	0.020	0.015	0.012	0.006	0.003	0.002	0.001
18	0.836	0.700	0.494	0.350	0.250	0.180	0.130	0.095	0.081	0.069	0.051	0.038	0.028	0.021	0.018	0.016	0.012	0.009	0.005	0.002	0.001	0.001
19	0.828	0.686	0.475	0.331	0.232	0.164	0.116	0.083	0.070	0.060	0.043	0.031	0.023	0.017	0.014	0.012	0.009	0.007	0.003	0.002	0.001	
20	0.820	0.673	0.456	0.312	0.215	0.149	0.104	0.073	0.061	0.051	0.037	0.026	0.019	0.014	0.012	0.010	0.007	0.005	0.002	0.001	0.001	
21	0.811	0.660	0.439	0.294	0.199	0.135	0.093	0.064	0.053	0.044	0.031	0.022	0.015	0.011	0.009	0.008	0.006	0.004	0.002	0.001		
22	0.803	0.647	0.422	0.278	0.184	0.123	0.083	0.056	0.046	0.038	0.026	0.018	0.013	0.009	0.007	0.006	0.004	0.003	0.001	0.001		
23	0.795	0.634	0.406	0.262	0.170	0.112	0.074	0.049	0.040	0.033	0.022	0.015	0.010	0.007	0.006	0.005	0.003	0.002	0.001			
24	0.788	0.622	0.390	0.247	0.158	0.102	0.066	0.043	0.035	0.028	0.019	0.013	0.008	0.006	0.005	0.004	0.003	0.002	0.001			
25	0.780	0.610	0.375	0.233	0.146	0.092	0.059	0.038	0.030	0.024	0.016	0.010	0.007	0.005	0.004	0.003	0.002	0.001	0.001			
26	0.772	0.598	0.361	0.220	0.135	0.084	0.053	0.033	0.026	0.021	0.014	0.009	0.006	0.004	0.003	0.002	0.002	0.001				
27	0.764	0.586	0.347	0.207	0.125	0.076	0.047	0.029	0.023	0.018	0.011	0.007	0.005	0.003	0.002	0.002	0.001	0.001				
28	0.757	0.574	0.333	0.196	0.116	0.069	0.042	0.026	0.020	0.016	0.010	0.006	0.004	0.002	0.002	0.002	0.001	0.001				
29	0.749	0.563	0.321	0.185	0.107	0.063	0.037	0.022	0.017	0.014	0.008	0.005	0.003	0.002	0.002	0.001	0.001	0.001				
30	0.742	0.552	0.308	0.174	0.099	0.057	0.033	0.020	0.015	0.012	0.007	0.004	0.003	0.002	0.001	0.001	0.001					
40	0.672	0.453	0.208	0.097	0.046	0.022	0.011	0.005	0.004	0.003	0.001	0.001										
50	0.608	0.372	0.141	0.054	0.021	0.009	0.003	0.001	0.001	0.001												

Table 4 Unit Normal Loss Integral

D	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.3989	.3940	.3890	.3841	.3793	.3744	.3697	.3649	.3602	.3556
.1	.3509	.3464	.3418	.3373	.3328	.3284	.3240	.3197	.3154	.3111
.2	.3069	.3027	.2986	.2944	.2904	.2863	.2824	.2784	.2745	.2706
.3	.2668	.2630	.2592	.2555	.2518	.2481	.2445	.2409	.2374	.2339
.4	.2304	.2270	.2236	.2203	.2169	.2137	.2104	.2072	.2040	.2009
.5	.1978	.1947	.1917	.1887	.1857	.1828	.1799	.1771	.1742	.1714
.6	.1687	.1659	.1633	.1606	.1580	.1554	.1528	.1503	.1478	.1453
.7	.1429	.1405	.1381	.1358	.1334	.1312	.1289	.1267	.1245	.1223
.8	.1202	.1181	.1160	.1140	.1120	.1100	.1080	.1061	.1042	.1023
.9	.1004	.09860	.09680	.09503	.09328	.09156	.08986	.08819	.08654	.08491
1.0	.08332	.08174	.08019	.07866	.07716	.07568	.07422	.07279	.07138	.06999
1.1	.06862	.06727	.06595	.06465	.06336	.06210	.06086	.05964	.05844	.05726
1.2	.05610	.05496	.05384	.05274	.05165	.05059	.04954	.04851	.04750	.04650
1.3	.04553	.04457	.04363	.04270	.04179	.04090	.04002	.03916	.03831	.03748
1.4	.03667	.03587	.03508	.03431	.03356	.03281	.03208	.03137	.03067	.02998
1.5	.02931	.02865	.02800	.02736	.02674	.02612	.02552	.02494	.02436	.02380
1.6	.02324	.02270	.02217	.02165	.02114	.02064	.02015	.01967	.01920	.01874
1.7	.01829	.01785	.01742	.01699	.01658	.01617	.01578	.01539	.01501	.01464
1.8	.01428	.01392	.01357	.01323	.01290	.01257	.01226	.01195	.01164	.01134
1.9	.01105	.01077	.01049	.01022	$.0^{2}9957$	$.0^{2}9698$	$.0^{2}9445$	$.0^{2}9198$	$.0^{2}8957$	$.0^{2}8721$
2.0	$.0^{2}8491$	$.0^{2}8266$	$.0^{2}8046$	$.0^{2}7832$	$.0^{2}7623$	$.0^{2}7418$	$.0^{2}7219$	$.0^{2}7024$	$.0^{2}6835$	$.0^{2}6649$
2.1	$.0^{2}6468$	$.0^{2}6292$	$.0^{2}6120$	$.0^{2}5952$	$.0^{2}5788$	$.0^{2}5628$	$.0^{2}5472$	$.0^{2}5320$	$.0^{2}5172$	$.0^{2}5028$
2.2	$.0^{2}4887$	$.0^{2}4750$	$.0^{2}4616$	$.0^{2}4486$	$.0^{2}4358$	$.0^{2}4235$	$.0^{2}4114$	$.0^{2}3996$	$.0^{2}3882$	$.0^{2}3770$
2.3	$.0^{2}3662$	$.0^{2}3556$	$.0^{2}3453$	$.0^{2}3352$	$.0^{2}3255$	$.0^{2}3159$	$.0^{2}3067$	$.0^{2}2977$	$.0^{2}2889$	$.0^{2}2804$
2.4	$.0^{2}2720$	$.0^{2}2640$	$.0^{2}2561$	$.0^{2}2484$	$.0^{2}2410$	$.0^{2}2337$	$.0^{2}2267$	$.0^{2}2199$	$.0^{2}2132$	$.0^{2}2067$
2.5	$.0^{2}2004$	$.0^{2}1943$	$.0^{2}1883$	$.0^{2}1826$	$.0^{2}1769$	$.0^{2}1715$	$.0^{2}1662$	$.0^{2}1610$	$.0^{2}1560$	$.0^{2}1511$
2.6	$.0^{2}1464$	$.0^{2}1418$	$.0^{2}1373$	$.0^{2}1330$	$.0^{2}1288$	$.0^{2}1247$	$.0^{2}1207$	$.0^{2}1169$	$.0^{2}1132$	$.0^{2}1095$
2.7	$.0^{2}1060$	$.0^{2}1026$	$.0^{3}9928$	$.0^{3}9607$	$.0^{3}9295$	$.0^{3}8992$	$.0^{3}8699$	$.0^{3}8414$	$.0^{3}8138$	$.0^{3}7870$
2.8	$.0^{3}7611$	$.0^{3}7359$	$.0^{3}7115$	$.0^{3}6879$	$.0^{3}6650$	$.0^{3}6428$	$.0^{3}6213$	$.0^{3}6004$	$.0^{3}5802$	$.0^{3}5606$
2.9	$.0^{3}5417$	$.0^{3}5233$	$.0^{3}5055$	$.0^{3}4883$	$.0^{3}4716$	$.0^{3}4555$	$.0^{3}4398$	$.0^{3}4247$	$.0^{3}4101$	$.0^{3}3959$
3.0	$.0^{3}3822$	$.0^{3}3689$	$.0^{3}3560$	$.0^{3}3436$	$.0^{3}3316$	$.0^{3}3199$	$.0^{3}3087$	$.0^{3}2978$	$.0^{3}2873$	$.0^{3}2771$
3.1	$.0^{3}2673$	$.0^{3}2577$	$.0^{3}2485$	$.0^{3}2396$	$.0^{3}2311$	$.0^{3}2227$	$.0^{3}2147$	$.0^{3}2070$	$.0^{3}1995$	$.0^{3}1922$
3.2	$.0^{3}1852$	$.0^{3}1785$	$.0^{3}1720$	$.0^{3}1657$	$.0^{3}1596$	$.0^{3}1537$	$.0^{3}1480$	$.0^{3}1426$	$.0^{3}1373$	$.0^{3}1322$
3.3	$.0^{3}1273$	$.0^{3}1225$	$.0^{3}1179$	$.0^{3}1135$	$.0^{3}1093$	$.0^{3}1051$	$.0^{3}1012$	$.0^{4}9734$	$.0^{4}9365$	$.0^{4}9009$
3.4	$.0^{4}8666$	$.0^{4}8335$	$.0^{4}8016$	$.0^{4}7709$	$.0^{4}7413$	$.0^{4}7127$	$.0^{4}6852$	$.0^{4}6587$	$.0^{4}6331$	$.0^{4}6085$
3.5	$.0^{4}5848$	$.0^{4}5620$	$.0^{4}5400$	$.0^{4}5188$	$.0^{4}4984$	$.0^{4}4788$	$.0^{4}4599$	$.0^{4}4417$	$.0^{4}4242$	$.0^{4}4073$
3.6	$.0^{4}3911$	$.0^{4}3755$	$.0^{4}3605$	$.0^{4}3460$	$.0^{4}3321$	$.0^{4}3188$	$.0^{4}3059$	$.0^{4}2935$	$.0^{4}2816$	$.0^{4}2702$
3.7	$.0^{4}2592$	$.0^{4}2486$	$.0^{4}2385$	$.0^{4}2287$	$.0^{4}2193$	$.0^{4}2103$	$.0^{4}2016$	$.0^{4}1933$	$.0^{4}1853$	$.0^{4}1776$
3.8	$.0^{4}1702$	$.0^{4}1632$	$.0^{4}1563$	$.0^{4}1498$	$.0^{4}1435$	$.0^{4}1375$	$.0^{4}1317$	$.0^{4}1262$	$.0^{4}1208$	$.0^{4}1157$
3.9	$.0^{4}1108$	$.0^{4}1061$	$.0^{4}1016$	$.0^{5}9723$	$.0^{5}9307$	$.0^{5}8908$	$.0^{5}8525$	$.0^{5}8158$	$.0^{5}7806$	$.0^{5}7469$
4.0	$.0^{5}7145$	$.0^{5}6835$	$.0^{5}6538$	$.0^{5}6253$	$.0^{5}5980$	$.0^{5}5718$	$.0^{5}5468$	$.0^{5}5227$	$.0^{5}4997$	$.0^{5}4777$
4.1	$.0^{5}4566$	$.0^{5}4364$	$.0^{5}4170$	$.0^{5}3985$	$.0^{5}3807$	$.0^{5}3637$	$.0^{5}3475$	$.0^{5}3319$	$.0^{5}3170$	$.0^{5}3027$
4.2	$.0^{5}2891$	$.0^{5}2760$	$.0^{5}2635$	$.0^{5}2516$	$.0^{5}2402$	$.0^{5}2292$	$.0^{5}2188$	$.0^{5}2088$	$.0^{5}1992$	$.0^{5}1901$
4.3	$.0^{5}1814$	$.0^{5}1730$	$.0^{5}1650$	$.0^{5}1574$	$.0^{5}1501$	$.0^{5}1431$	$.0^{5}1365$	$.0^{5}1301$	$.0^{5}1241$	$.0^{5}1183$
4.4	$.0^{5}1127$	$.0^{5}1074$	$.0^{5}1024$	$.0^{6}9756$	$.0^{6}9296$	$.0^{6}8857$	$.0^{6}8437$	$.0^{6}8037$	$.0^{6}7655$	$.0^{6}7290$
4.5	$.0^{6}6942$	$.0^{6}6610$	$.0^{6}6294$	$.0^{6}5992$	$.0^{6}5704$	$.0^{6}5429$	$.0^{6}5167$	$.0^{6}4917$	$.0^{6}4679$	$.0^{6}4452$
4.6	$.0^{6}4236$	$.0^{6}4029$	$.0^{6}3833$	$.0^{6}3645$	$.0^{6}3467$	$.0^{6}3297$	$.0^{6}3135$	$.0^{6}2981$	$.0^{6}2834$	$.0^{6}2694$
4.7	$.0^{6}2560$	$.0^{6}2433$	$.0^{6}2313$	$.0^{6}2197$	$.0^{6}2088$	$.0^{6}1984$	$.0^{6}1884$	$.0^{6}1790$	$.0^{6}1700$	$.0^{6}1615$
4.8	$.0^{6}1533$	$.0^{6}1456$	$.0^{6}1382$	$.0^{6}1312$	$.0^{6}1246$	$.0^{6}1182$	$.0^{6}1122$	$.0^{6}1065$	$.0^{6}1011$	$.0^{7}9588$
4.9	$.0^{7}9096$	$.0^{7}8629$	$.0^{7}8185$	$.0^{7}7763$	$.0^{7}7362$	$.0^{7}6982$	$.0^{7}6620$	$.0^{7}6276$	$.0^{7}5950$	$.0^{7}5640$

Example of table notation: $.0^{4}5848 = .00005848$.

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Table 5 Binomial Distribution

$$P_b(\bar{r} \geq r|n,p)$$

N = 1

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9900	.9800	.9700	.9600	.9500	.9400	.9300	.9200	.9100	.9000		1
1		.0100	.0200	.0300	.0400	.0500	.0600	.0700	.0800	.0900	.1000		0
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.8900	.8800	.8700	.8600	.8500	.8400	.8300	.8200	.8100	.8000		1
1		.1100	.1200	.1300	.1400	.1500	.1600	.1700	.1800	.1900	.2000		0
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.7900	.7800	.7700	.7600	.7500	.7400	.7300	.7200	.7100	.7000		1
1		.2100	.2200	.2300	.2400	.2500	.2600	.2700	.2800	.2900	.3000		0
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.6900	.6800	.6700	.6600	.6500	.6400	.6300	.6200	.6100	.6000		1
1		.3100	.3200	.3300	.3400	.3500	.3600	.3700	.3800	.3900	.4000		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.5900	.5800	.5700	.5600	.5500	.5400	.5300	.5200	.5100	.5000		1
1		.4100	.4200	.4300	.4400	.4500	.4600	.4700	.4800	.4900	.5000		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 2

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9801	.9604	.9409	.9216	.9025	.8836	.8649	.8464	.8281	.8100		2
1		.0198	.0392	.0582	.0768	.0950	.1128	.1302	.1472	.1638	.1800		1
2		.0001	.0004	.0009	.0016	.0025	.0036	.0049	.0064	.0081	.0100		0
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.7921	.7744	.7569	.7396	.7225	.7056	.6889	.6724	.6561	.6400		2
1		.1958	.2112	.2262	.2408	.2550	.2688	.2822	.2952	.3078	.3200		1
2		.0121	.0144	.0169	.0196	.0225	.0256	.0289	.0324	.0361	.0400		0
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.6241	.6084	.5929	.5776	.5625	.5476	.5329	.5184	.5041	.4900		2
1		.3318	.3432	.3542	.3648	.3750	.3848	.3942	.4032	.4118	.4200		1
2		.0441	.0484	.0529	.0576	.0625	.0676	.0729	.0784	.0841	.0900		0
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.4761	.4624	.4489	.4356	.4225	.4096	.3969	.3844	.3721	.3600		2
1		.4278	.4352	.4422	.4488	.4550	.4608	.4662	.4712	.4758	.4800		1
2		.0961	.1024	.1089	.1156	.1225	.1296	.1369	.1444	.1521	.1600		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.3481	.3364	.3249	.3136	.3025	.2916	.2809	.2704	.2601	.2500		2
1		.4838	.4872	.4902	.4928	.4950	.4968	.4982	.4992	.4998	.5000		1
2		.1681	.1764	.1849	.1936	.2025	.2116	.2209	.2304	.2401	.2500		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

From Robert A. Parsons, *Statistical Analysis: A Decision Making Approach*, Harper & Row, Publishers, Incorporated, New York, 1974.

Table 5 Binomial Distribution *(Continued)*

N = 3

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.9703	.9412	.9127	.8847	.8574	.8306	.8044	.7787	.7536	.7290	3
1	.0294	.0576	.0847	.1106	.1354	.1590	.1816	.2031	.2236	.2430	2
2	.0003	.0012	.0026	.0046	.0071	.0102	.0137	.0177	.0221	.0270	1
3	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0007	.0010	0
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.7050	.6815	.6585	.6361	.6141	.5927	.5718	.5514	.5314	.5120	3
1	.2614	.2788	.2952	.3106	.3251	.3387	.3513	.3631	.3740	.3840	2
2	.0323	.0380	.0441	.0506	.0574	.0645	.0720	.0797	.0877	.0960	1
3	.0013	.0017	.0022	.0027	.0034	.0041	.0049	.0058	.0069	.0080	0
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.4930	.4746	.4565	.4390	.4219	.4052	.3890	.3732	.3579	.3430	3
1	.3932	.4015	.4091	.4159	.4219	.4271	.4316	.4355	.4386	.4410	2
2	.1045	.1133	.1222	.1313	.1406	.1501	.1597	.1693	.1791	.1890	1
3	.0093	.0106	.0122	.0138	.0156	.0175	.0197	.0220	.0244	.0270	0
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.3285	.3144	.3008	.2875	.2746	.2621	.2500	.2383	.2270	.2160	3
1	.4428	.4439	.4444	.4443	.4436	.4424	.4406	.4382	.4354	.4320	2
2	.1989	.2089	.2189	.2289	.2389	.2488	.2587	.2686	.2783	.2880	1
3	.0298	.0328	.0359	.0393	.0429	.0467	.0507	.0549	.0593	.0640	0
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.2054	.1951	.1852	.1756	.1664	.1575	.1489	.1406	.1327	.1250	3
1	.4282	.4239	.4191	.4140	.4084	.4024	.3961	.3894	.3823	.3750	2
2	.2975	.3069	.3162	.3252	.3341	.3428	.3512	.3594	.3674	.3750	1
3	.0689	.0741	.0795	.0852	.0911	.0973	.1038	.1106	.1176	.1250	0
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

N = 4

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.9606	.9224	.8853	.8493	.8145	.7807	.7481	.7164	.6857	.6561	4
1	.0388	.0753	.1095	.1416	.1715	.1993	.2252	.2492	.2713	.2916	3
2	.0006	.0023	.0051	.0088	.0135	.0191	.0254	.0325	.0402	.0486	2
3	.0000	.0000	.0001	.0002	.0005	.0008	.0013	.0019	.0027	.0036	1
4	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	0
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.6274	.5997	.5729	.5470	.5220	.4979	.4746	.4521	.4305	.4096	4
1	.3102	.3271	.3424	.3562	.3685	.3793	.3888	.3970	.4039	.4096	3
2	.0575	.0669	.0767	.0870	.0975	.1084	.1195	.1307	.1421	.1536	2
3	.0047	.0061	.0076	.0094	.0115	.0138	.0163	.0191	.0222	.0256	1
4	.0001	.0002	.0003	.0004	.0005	.0007	.0008	.0010	.0013	.0016	0
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.3895	.3702	.3515	.3336	.3164	.2999	.2840	.2687	.2541	.2401	4
1	.4142	.4176	.4200	.4214	.4219	.4214	.4201	.4180	.4152	.4116	3
2	.1651	.1767	.1882	.1996	.2109	.2221	.2331	.2439	.2544	.2646	2
3	.0293	.0332	.0375	.0420	.0469	.0520	.0575	.0632	.0693	.0756	1
4	.0019	.0023	.0028	.0033	.0039	.0046	.0053	.0061	.0071	.0081	0
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

Table 5 Binomial Distribution *(Continued)*

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.2267	.2138	.2015	.1897	.1785	.1678	.1575	.1478	.1385	.1296		4
1		.4074	.4025	.3970	.3910	.3845	.3775	.3701	.3623	.3541	.3456		3
2		.2745	.2841	.2933	.3021	.3105	.3185	.3260	.3330	.3396	.3456		2
3		.0822	.0891	.0963	.1038	.1115	.1194	.1276	.1361	.1447	.1536		1
4		.0092	.0105	.0119	.0134	.0150	.0168	.0187	.0209	.0231	.0256		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.1212	.1132	.1056	.0983	.0915	.0850	.0789	.0731	.0677	.0625		4
1		.3368	.3278	.3185	.3091	.2995	.2897	.2799	.2700	.2600	.2500		3
2		.3511	.3560	.3604	.3643	.3675	.3702	.3723	.3738	.3747	.3750		2
3		.1627	.1719	.1813	.1908	.2005	.2102	.2201	.2300	.2400	.2500		1
4		.0283	.0311	.0342	.0375	.0410	.0448	.0488	.0531	.0576	.0625		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 5

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9510	.9039	.8587	.8154	.7738	.7339	.6957	.6591	.6240	.5905		5
1		.0480	.0922	.1328	.1699	.2036	.2342	.2618	.2866	.3086	.3281		4
2		.0010	.0038	.0082	.0142	.0214	.0299	.0394	.0498	.0610	.0729		3
3		.0000	.0001	.0003	.0006	.0011	.0019	.0030	.0043	.0060	.0081		2
4		.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005		1
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.5584	.5277	.4984	.4704	.4437	.4182	.3939	.3707	.3487	.3277		5
1		.3451	.3598	.3724	.3829	.3915	.3983	.4034	.4069	.4089	.4096		4
2		.0853	.0981	.1113	.1247	.1382	.1517	.1652	.1786	.1919	.2048		3
3		.0105	.0134	.0166	.0203	.0244	.0289	.0338	.0392	.0450	.0512		2
4		.0007	.0009	.0012	.0017	.0022	.0028	.0035	.0043	.0053	.0064		1
5		.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003		0
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.3077	.2887	.2707	.2536	.2373	.2219	.2073	.1935	.1804	.1681		5
1		.4090	.4072	.4043	.4003	.3955	.3898	.3834	.3762	.3685	.3602		4
2		.2174	.2297	.2415	.2529	.2637	.2739	.2836	.2926	.3010	.3087		3
3		.0578	.0648	.0721	.0798	.0879	.0962	.1049	.1138	.1229	.1323		2
4		.0077	.0091	.0108	.0126	.0146	.0169	.0194	.0221	.0251	.0284		1
5		.0004	.0005	.0006	.0008	.0010	.0012	.0014	.0017	.0021	.0024		0
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.1564	.1454	.1350	.1252	.1160	.1074	.0992	.0916	.0845	.0778		5
1		.3513	.3421	.3325	.3226	.3124	.3020	.2914	.2808	.2700	.2592		4
2		.3157	.3220	.3275	.3323	.3364	.3397	.3423	.3441	.3452	.3456		3
3		.1418	.1515	.1613	.1712	.1811	.1911	.2010	.2109	.2207	.2304		2
4		.0319	.0357	.0397	.0441	.0488	.0537	.0590	.0646	.0706	.0768		1
5		.0029	.0034	.0039	.0045	.0053	.0060	.0069	.0079	.0090	.0102		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0715	.0656	.0602	.0551	.0503	.0459	.0418	.0380	.0345	.0313		5
1		.2484	.2376	.2270	.2164	.2059	.1956	.1854	.1755	.1657	.1563		4
2		.3452	.3442	.3424	.3400	.3369	.3332	.3289	.3240	.3185	.3125		3
3		.2399	.2492	.2583	.2671	.2757	.2838	.2916	.2990	.3060	.3125		2
4		.0834	.0902	.0974	.1049	.1128	.1209	.1293	.1380	.1470	.1563		1
5		.0116	.0131	.0147	.0165	.0185	.0206	.0229	.0255	.0282	.0313		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

Table 5 Binomial Distribution *(Continued)*

N = 6

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9415	.8858	.8330	.7828	.7351	.6899	.6470	.6064	.5679	.5314		6
1		.0571	.1085	.1546	.1957	.2321	.2642	.2922	.3164	.3370	.3543		5
2		.0014	.0055	.0120	.0204	.0305	.0422	.0550	.0688	.0833	.0984		4
3		.0000	.0002	.0005	.0011	.0021	.0036	.0055	.0080	.0110	.0146		3
4		.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005	.0008	.0012		2
5		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		1
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.4970	.4644	.4336	.4046	.3771	.3513	.3269	.3040	.2824	.2621		6
1		.3685	.3800	.3888	.3952	.3993	.4015	.4018	.4004	.3975	.3932		5
2		.1139	.1295	.1452	.1608	.1762	.1912	.2057	.2197	.2331	.2458		4
3		.0188	.0236	.0289	.0349	.0415	.0486	.0562	.0643	.0729	.0819		3
4		.0017	.0024	.0032	.0043	.0055	.0069	.0086	.0106	.0128	.0154		2
5		.0001	.0001	.0002	.0003	.0004	.0005	.0007	.0009	.0012	.0015		1
6		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		0
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.2431	.2252	.2084	.1927	.1780	.1642	.1513	.1393	.1281	.1176		6
1		.3877	.3811	.3735	.3651	.3560	.3462	.3358	.3251	.3139	.3025		5
2		.2577	.2687	.2789	.2882	.2966	.3041	.3105	.3160	.3206	.3241		4
3		.0913	.1011	.1111	.1214	.1318	.1424	.1531	.1639	.1746	.1852		3
4		.0182	.0214	.0249	.0287	.0330	.0375	.0425	.0478	.0535	.0595		2
5		.0019	.0024	.0030	.0036	.0044	.0053	.0063	.0074	.0087	.0102		1
6		.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0006	.0007		0
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.1079	.0989	.0905	.0827	.0754	.0687	.0625	.0568	.0515	.0467		6
1		.2909	.2792	.2673	.2555	.2437	.2319	.2203	.2089	.1976	.1866		5
2		.3267	.3284	.3292	.3290	.3280	.3261	.3235	.3201	.3159	.3110		4
3		.1957	.2061	.2162	.2260	.2355	.2446	.2533	.2616	.2693	.2765		3
4		.0660	.0727	.0799	.0873	.0951	.1032	.1116	.1202	.1291	.1382		2
5		.0119	.0137	.0157	.0180	.0205	.0232	.0262	.0295	.0330	.0369		1
6		.0009	.0011	.0013	.0015	.0018	.0022	.0026	.0030	.0035	.0041		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0422	.0381	.0343	.0308	.0277	.0248	.0222	.0198	.0176	.0156		6
1		.1759	.1654	.1552	.1454	.1359	.1267	.1179	.1095	.1014	.0938		5
2		.3055	.2994	.2928	.2856	.2780	.2699	.2615	.2527	.2436	.2344		4
3		.2831	.2891	.2945	.2992	.3032	.3065	.3091	.3110	.3121	.3125		3
4		.1475	.1570	.1666	.1763	.1861	.1958	.2056	.2153	.2249	.2344		2
5		.0410	.0455	.0503	.0554	.0609	.0667	.0729	.0795	.0864	.0938		1
6		.0048	.0055	.0063	.0073	.0083	.0095	.0108	.0122	.0138	.0156		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 7

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9321	.8681	.8080	.7514	.6983	.6485	.6017	.5578	.5168	.4783		7
1		.0659	.1240	.1749	.2192	.2573	.2897	.3170	.3396	.3578	.3720		6
2		.0020	.0076	.0162	.0274	.0406	.0555	.0716	.0886	.1061	.1240		5
3		.0000	.0003	.0008	.0019	.0036	.0059	.0090	.0128	.0175	.0230		4
4		.0000	.0000	.0000	.0001	.0002	.0004	.0007	.0011	.0017	.0026		3
5		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002		2
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

Table 5 Binomial Distribution *(Continued)*

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.4423	.4087	.3773	.3479	.3206	.2951	.2714	.2493	.2288	.2097		7
1		.3827	.3901	.3946	.3965	.3960	.3935	.3891	.3830	.3756	.3670		6
2		.1419	.1596	.1769	.1936	.2097	.2248	.2391	.2523	.2643	.2753		5
3		.0292	.0363	.0441	.0525	.0617	.0714	.0816	.0923	.1033	.1147		4
4		.0036	.0049	.0066	.0086	.0109	.0136	.0167	.0203	.0242	.0287		3
5		.0003	.0004	.0006	.0008	.0012	.0016	.0021	.0027	.0034	.0043		2
6		.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0004		1
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.1920	.1757	.1605	.1465	.1335	.1215	.1105	.1003	.0910	.0824		7
1		.3573	.3468	.3356	.3237	.3115	.2989	.2860	.2731	.2600	.2471		6
2		.2850	.2935	.3007	.3067	.3115	.3150	.3174	.3186	.3186	.3177		5
3		.1263	.1379	.1497	.1614	.1730	.1845	.1956	.2065	.2169	.2269		4
4		.0336	.0389	.0447	.0510	.0577	.0648	.0724	.0803	.0886	.0972		3
5		.0054	.0066	.0080	.0097	.0115	.0137	.0161	.0187	.0217	.0250		2
6		.0005	.0006	.0008	.0010	.0013	.0016	.0020	.0024	.0030	.0036		1
7		.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002		0
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.0745	.0672	.0606	.0546	.0490	.0440	.0394	.0352	.0314	.0280		7
1		.2342	.2215	.2090	.1967	.1848	.1732	.1619	.1511	.1407	.1306		6
2		.3156	.3127	.3088	.3040	.2985	.2922	.2853	.2778	.2698	.2613		5
3		.2363	.2452	.2535	.2610	.2679	.2740	.2793	.2838	.2875	.2903		4
4		.1062	.1154	.1248	.1345	.1442	.1541	.1640	.1739	.1838	.1935		3
5		.0286	.0326	.0369	.0416	.0466	.0520	.0578	.0640	.0705	.0774		2
6		.0043	.0051	.0061	.0071	.0084	.0098	.0113	.0131	.0150	.0172		1
7		.0003	.0003	.0004	.0005	.0006	.0008	.0009	.0011	.0014	.0016		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0249	.0221	.0195	.0173	.0152	.0134	.0117	.0103	.0090	.0078		7
1		.1211	.1119	.1032	.0950	.0872	.0798	.0729	.0664	.0604	.0547		6
2		.2524	.2431	.2336	.2239	.2140	.2040	.1940	.1840	.1740	.1641		5
3		.2923	.2934	.2937	.2932	.2918	.2897	.2867	.2830	.2786	.2734		4
4		.2031	.2125	.2216	.2304	.2388	.2468	.2543	.2612	.2676	.2734		3
5		.0847	.0923	.1003	.1086	.1172	.1261	.1353	.1447	.1543	.1641		2
6		.0196	.0223	.0252	.0284	.0320	.0358	.0400	.0445	.0494	.0547		1
7		.0019	.0023	.0027	.0032	.0037	.0044	.0051	.0059	.0068	.0078		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 8

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9227	.8508	.7837	.7214	.6634	.6096	.5596	.5132	.4703	.4305		8
1		.0746	.1389	.1939	.2405	.2793	.3113	.3370	.3570	.3721	.3826		7
2		.0026	.0099	.0210	.0351	.0515	.0695	.0888	.1087	.1288	.1488		6
3		.0001	.0004	.0013	.0029	.0054	.0089	.0134	.0189	.0255	.0331		5
4		.0000	.0000	.0001	.0002	.0004	.0007	.0013	.0021	.0031	.0046		4
5		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0004		3
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.3937	.3596	.3282	.2992	.2725	.2479	.2252	.2044	.1853	.1678		8
1		.3892	.3923	.3923	.3897	.3847	.3777	.3691	.3590	.3477	.3355		7
2		.1684	.1872	.2052	.2220	.2376	.2518	.2646	.2758	.2855	.2936		6
3		.0416	.0511	.0613	.0723	.0839	.0959	.1084	.1211	.1339	.1468		5
4		.0064	.0087	.0115	.0147	.0185	.0228	.0277	.0332	.0393	.0459		4
5		.0006	.0009	.0014	.0019	.0026	.0035	.0045	.0058	.0074	.0092		3
6		.0000	.0001	.0001	.0002	.0002	.0003	.0005	.0006	.0009	.0011		2
7		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001		1
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

Table 5 Binomial Distribution *(Continued)*

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.1517	.1370	.1236	.1113	.1001	.0899	.0806	.0722	.0646	.0576	8
1	.3226	.3092	.2953	.2812	.2670	.2527	.2386	.2247	.2110	.1977	7
2	.3002	.3052	.3087	.3108	.3115	.3108	.3089	.3058	.3017	.2965	6
3	.1596	.1722	.1844	.1963	.2076	.2184	.2285	.2379	.2464	.2541	5
4	.0530	.0607	.0689	.0775	.0865	.0959	.1056	.1156	.1258	.1361	4
5	.0113	.0137	.0165	.0196	.0231	.0270	.0313	.0360	.0411	.0467	3
6	.0015	.0019	.0025	.0031	.0038	.0047	.0058	.0070	.0084	.0100	2
7	.0001	.0002	.0002	.0003	.0004	.0005	.0006	.0008	.0010	.0012	1
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	0
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0514	.0457	.0406	.0360	.0319	.0281	.0248	.0218	.0192	.0168	8
1	.1847	.1721	.1600	.1484	.1373	.1267	.1166	.1071	.0981	.0896	7
2	.2904	.2835	.2758	.2675	.2587	.2494	.2397	.2297	.2194	.2090	6
3	.2609	.2668	.2717	.2756	.2786	.2805	.2815	.2815	.2806	.2787	5
4	.1465	.1569	.1673	.1775	.1875	.1973	.2067	.2157	.2242	.2322	4
5	.0527	.0591	.0659	.0732	.0808	.0888	.0971	.1058	.1147	.1239	3
6	.0118	.0139	.0162	.0188	.0217	.0250	.0285	.0324	.0367	.0413	2
7	.0015	.0019	.0023	.0028	.0033	.0040	.0048	.0057	.0067	.0079	1
8	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0004	.0005	.0007	0
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.0147	.0128	.0111	.0097	.0084	.0072	.0062	.0053	.0046	.0039	8
1	.0816	.0742	.0672	.0608	.0548	.0493	.0442	.0395	.0352	.0313	7
2	.1985	.1880	.1776	.1672	.1569	.1469	.1371	.1275	.1183	.1094	6
3	.2759	.2723	.2679	.2627	.2568	.2503	.2431	.2355	.2273	.2188	5
4	.2397	.2465	.2526	.2580	.2627	.2665	.2695	.2717	.2730	.2734	4
5	.1332	.1428	.1525	.1622	.1719	.1816	.1912	.2006	.2098	.2188	3
6	.0463	.0517	.0575	.0637	.0703	.0774	.0848	.0926	.1008	.1094	2
7	.0092	.0107	.0124	.0143	.0164	.0188	.0215	.0244	.0277	.0313	1
8	.0008	.0010	.0012	.0014	.0017	.0020	.0024	.0028	.0033	.0039	0
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

N = 9

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.9135	.8337	.7602	.6925	.6302	.5730	.5204	.4722	.4279	.3874	9
1	.0830	.1531	.2116	.2597	.2985	.3292	.3525	.3695	.3809	.3874	8
2	.0034	.0125	.0262	.0433	.0629	.0840	.1061	.1285	.1507	.1722	7
3	.0001	.0006	.0019	.0042	.0077	.0125	.0186	.0261	.0348	.0446	6
4	.0000	.0000	.0001	.0003	.0006	.0012	.0021	.0034	.0052	.0074	5
5	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005	.0008	4
6	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	3
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.3504	.3165	.2855	.2573	.2316	.2082	.1869	.1676	.1501	.1342	9
1	.3897	.3884	.3840	.3770	.3679	.3569	.3446	.3312	.3169	.3020	8
2	.1927	.2119	.2295	.2455	.2597	.2720	.2823	.2908	.2973	.3020	7
3	.0556	.0674	.0800	.0933	.1069	.1209	.1349	.1489	.1627	.1762	6
4	.0103	.0138	.0179	.0228	.0283	.0345	.0415	.0490	.0573	.0661	5
5	.0013	.0019	.0027	.0037	.0050	.0066	.0085	.0108	.0134	.0165	4
6	.0001	.0002	.0003	.0004	.0006	.0008	.0012	.0016	.0021	.0028	3
7	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	2
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.1199	.1069	.0952	.0846	.0751	.0665	.0589	.0520	.0458	.0404	9
1	.2867	.2713	.2558	.2404	.2253	.2104	.1960	.1820	.1685	.1556	8
2	.3049	.3061	.3056	.3037	.3003	.2957	.2899	.2831	.2754	.2668	7
3	.1891	.2014	.2130	.2238	.2336	.2424	.2502	.2569	.2624	.2668	6

Table 5 Binomial Distribution *(Continued)*

4		.0754	.0852	.0954	.1060	.1169	.1278	.1388	.1499	.1608	.1715		5
5		.0200	.0240	.0285	.0335	.0389	.0449	.0513	.0583	.0657	.0735		4
6		.0036	.0045	.0057	.0070	.0087	.0105	.0127	.0151	.0179	.0210		3
7		.0004	.0005	.0007	.0010	.0012	.0016	.0020	.0025	.0031	.0039		2
8		.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0004		1
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.0355	.0311	.0272	.0238	.0207	.0180	.0156	.0135	.0117	.0101		9
1		.1433	.1317	.1206	.1102	.1004	.0912	.0826	.0747	.0673	.0605		8
2		.2576	.2478	.2376	.2270	.2162	.2052	.1941	.1831	.1721	.1612		7
3		.2701	.2721	.2731	.2729	.2716	.2693	.2660	.2618	.2567	.2508		6
4		.1820	.1921	.2017	.2109	.2194	.2272	.2344	.2407	.2462	.2508		5
5		.0818	.0904	.0994	.1086	.1181	.1278	.1376	.1475	.1574	.1672		4
6		.0245	.0284	.0326	.0373	.0424	.0479	.0539	.0603	.0671	.0743		3
7		.0047	.0057	.0069	.0082	.0098	.0115	.0136	.0156	.0184	.0212		2
8		.0005	.0007	.0008	.0011	.0013	.0016	.0020	.0024	.0029	.0035		1
9		.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003		0
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0087	.0074	.0064	.0054	.0046	.0039	.0033	.0028	.0023	.0020		9
1		.0542	.0484	.0431	.0383	.0339	.0299	.0263	.0231	.0202	.0176		8
2		.1506	.1402	.1301	.1204	.1110	.1020	.0934	.0853	.0776	.0703		7
3		.2442	.2369	.2291	.2207	.2119	.2027	.1933	.1837	.1739	.1641		6
4		.2545	.2573	.2592	.2601	.2600	.2590	.2571	.2543	.2506	.2461		5
5		.1769	.1863	.1955	.2044	.2128	.2207	.2280	.2347	.2408	.2461		4
6		.0819	.0900	.0983	.1070	.1160	.1253	.1348	.1445	.1542	.1641		3
7		.0244	.0279	.0318	.0360	.0407	.0458	.0512	.0571	.0635	.0703		2
8		.0042	.0051	.0060	.0071	.0083	.0097	.0114	.0132	.0153	.0176		1
9		.0003	.0004	.0005	.0006	.0008	.0009	.0011	.0014	.0016	.0020		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 10

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.9044	.8171	.7374	.6648	.5987	.5386	.4840	.4344	.3894	.3487		10
1		.0914	.1667	.2281	.2770	.3151	.3438	.3643	.3777	.3851	.3874		9
2		.0042	.0153	.0317	.0519	.0746	.0988	.1234	.1478	.1714	.1937		8
3		.0001	.0008	.0026	.0058	.0105	.0168	.0248	.0343	.0452	.0574		7
4		.0000	.0000	.0001	.0004	.0010	.0019	.0033	.0052	.0078	.0112		6
5		.0000	.0000	.0000	.0000	.0001	.0001	.0003	.0005	.0009	.0015		5
6		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001		4
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.3118	.2785	.2484	.2213	.1969	.1749	.1552	.1374	.1216	.1074		10
1		.3854	.3798	.3712	.3603	.3474	.3331	.3178	.3017	.2852	.2684		9
2		.2143	.2330	.2496	.2639	.2759	.2856	.2929	.2980	.3010	.3020		8
3		.0706	.0847	.0995	.1146	.1298	.1450	.1600	.1745	.1883	.2013		7
4		.0153	.0202	.0260	.0326	.0401	.0483	.0573	.0670	.0773	.0881		6
5		.0023	.0033	.0047	.0064	.0085	.0111	.0141	.0177	.0218	.0264		5
6		.0002	.0004	.0006	.0009	.0012	.0018	.0024	.0032	.0043	.0055		4
7		.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0004	.0006	.0008		3
8		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001		2
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.0947	.0834	.0733	.0643	.0563	.0492	.0430	.0374	.0326	.0282		10
1		.2517	.2351	.2188	.2030	.1877	.1730	.1590	.1456	.1330	.1211		9
2		.3011	.2984	.2942	.2885	.2816	.2735	.2646	.2548	.2444	.2335		8
3		.2134	.2244	.2343	.2429	.2503	.2563	.2609	.2642	.2662	.2668		7
4		.0993	.1108	.1225	.1343	.1460	.1576	.1689	.1798	.1903	.2001		6
5		.0317	.0375	.0439	.0509	.0584	.0664	.0750	.0839	.0933	.1029		5

Table 5 Binomial Distribution *(Continued)*

6	.0070	.0088	.0109	.0134	.0162	.0195	.0231	.0272	.0317	.0368	4
7	.0011	.0014	.0019	.0024	.0031	.0039	.0049	.0060	.0074	.0090	3
8	.0001	.0002	.0002	.0003	.0004	.0005	.0007	.0009	.0011	.0014	2
9	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	1
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0245	.0211	.0182	.0157	.0135	.0115	.0098	.0084	.0071	.0060	10
1	.1099	.0995	.0898	.0808	.0725	.0649	.0578	.0514	.0456	.0403	9
2	.2222	.2107	.1990	.1873	.1757	.1642	.1529	.1419	.1312	.1209	8
3	.2662	.2644	.2614	.2573	.2522	.2462	.2394	.2319	.2237	.2150	7
4	.2093	.2177	.2253	.2320	.2377	.2424	.2461	.2487	.2503	.2508	6
5	.1128	.1229	.1332	.1434	.1536	.1636	.1734	.1829	.1920	.2007	5
6	.0422	.0482	.0547	.0616	.0689	.0767	.0849	.0934	.1023	.1115	4
7	.0108	.0130	.0154	.0181	.0212	.0247	.0285	.0327	.0374	.0425	3
8	.0018	.0023	.0028	.0035	.0043	.0052	.0063	.0075	.0090	.0106	2
9	.0002	.0002	.0003	.0004	.0005	.0007	.0008	.0010	.0013	.0016	1
10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	0
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.0051	.0043	.0036	.0030	.0025	.0021	.0017	.0014	.0012	.0010	10
1	.0355	.0312	.0273	.0238	.0207	.0180	.0155	.0133	.0114	.0098	9
2	.1111	.1017	.0927	.0843	.0763	.0688	.0619	.0554	.0495	.0439	8
3	.2058	.1963	.1865	.1765	.1665	.1564	.1454	.1364	.1267	.1172	7
4	.2503	.2488	.2462	.2427	.2384	.2331	.2271	.2204	.2130	.2051	6
5	.2087	.2162	.2229	.2289	.2340	.2383	.2417	.2441	.2456	.2461	5
6	.1209	.1304	.1401	.1499	.1596	.1692	.1786	.1878	.1966	.2051	4
7	.0480	.0540	.0604	.0673	.0746	.0824	.0905	.0991	.1080	.1172	3
8	.0125	.0147	.0171	.0198	.0229	.0263	.0301	.0343	.0389	.0439	2
9	.0019	.0024	.0029	.0035	.0042	.0050	.0059	.0070	.0083	.0098	1
10	.0001	.0002	.0002	.0003	.0003	.0004	.0005	.0006	.0008	.0010	0
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	R

N = 11

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.8953	.8007	.7153	.6382	.5688	.5063	.4501	.3996	.3544	.3138	11
1	.0995	.1798	.2434	.2925	.3293	.3555	.3727	.3823	.3855	.3835	10
2	.0050	.0183	.0376	.0609	.0867	.1135	.1403	.1662	.1906	.2131	9
3	.0002	.0011	.0035	.0076	.0137	.0217	.0317	.0434	.0566	.0710	8
4	.0000	.0000	.0002	.0006	.0014	.0028	.0048	.0075	.0112	.0158	7
5	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0009	.0015	.0025	6
6	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	5
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.2775	.2451	.2161	.1903	.1673	.1469	.1288	.1127	.0985	.0859	11
1	.3773	.3676	.3552	.3408	.3248	.3078	.2902	.2721	.2541	.2362	10
2	.2332	.2507	.2654	.2774	.2866	.2932	.2971	.2987	.2980	.2953	9
3	.0865	.1025	.1190	.1355	.1517	.1675	.1826	.1967	.2097	.2215	8
4	.0214	.0280	.0356	.0441	.0536	.0638	.0748	.0864	.0984	.1107	7
5	.0037	.0053	.0074	.0101	.0132	.0170	.0214	.0265	.0323	.0388	6
6	.0005	.0007	.0011	.0016	.0023	.0032	.0044	.0058	.0076	.0097	5
7	.0000	.0001	.0001	.0002	.0003	.0004	.0006	.0009	.0013	.0017	4
8	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	3
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.0748	.0650	.0564	.0489	.0422	.0364	.0314	.0270	.0231	.0198	11
1	.2187	.2017	.1854	.1697	.1549	.1408	.1276	.1153	.1038	.0932	10
2	.2907	.2845	.2768	.2680	.2581	.2474	.2360	.2242	.2121	.1998	9
3	.2318	.2407	.2481	.2539	.2581	.2608	.2619	.2616	.2599	.2568	8
4	.1232	.1358	.1482	.1603	.1721	.1832	.1937	.2035	.2123	.2201	7

Table 5 Binomial Distribution *(Continued)*

R	P											
5		.0459	.0536	.0620	.0709	.0803	.0901	.1003	.1108	.1214	.1321		6
6		.0122	.0151	.0185	.0224	.0268	.0317	.0371	.0431	.0496	.0566		5
7		.0023	.0030	.0039	.0050	.0064	.0079	.0098	.0120	.0145	.0173		4
8		.0003	.0004	.0006	.0008	.0011	.0014	.0018	.0023	.0030	.0037		3
9		.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005		2
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.0169	.0144	.0122	.0104	.0088	.0074	.0062	.0052	.0044	.0036		11
1		.0834	.0744	.0662	.0587	.0518	.0457	.0401	.0351	.0306	.0266		10
2		.1874	.1751	.1630	.1511	.1395	.1284	.1177	.1075	.0978	.0887		9
3		.2526	.2472	.2408	.2335	.2254	.2167	.2074	.1977	.1876	.1774		8
4		.2269	.2326	.2372	.2406	.2428	.2438	.2436	.2423	.2399	.2365		7
5		.1427	.1533	.1636	.1735	.1830	.1920	.2003	.2079	.2148	.2207		6
6		.0641	.0721	.0806	.0894	.0985	.1080	.1176	.1274	.1373	.1471		5
7		.0206	.0242	.0283	.0329	.0379	.0434	.0494	.0558	.0627	.0701		4
8		.0046	.0057	.0070	.0085	.0102	.0122	.0145	.0171	.0200	.0234		3
9		.0007	.0009	.0011	.0015	.0018	.0023	.0028	.0035	.0043	.0052		2
10		.0001	.0001	.0001	.0001	.0002	.0003	.0003	.0004	.0005	.0007		1
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0030	.0025	.0021	.0017	.0014	.0011	.0009	.0008	.0006	.0005		11
1		.0231	.0199	.0171	.0147	.0125	.0107	.0090	.0076	.0064	.0054		10
2		.0801	.0721	.0646	.0577	.0513	.0454	.0401	.0352	.0308	.0269		9
3		.1670	.1566	.1462	.1359	.1259	.1161	.1067	.0976	.0888	.0806		8
4		.2321	.2267	.2206	.2136	.2060	.1979	.1892	.1801	.1707	.1611		7
5		.2258	.2299	.2329	.2350	.2360	.2359	.2348	.2327	.2296	.2256		6
6		.1569	.1664	.1757	.1846	.1931	.2010	.2083	.2148	.2206	.2256		5
7		.0779	.0861	.0947	.1036	.1128	.1223	.1319	.1416	.1514	.1611		4
8		.0271	.0312	.0357	.0407	.0462	.0521	.0585	.0654	.0727	.0806		3
9		.0063	.0075	.0090	.0107	.0126	.0148	.0173	.0201	.0233	.0269		2
10		.0009	.0011	.0014	.0017	.0021	.0025	.0031	.0037	.0045	.0054		1
11		.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0003	.0004	.0005		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 12

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.8864	.7847	.6938	.6127	.5404	.4759	.4186	.3677	.3225	.2824		12
1		.1074	.1922	.2575	.3064	.3413	.3645	.3781	.3837	.3827	.3766		11
2		.0060	.0216	.0438	.0702	.0988	.1280	.1565	.1835	.2082	.2301		10
3		.0002	.0015	.0045	.0098	.0173	.0272	.0393	.0532	.0686	.0852		9
4		.0000	.0001	.0003	.0009	.0021	.0039	.0067	.0104	.0153	.0213		8
5		.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0014	.0024	.0038		7
6		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	.0005		6
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.2470	.2157	.1880	.1637	.1422	.1234	.1069	.0924	.0798	.0687		12
1		.3663	.3529	.3372	.3197	.3012	.2821	.2627	.2434	.2245	.2062		11
2		.2490	.2647	.2771	.2863	.2924	.2955	.2960	.2939	.2897	.2835		10
3		.1026	.1203	.1380	.1553	.1720	.1876	.2021	.2151	.2265	.2362		9
4		.0285	.0369	.0464	.0569	.0683	.0804	.0931	.1062	.1195	.1329		8
5		.0056	.0081	.0111	.0148	.0193	.0245	.0305	.0373	.0449	.0532		7
6		.0008	.0013	.0019	.0028	.0040	.0054	.0073	.0096	.0123	.0155		6
7		.0001	.0001	.0002	.0004	.0006	.0009	.0013	.0018	.0025	.0033		5
8		.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0004	.0005		4
9		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		3
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.0591	.0507	.0434	.0371	.0317	.0270	.0229	.0194	.0164	.0138		12
1		.1885	.1717	.1557	.1407	.1267	.1137	.1016	.0906	.0804	.0712		11

Table 5 Binomial Distribution *(Continued)*

2	.2756	.2663	.2558	.2444	.2323	.2197	.2068	.1937	.1807	.1678	10
3	.2442	.2503	.2547	.2573	.2581	.2573	.2549	.2511	.2460	.2397	9
4	.1460	.1589	.1712	.1828	.1936	.2034	.2122	.2197	.2261	.2311	8
5	.0621	.0717	.0818	.0924	.1032	.1143	.1255	.1367	.1477	.1585	7
6	.0193	.0236	.0285	.0340	.0401	.0469	.0542	.0620	.0704	.0792	6
7	.0044	.0057	.0073	.0092	.0115	.0141	.0172	.0207	.0246	.0291	5
8	.0007	.0010	.0014	.0018	.0024	.0031	.0040	.0050	.0063	.0078	4
9	.0001	.0001	.0002	.0003	.0004	.0005	.0007	.0009	.0011	.0015	3
10	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	2
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0116	.0098	.0082	.0068	.0057	.0047	.0039	.0032	.0027	.0022	12
1	.0628	.0552	.0484	.0422	.0368	.0319	.0276	.0237	.0204	.0174	11
2	.1552	.1429	.1310	.1197	.1088	.0986	.0890	.0800	.0716	.0639	10
3	.2324	.2241	.2151	.2055	.1954	.1849	.1742	.1634	.1526	.1419	9
4	.2349	.2373	.2384	.2382	.2367	.2340	.2302	.2254	.2195	.2128	8
5	.1688	.1787	.1879	.1963	.2039	.2106	.2163	.2210	.2246	.2270	7
6	.0885	.0981	.1079	.1180	.1281	.1382	.1482	.1580	.1675	.1766	6
7	.0341	.0396	.0456	.0521	.0591	.0666	.0746	.0830	.0918	.1009	5
8	.0096	.0116	.0140	.0168	.0199	.0234	.0274	.0318	.0367	.0420	4
9	.0019	.0024	.0031	.0038	.0048	.0059	.0071	.0087	.0104	.0125	3
10	.0003	.0003	.0005	.0006	.0008	.0010	.0013	.0016	.0020	.0025	2
11	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	1
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.0018	.0014	.0012	.0010	.0008	.0006	.0005	.0004	.0003	.0002	12
1	.0148	.0126	.0106	.0090	.0075	.0063	.0052	.0043	.0036	.0029	11
2	.0567	.0502	.0442	.0388	.0339	.0294	.0255	.0220	.0189	.0161	10
3	.1314	.1211	.1111	.1015	.0923	.0836	.0754	.0676	.0604	.0537	9
4	.2054	.1973	.1886	.1794	.1700	.1602	.1504	.1405	.1306	.1208	8
5	.2284	.2285	.2276	.2256	.2225	.2184	.2134	.2075	.2008	.1934	7
6	.1851	.1931	.2003	.2068	.2124	.2171	.2208	.2234	.2250	.2256	6
7	.1103	.1198	.1295	.1393	.1489	.1585	.1678	.1768	.1853	.1934	5
8	.0479	.0542	.0611	.0684	.0762	.0844	.0930	.1020	.1113	.1208	4
9	.0148	.0175	.0205	.0239	.0277	.0319	.0367	.0418	.0475	.0537	3
10	.0031	.0038	.0046	.0056	.0068	.0082	.0098	.0116	.0137	.0161	2
11	.0004	.0005	.0006	.0008	.0010	.0013	.0016	.0019	.0024	.0029	1
12	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0002	.0002	0
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

N = 13

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.8775	.7690	.6730	.5882	.5133	.4474	.3893	.3383	.2935	.2542	13
1	.1152	.2040	.2706	.3186	.3512	.3712	.3809	.3824	.3773	.3672	12
2	.0070	.0250	.0502	.0797	.1109	.1422	.1720	.1995	.2239	.2448	11
3	.0003	.0019	.0057	.0122	.0214	.0333	.0475	.0636	.0812	.0997	10
4	.0000	.0001	.0004	.0013	.0028	.0053	.0089	.0138	.0201	.0277	9
5	.0000	.0000	.0000	.0001	.0003	.0006	.0012	.0022	.0036	.0055	8
6	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	.0005	.0008	7
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	6
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.2198	.1898	.1636	.1408	.1209	.1037	.0887	.0758	.0646	.0550	13
1	.3532	.3364	.3178	.2979	.2774	.2567	.2362	.2163	.1970	.1787	12
2	.2619	.2753	.2849	.2910	.2937	.2934	.2903	.2848	.2773	.2680	11
3	.1187	.1376	.1561	.1737	.1900	.2049	.2180	.2293	.2385	.2457	10
4	.0367	.0469	.0583	.0707	.0838	.0976	.1116	.1258	.1399	.1535	9
5	.0082	.0115	.0157	.0207	.0266	.0335	.0412	.0497	.0591	.0691	8
6	.0013	.0021	.0031	.0045	.0063	.0085	.0112	.0149	.0185	.0230	7
7	.0002	.0003	.0005	.0007	.0011	.0016	.0023	.0032	.0043	.0058	6

Table 5 Binomial Distribution *(Continued)*

8		.0000	.0000	.0001	.0001	.0001	.0002	.0004	.0005	.0008	.0011		5
9		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002		4
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.0467	.0396	.0334	.0282	.0238	.0200	.0167	.0140	.0117	.0097		13
1		.1613	.1450	.1299	.1159	.1029	.0911	.0804	.0706	.0619	.0540		12
2		.2573	.2455	.2328	.2195	.2059	.1921	.1784	.1648	.1516	.1388		11
3		.2508	.2539	.2550	.2542	.2517	.2475	.2419	.2351	.2271	.2181		10
4		.1667	.1790	.1904	.2007	.2097	.2174	.2237	.2285	.2319	.2337		9
5		.0797	.0909	.1024	.1141	.1258	.1375	.1489	.1600	.1705	.1803		8
6		.0283	.0342	.0408	.0480	.0559	.0644	.0734	.0829	.0928	.1030		7
7		.0075	.0096	.0122	.0152	.0186	.0226	.0272	.0323	.0379	.0442		6
8		.0015	.0020	.0027	.0036	.0047	.0060	.0075	.0094	.0116	.0142		5
9		.0002	.0003	.0005	.0006	.0009	.0012	.0015	.0020	.0026	.0034		4
10		.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0006		3
11		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		2
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.0080	.0066	.0055	.0045	.0037	.0030	.0025	.0020	.0016	.0013		13
1		.0469	.0407	.0351	.0302	.0259	.0221	.0188	.0159	.0135	.0113		12
2		.1265	.1148	.1037	.0933	.0836	.0746	.0663	.0586	.0516	.0453		11
3		.2084	.1981	.1874	.1763	.1651	.1538	.1427	.1317	.1210	.1107		10
4		.2341	.2331	.2307	.2270	.2222	.2163	.2095	.2018	.1934	.1845		9
5		.1893	.1974	.2045	.2105	.2154	.2190	.2215	.2227	.2226	.2214		8
6		.1134	.1239	.1343	.1446	.1546	.1643	.1734	.1820	.1898	.1968		7
7		.0510	.0583	.0662	.0745	.0833	.0924	.1019	.1115	.1213	.1312		6
8		.0172	.0206	.0244	.0288	.0336	.0390	.0449	.0513	.0582	.0656		5
9		.0043	.0054	.0067	.0082	.0101	.0122	.0146	.0175	.0207	.0243		4
10		.0008	.0010	.0013	.0017	.0022	.0027	.0034	.0043	.0053	.0065		3
11		.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0007	.0009	.0012		2
12		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001		1
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0010	.0008	.0007	.0005	.0004	.0003	.0003	.0002	.0002	.0001		13
1		.0095	.0079	.0066	.0054	.0045	.0037	.0030	.0024	.0020	.0016		12
2		.0395	.0344	.0298	.0256	.0220	.0188	.0160	.0135	.0114	.0095		11
3		.1007	.0913	.0823	.0739	.0660	.0587	.0519	.0457	.0401	.0349		10
4		.1750	.1653	.1553	.1451	.1350	.1250	.1151	.1055	.0962	.0873		9
5		.2189	.2154	.2108	.2053	.1989	.1917	.1838	.1753	.1664	.1571		8
6		.2029	.2080	.2121	.2151	.2169	.2177	.2173	.2158	.2131	.2095		7
7		.1410	.1506	.1600	.1690	.1775	.1854	.1927	.1992	.2048	.2095		6
8		.0735	.0818	.0905	.0996	.1089	.1185	.1282	.1379	.1476	.1571		5
9		.0284	.0329	.0379	.0435	.0495	.0561	.0631	.0707	.0788	.0873		4
10		.0079	.0095	.0114	.0137	.0162	.0191	.0224	.0261	.0303	.0349		3
11		.0015	.0019	.0024	.0029	.0036	.0044	.0054	.0066	.0079	.0095		2
12		.0002	.0002	.0003	.0004	.0005	.0006	.0008	.0010	.0013	.0016		1
13		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001		0
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 14

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.8687	.7536	.6528	.5647	.4877	.4205	.3620	.3112	.2670	.2288		14
1		.1229	.2153	.2827	.3294	.3593	.3758	.3815	.3788	.3698	.3559		13
2		.0081	.0286	.0568	.0892	.1229	.1559	.1867	.2141	.2377	.2570		12
3		.0003	.0023	.0070	.0149	.0259	.0398	.0562	.0745	.0940	.1142		11
4		.0000	.0001	.0006	.0017	.0037	.0070	.0116	.0178	.0256	.0349		10
5		.0000	.0000	.0000	.0001	.0004	.0009	.0018	.0031	.0051	.0078		9
6		.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0013		8
7		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002		7
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

Table 5 Binomial Distribution *(Continued)*

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.1956	.1670	.1423	.1211	.1028	.0871	.0736	.0621	.0523	.0440	14
1	.3385	.3188	.2977	.2759	.2539	.2322	.2112	.1910	.1719	.1539	13
2	.2720	.2826	.2892	.2919	.2912	.2875	.2811	.2725	.2620	.2501	12
3	.1345	.1542	.1728	.1901	.2056	.2190	.2303	.2393	.2459	.2501	11
4	.0457	.0578	.0710	.0851	.0998	.1147	.1297	.1444	.1586	.1720	10
5	.0113	.0158	.0212	.0277	.0352	.0437	.0531	.0634	.0744	.0860	9
6	.0021	.0032	.0048	.0068	.0093	.0125	.0163	.0209	.0262	.0322	8
7	.0003	.0005	.0008	.0013	.0019	.0027	.0038	.0052	.0070	.0092	7
8	.0000	.0001	.0001	.0002	.0003	.0005	.0007	.0010	.0014	.0020	6
9	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	5
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.0369	.0309	.0258	.0214	.0178	.0148	.0122	.0101	.0083	.0068	14
1	.1372	.1218	.1077	.0948	.0832	.0726	.0632	.0548	.0473	.0407	13
2	.2371	.2234	.2091	.1946	.1802	.1659	.1519	.1385	.1256	.1134	12
3	.2521	.2520	.2499	.2459	.2402	.2331	.2248	.2154	.2052	.1943	11
4	.1843	.1955	.2052	.2135	.2202	.2252	.2236	.2304	.2305	.2290	10
5	.0980	.1103	.1226	.1348	.1468	.1583	.1691	.1792	.1883	.1963	9
6	.0391	.0466	.0549	.0639	.0734	.0834	.0938	.1045	.1153	.1262	8
7	.0119	.0150	.0188	.0231	.0280	.0335	.0397	.0464	.0538	.0618	7
8	.0028	.0037	.0049	.0064	.0082	.0103	.0128	.0158	.0192	.0232	6
9	.0005	.0007	.0010	.0013	.0018	.0024	.0032	.0041	.0052	.0066	5
10	.0001	.0001	.0001	.0002	.0003	.0004	.0006	.0008	.0011	.0014	4
11	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002	3
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0055	.0045	.0037	.0030	.0024	.0019	.0016	.0012	.0010	.0008	14
1	.0349	.0298	.0253	.0215	.0181	.0152	.0128	.0106	.0088	.0073	13
2	.1018	.0911	.0811	.0719	.0634	.0557	.0487	.0424	.0367	.0317	12
3	.1830	.1715	.1598	.1481	.1366	.1253	.1144	.1039	.0940	.0845	11
4	.2261	.2219	.2164	.2098	.2022	.1938	.1848	.1752	.1652	.1549	10
5	.2032	.2088	.2132	.2161	.2178	.2181	.2170	.2147	.2112	.2066	9
6	.1369	.1474	.1575	.1670	.1759	.1840	.1912	.1974	.2026	.2066	8
7	.0703	.0793	.0886	.0983	.1082	.1183	.1283	.1383	.1480	.1574	7
8	.0276	.0326	.0382	.0443	.0510	.0582	.0659	.0742	.0828	.0918	6
9	.0083	.0102	.0125	.0152	.0183	.0218	.0258	.0303	.0353	.0408	5
10	.0019	.0024	.0031	.0039	.0049	.0061	.0076	.0093	.0113	.0136	4
11	.0003	.0004	.0006	.0007	.0010	.0013	.0016	.0021	.0026	.0033	3
12	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	2
13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	1
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.0006	.0005	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	14
1	.0060	.0049	.0040	.0033	.0027	.0021	.0017	.0014	.0011	.0009	13
2	.0272	.0233	.0198	.0168	.0141	.0118	.0099	.0082	.0068	.0056	12
3	.0757	.0674	.0597	.0527	.0462	.0403	.0350	.0303	.0260	.0222	11
4	.1446	.1342	.1239	.1138	.1040	.0945	.0854	.0768	.0687	.0611	10
5	.2009	.1943	.1869	.1788	.1701	.1610	.1515	.1418	.1320	.1222	9
6	.2094	.2111	.2115	.2108	.2088	.2057	.2015	.1963	.1902	.1833	8
7	.1663	.1747	.1824	.1892	.1952	.2003	.2043	.2071	.2089	.2095	7
8	.1011	.1107	.1204	.1301	.1398	.1493	.1585	.1673	.1756	.1833	6
9	.0469	.0534	.0605	.0682	.0762	.0849	.0937	.1030	.1125	.1222	5
10	.0163	.0193	.0228	.0268	.0312	.0361	.0415	.0475	.0540	.0611	4
11	.0041	.0051	.0063	.0076	.0093	.0112	.0134	.0160	.0189	.0222	3
12	.0007	.0009	.0012	.0015	.0019	.0024	.0030	.0037	.0045	.0056	2
13	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0007	.0009	1
14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	0
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

Table 5 Binomial Distribution *(Continued)*

N = 15

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.8601	.7386	.6333	.5421	.4633	.3953	.3367	.2863	.2430	.2059	15
1	.1303	.2261	.2938	.3388	.3658	.3785	.3801	.3734	.3605	.3432	14
2	.0092	.0323	.0636	.0988	.1348	.1691	.2003	.2273	.2496	.2669	13
3	.0004	.0029	.0085	.0178	.0307	.0468	.0653	.0857	.1070	.1285	12
4	.0000	.0002	.0008	.0022	.0049	.0090	.0148	.0223	.0317	.0428	11
5	.0000	.0000	.0001	.0002	.0006	.0013	.0024	.0043	.0069	.0105	10
6	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0006	.0011	.0019	9
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	8
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.1741	.1470	.1238	.1041	.0874	.0731	.0511	.0510	.0424	.0352	15
1	.3228	.3006	.2775	.2542	.2312	.2090	.1878	.1676	.1492	.1319	14
2	.2793	.2870	.2903	.2897	.2856	.2787	.2692	.2578	.2449	.2309	13
3	.1496	.1696	.1880	.2044	.2184	.2300	.2389	.2452	.2489	.2501	12
4	.0555	.0694	.0843	.0998	.1156	.1314	.1468	.1615	.1752	.1876	11
5	.0151	.0208	.0277	.0357	.0449	.0551	.0662	.0780	.0904	.1032	10
6	.0031	.0047	.0069	.0097	.0132	.0175	.0226	.0285	.0353	.0430	9
7	.0005	.0008	.0013	.0020	.0030	.0043	.0059	.0081	.0107	.0138	8
8	.0001	.0001	.0002	.0003	.0005	.0008	.0012	.0018	.0025	.0035	7
9	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0007	6
10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	5
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.0291	.0241	.0198	.0163	.0134	.0109	.0089	.0072	.0059	.0047	15
1	.1162	.1018	.0889	.0772	.0668	.0576	.0494	.0423	.0360	.0305	14
2	.2162	.2010	.1858	.1707	.1559	.1416	.1280	.1150	.1029	.0916	13
3	.2490	.2457	.2405	.2336	.2252	.2156	.2051	.1939	.1821	.1700	12
4	.1986	.2079	.2155	.2213	.2252	.2273	.2276	.2262	.2231	.2186	11
5	.1161	.1290	.1416	.1537	.1651	.1757	.1852	.1935	.2005	.2061	10
6	.0514	.0606	.0705	.0809	.0917	.1023	.1142	.1254	.1365	.1472	9
7	.0176	.0220	.0271	.0329	.0393	.0465	.0543	.0627	.0717	.0811	8
8	.0047	.0062	.0081	.0104	.0131	.0163	.0201	.0244	.0293	.0348	7
9	.0010	.0014	.0019	.0025	.0034	.0045	.0058	.0074	.0093	.0116	6
10	.0002	.0002	.0003	.0005	.0007	.0009	.0013	.0017	.0023	.0030	5
11	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0006	4
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	3
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0038	.0031	.0025	.0020	.0016	.0012	.0010	.0008	.0006	.0005	15
1	.0258	.0217	.0182	.0152	.0126	.0104	.0086	.0071	.0058	.0047	14
2	.0811	.0715	.0627	.0547	.0476	.0411	.0354	.0303	.0259	.0219	13
3	.1579	.1457	.1378	.1222	.1110	.1002	.0901	.0805	.0716	.0634	12
4	.2128	.2057	.1977	.1888	.1792	.1692	.1587	.1481	.1374	.1268	11
5	.2103	.2130	.2142	.2140	.2123	.2093	.2051	.1997	.1933	.1859	10
6	.1575	.1671	.1759	.1837	.1906	.1963	.2008	.2040	.2059	.2066	9
7	.0910	.1011	.1114	.1217	.1319	.1419	.1516	.1608	.1693	.1771	8
8	.0409	.0476	.0549	.0627	.0710	.0798	.0890	.0985	.1082	.1181	7
9	.0143	.0174	.0210	.0251	.0298	.0349	.0407	.0470	.0538	.0612	6
10	.0038	.0049	.0062	.0078	.0096	.0118	.0143	.0173	.0206	.0245	5
11	.0008	.0011	.0014	.0018	.0024	.0030	.0038	.0048	.0060	.0074	4
12	.0001	.0002	.0002	.0003	.0004	.0006	.0007	.0010	.0013	.0016	3
13	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0003	2
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

Table 5 Binomial Distribution *(Continued)*

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0		.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0000	.0000		15
1		.0038	.0031	.0025	.0020	.0016	.0012	.0010	.0008	.0006	.0005		14
2		.0185	.0156	.0130	.0108	.0090	.0074	.0060	.0049	.0040	.0032		13
3		.0558	.0489	.0426	.0369	.0318	.0272	.0232	.0197	.0166	.0139		12
4		.1163	.1061	.0963	.0869	.0780	.0696	.0617	.0545	.0478	.0417		11
5		.1778	.1691	.1598	.1502	.1404	.1304	.1204	.1106	.1010	.0916		10
6		.2060	.2041	.2010	.1967	.1914	.1851	.1780	.1702	.1617	.1527		9
7		.1840	.1900	.1949	.1987	.2013	.2028	.2030	.2020	.1997	.1964		8
8		.1279	.1376	.1470	.1561	.1647	.1727	.1800	.1864	.1919	.1964		7
9		.0691	.0775	.0863	.0954	.1048	.1144	.1241	.1338	.1434	.1527		6
10		.0288	.0337	.0390	.0450	.0515	.0585	.0661	.0741	.0827	.0916		5
11		.0091	.0111	.0134	.0161	.0191	.0226	.0266	.0311	.0361	.0417		4
12		.0021	.0027	.0034	.0042	.0052	.0064	.0079	.0096	.0116	.0139		3
13		.0003	.0004	.0006	.0008	.0010	.0013	.0016	.0020	.0026	.0032		2
14		.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005		1
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 16

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.8515	.7238	.6143	.5204	.4401	.3716	.3131	.2634	.2211	.1853		16
1		.1376	.2363	.3040	.3469	.3705	.3795	.3771	.3665	.3499	.3294		15
2		.0104	.0362	.0705	.1084	.1463	.1817	.2129	.2390	.2596	.2745		14
3		.0005	.0034	.0102	.0211	.0359	.0541	.0748	.0970	.1198	.1423		13
4		.0000	.0002	.0010	.0029	.0061	.0112	.0193	.0274	.0385	.0514		12
5		.0000	.0000	.0001	.0003	.0008	.0017	.0033	.0057	.0091	.0137		11
6		.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0009	.0017	.0028		10
7		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004		9
8		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		8
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.1550	.1293	.1077	.0895	.0743	.0614	.0507	.0418	.0343	.0281		16
1		.3065	.2822	.2575	.2332	.2097	.1873	.1662	.1468	.1289	.1126		15
2		.2841	.2886	.2886	.2847	.2775	.2675	.2554	.2416	.2267	.2111		14
3		.1638	.1837	.2013	.2163	.2285	.2378	.2441	.2475	.2482	.2463		13
4		.0658	.0814	.0977	.1144	.1311	.1472	.1625	.1766	.1892	.2001		12
5		.0195	.0266	.0351	.0447	.0555	.0673	.0799	.0930	.1065	.1201		11
6		.0044	.0067	.0096	.0133	.0180	.0235	.0300	.0374	.0458	.0550		10
7		.0008	.0013	.0020	.0031	.0045	.0064	.0088	.0117	.0153	.0197		9
8		.0001	.0002	.0003	.0006	.0009	.0014	.0020	.0029	.0041	.0055		8
9		.0000	.0000	.0000	.0001	.0001	.0002	.0004	.0006	.0008	.0012		7
10		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002		6
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30		
0		.0230	.0188	.0153	.0124	.0100	.0081	.0065	.0052	.0042	.0033		16
1		.0979	.0847	.0730	.0626	.0535	.0455	.0385	.0325	.0273	.0228		15
2		.1952	.1792	.1635	.1482	.1336	.1198	.1068	.0947	.0835	.0732		14
3		.2421	.2359	.2279	.2185	.2079	.1964	.1843	.1718	.1591	.1465		13
4		.2092	.2162	.2212	.2242	.2252	.2243	.2215	.2171	.2112	.2040		12
5		.1334	.1464	.1586	.1699	.1802	.1891	.1966	.2026	.2071	.2099		11
6		.0650	.0757	.0869	.0984	.1101	.1219	.1333	.1445	.1551	.1649		10
7		.0247	.0305	.0371	.0444	.0524	.0611	.0704	.0803	.0905	.1010		9
8		.0074	.0097	.0125	.0158	.0197	.0242	.0293	.0351	.0416	.0487		8
9		.0017	.0024	.0033	.0044	.0058	.0075	.0096	.0121	.0151	.0185		7
10		.0003	.0005	.0007	.0010	.0014	.0019	.0025	.0033	.0043	.0056		6
11		.0000	.0001	.0001	.0002	.0002	.0004	.0005	.0007	.0010	.0013		5
12		.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002		4
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P	R

Table 5 Binomial Distribution *(Continued)*

R	P .31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0	.0026	.0021	.0016	.0013	.0010	.0008	.0006	.0005	.0004	.0003		16
1	.0190	.0157	.0130	.0107	.0087	.0071	.0058	.0047	.0038	.0030		15
2	.0639	.0555	.0480	.0413	.0353	.0301	.0255	.0215	.0180	.0150		14
3	.1341	.1220	.1103	.0992	.0888	.0790	.0699	.0615	.0538	.0468		13
4	.1958	.1865	.1766	.1662	.1553	.1444	.1333	.1224	.1118	.1014		12
5	.2111	.2107	.2088	.2054	.2008	.1949	.1879	.1801	.1715	.1623		11
6	.1739	.1818	.1885	.1940	.1982	.2010	.2024	.2024	.2010	.1983		10
7	.1116	.1222	.1326	.1428	.1524	.1615	.1698	.1772	.1836	.1889		9
8	.0564	.0647	.0735	.0827	.0923	.1022	.1122	.1222	.1320	.1417		8
9	.0225	.0271	.0322	.0379	.0442	.0511	.0586	.0666	.0750	.0840		7
10	.0071	.0089	.0111	.0137	.0167	.0201	.0241	.0286	.0336	.0392		6
11	.0017	.0023	.0030	.0038	.0049	.0062	.0077	.0095	.0117	.0142		5
12	.0003	.0004	.0006	.0008	.0011	.0014	.0019	.0024	.0031	.0040		4
13	.0000	.0001	.0001	.0001	.0002	.0003	.0003	.0005	.0006	.0008		3
14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001		2
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P .41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
0	.0002	.0002	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000		16
1	.0024	.0019	.0015	.0012	.0009	.0007	.0006	.0004	.0003	.0002		15
2	.0125	.0103	.0085	.0069	.0056	.0046	.0037	.0029	.0023	.0018		14
3	.0405	.0349	.0299	.0254	.0215	.0181	.0151	.0126	.0104	.0085		13
4	.0915	.0821	.0732	.0649	.0572	.0501	.0436	.0378	.0325	.0278		12
5	.1526	.1426	.1325	.1224	.1123	.1024	.0929	.0837	.0749	.0667		11
6	.1944	.1894	.1833	.1762	.1684	.1600	.1510	.1416	.1319	.1222		10
7	.1930	.1959	.1975	.1978	.1969	.1947	.1912	.1867	.1811	.1746		9
8	.1509	.1596	.1676	.1749	.1812	.1865	.1908	.1939	.1958	.1964		8
9	.0932	.1027	.1124	.1221	.1318	.1413	.1504	.1591	.1672	.1746		7
10	.0453	.0521	.0594	.0672	.0755	.0842	.0934	.1028	.1124	.1222		6
11	.0172	.0206	.0244	.0288	.0337	.0391	.0452	.0518	.0589	.0667		5
12	.0050	.0062	.0077	.0094	.0115	.0139	.0167	.0199	.0236	.0278		4
13	.0011	.0014	.0018	.0023	.0029	.0036	.0046	.0057	.0070	.0085		3
14	.0002	.0002	.0003	.0004	.0005	.0007	.0009	.0011	.0014	.0018		2
15	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002		1
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 17

R	P .01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0	.8429	.7093	.5958	.4996	.4181	.3493	.2912	.2423	.2012	.1668		17
1	.1447	.2461	.3133	.3539	.3741	.3790	.3726	.3582	.3383	.3150		16
2	.0117	.0402	.0775	.1180	.1575	.1935	.2244	.2492	.2677	.2800		15
3	.0006	.0041	.0120	.0246	.0415	.0613	.0844	.1083	.1324	.1556		14
4	.0000	.0003	.0013	.0036	.0076	.0138	.0222	.0330	.0458	.0605		13
5	.0000	.0000	.0001	.0004	.0010	.0023	.0044	.0075	.0118	.0175		12
6	.0000	.0000	.0000	.0000	.0001	.0003	.0007	.0013	.0023	.0039		11
7	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0007		10
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		9
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P .11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0	.1379	.1138	.0937	.0770	.0631	.0516	.0421	.0343	.0278	.0225		17
1	.2898	.2638	.2381	.2131	.1893	.1671	.1466	.1279	.1109	.0957		16
2	.2865	.2878	.2846	.2775	.2673	.2547	.2402	.2245	.2081	.1914		15
3	.1771	.1963	.2126	.2259	.2359	.2425	.2460	.2464	.2441	.2393		14
4	.0766	.0937	.1112	.1287	.1457	.1617	.1764	.1893	.2004	.2093		13
5	.0246	.0332	.0432	.0545	.0668	.0801	.0939	.1091	.1222	.1361		12
6	.0061	.0091	.0129	.0177	.0236	.0305	.0385	.0474	.0573	.0680		11
7	.0012	.0019	.0030	.0045	.0065	.0091	.0124	.0164	.0211	.0267		10
8	.0002	.0003	.0006	.0009	.0014	.0022	.0032	.0045	.0062	.0084		9
9	.0000	.0000	.0001	.0002	.0003	.0004	.0006	.0010	.0015	.0021		8
10	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0004		7
11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		6
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P	R

Table 5 Binomial Distribution *(Continued)*

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0		.0182	.0146	.0118	.0094	.0075	.0060	.0047	.0038	.0030	.0023	17
1		.0822	.0702	.0597	.0505	.0426	.0357	.0299	.0248	.0206	.0169	16
2		.1747	.1584	.1427	.1277	.1136	.1005	.0883	.0772	.0672	.0581	15
3		.2322	.2234	.2131	.2016	.1893	.1765	.1634	.1502	.1372	.1245	14
4		.2161	.2205	.2228	.2228	.2209	.2170	.2115	.2044	.1961	.1868	13
5		.1493	.1617	.1730	.1830	.1914	.1982	.2033	.2067	.2083	.2081	12
6		.0794	.0912	.1034	.1156	.1276	.1393	.1504	.1608	.1701	.1784	11
7		.0332	.0404	.0485	.0573	.0668	.0769	.0874	.0982	.1092	.1201	10
8		.0110	.0143	.0181	.0226	.0279	.0338	.0404	.0478	.0558	.0644	9
9		.0029	.0040	.0054	.0071	.0093	.0119	.0150	.0186	.0228	.0276	8
10		.0006	.0009	.0013	.0018	.0025	.0033	.0044	.0058	.0074	.0095	7
11		.0001	.0002	.0002	.0004	.0005	.0007	.0010	.0014	.0019	.0026	6
12		.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0004	.0006	5
13		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	4
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0		.0018	.0014	.0011	.0009	.0007	.0005	.0004	.0003	.0002	.0002	17
1		.0139	.0114	.0093	.0075	.0060	.0048	.0039	.0031	.0024	.0019	16
2		.0500	.0428	.0364	.0309	.0260	.0218	.0182	.0151	.0125	.0102	15
3		.1123	.1007	.0898	.0795	.0701	.0614	.0534	.0463	.0398	.0341	14
4		.1766	.1659	.1547	.1434	.1320	.1209	.1099	.0993	.0892	.0796	13
5		.2063	.2030	.1982	.1921	.1849	.1767	.1677	.1582	.1482	.1379	12
6		.1854	.1910	.1952	.1979	.1991	.1989	.1970	.1939	.1895	.1839	11
7		.1309	.1413	.1511	.1602	.1685	.1757	.1818	.1868	.1904	.1927	10
8		.0735	.0831	.0930	.1032	.1134	.1235	.1335	.1431	.1521	.1606	9
9		.0330	.0391	.0458	.0531	.0611	.0695	.0784	.0877	.0973	.1070	8
10		.0119	.0147	.0181	.0219	.0263	.0313	.0368	.0430	.0498	.0571	7
11		.0034	.0044	.0057	.0072	.0090	.0112	.0138	.0168	.0202	.0242	6
12		.0008	.0010	.0014	.0018	.0024	.0031	.0040	.0051	.0065	.0081	5
13		.0001	.0002	.0003	.0004	.0005	.0007	.0009	.0012	.0016	.0021	4
14		.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	3
15		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	2
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0		.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	17
1		.0015	.0012	.0009	.0007	.0005	.0004	.0003	.0002	.0002	.0001	16
2		.0084	.0068	.0055	.0044	.0035	.0029	.0022	.0017	.0013	.0010	15
3		.0290	.0246	.0207	.0173	.0144	.0119	.0097	.0079	.0064	.0052	14
4		.0706	.0622	.0546	.0475	.0411	.0354	.0302	.0257	.0217	.0182	13
5		.1276	.1172	.1070	.0971	.0875	.0784	.0697	.0616	.0541	.0472	12
6		.1773	.1697	.1614	.1525	.1432	.1335	.1237	.1138	.1040	.0944	11
7		.1936	.1932	.1914	.1883	.1841	.1787	.1723	.1650	.1570	.1484	10
8		.1682	.1748	.1805	.1850	.1883	.1903	.1910	.1904	.1886	.1855	9
9		.1169	.1266	.1361	.1453	.1540	.1621	.1694	.1758	.1812	.1855	8
10		.0650	.0733	.0822	.0914	.1008	.1105	.1202	.1298	.1393	.1484	7
11		.0287	.0338	.0394	.0457	.0525	.0599	.0678	.0763	.0851	.0944	6
12		.0100	.0122	.0149	.0179	.0215	.0255	.0301	.0352	.0409	.0472	5
13		.0027	.0034	.0043	.0054	.0068	.0084	.0103	.0125	.0151	.0182	4
14		.0005	.0007	.0009	.0012	.0016	.0020	.0026	.0033	.0041	.0052	3
15		.0001	.0001	.0001	.0002	.0003	.0003	.0005	.0006	.0008	.0010	2
16		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	1
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	R

N = 18

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0		.8345	.6951	.5780	.4796	.3972	.3283	.2708	.2229	.1831	.1501	18

Table 5 Binomial Distribution *(Continued)*

1		.1517	.2554	.3217	.3597	.3763	.3772	.3669	.3489	.3260	.3002	17
2		.0130	.0443	.0846	.1274	.1683	.2047	.2348	.2579	.2741	.2835	16
3		.0007	.0048	.0140	.0283	.0473	.0697	.0942	.1196	.1446	.1680	15
4		.0000	.0004	.0016	.0044	.0093	.0167	.0266	.0390	.0536	.0700	14
5		.0000	.0000	.0001	.0005	.0014	.0030	.0056	.0095	.0148	.0218	13
6		.0000	.0000	.0000	.0000	.0002	.0004	.0009	.0018	.0032	.0052	12
7		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0005	.0010	11
8		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	10
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0		.1227	.1002	.0815	.0662	.0536	.0434	.0349	.0281	.0225	.0180	18
1		.2731	.2458	.2193	.1940	.1704	.1486	.1288	.1110	.0951	.0811	17
2		.2869	.2850	.2785	.2685	.2556	.2407	.2243	.2071	.1897	.1723	16
3		.1891	.2072	.2220	.2331	.2406	.2445	.2450	.2425	.2373	.2297	15
4		.0877	.1060	.1244	.1423	.1592	.1746	.1882	.1996	.2087	.2153	14
5		.0303	.0405	.0520	.0649	.0787	.0931	.1079	.1227	.1371	.1507	13
6		.0081	.0120	.0168	.0229	.0301	.0384	.0479	.0584	.0697	.0816	12
7		.0017	.0028	.0043	.0064	.0091	.0126	.0168	.0220	.0280	.0350	11
8		.0003	.0005	.0009	.0014	.0022	.0033	.0047	.0066	.0090	.0120	10
9		.0000	.0001	.0001	.0003	.0004	.0007	.0011	.0016	.0024	.0033	9
10		.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0008	8
11		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	7
		.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R	P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0		.0144	.0114	.0091	.0072	.0056	.0044	.0035	.0027	.0021	.0016	18
1		.0687	.0580	.0487	.0407	.0338	.0280	.0231	.0189	.0155	.0126	17
2		.1553	.1390	.1236	.1092	.0958	.0836	.0725	.0626	.0537	.0458	16
3		.2202	.2091	.1969	.1839	.1704	.1567	.1431	.1298	.1169	.1046	15
4		.2195	.2212	.2205	.2177	.2130	.2065	.1985	.1892	.1790	.1681	14
5		.1634	.1747	.1845	.1925	.1988	.2031	.2055	.2061	.2048	.2017	13
6		.0941	.1067	.1194	.1317	.1436	.1546	.1647	.1736	.1812	.1873	12
7		.0429	.0516	.0611	.0713	.0820	.0931	.1044	.1157	.1269	.1376	11
8		.0157	.0200	.0251	.0310	.0376	.0450	.0531	.0619	.0713	.0811	10
9		.0046	.0063	.0083	.0109	.0139	.0176	.0218	.0267	.0323	.0386	9
10		.0011	.0016	.0022	.0031	.0042	.0056	.0073	.0094	.0119	.0149	8
11		.0002	.0003	.0005	.0007	.0010	.0014	.0020	.0026	.0035	.0046	7
12		.0000	.0001	.0001	.0001	.0002	.0003	.0004	.0006	.0008	.0012	6
13		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	5
		.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0		.0013	.0010	.0007	.0006	.0004	.0003	.0002	.0002	.0001	.0001	18
1		.0102	.0082	.0066	.0052	.0042	.0033	.0026	.0020	.0016	.0012	17
2		.0388	.0327	.0275	.0229	.0190	.0157	.0129	.0105	.0086	.0069	16
3		.0930	.0822	.0722	.0630	.0547	.0471	.0404	.0344	.0292	.0246	15
4		.1567	.1450	.1333	.1217	.1104	.0994	.0890	.0791	.0699	.0614	14
5		.1971	.1911	.1878	.1755	.1664	.1566	.1463	.1358	.1252	.1146	13
6		.1919	.1948	.1962	.1959	.1941	.1908	.1852	.1803	.1734	.1655	12
7		.1478	.1572	.1656	.1730	.1792	.1840	.1875	.1895	.1900	.1892	11
8		.0913	.1017	.1122	.1226	.1327	.1423	.1514	.1597	.1671	.1734	10
9		.0456	.0532	.0614	.0701	.0794	.0890	.0988	.1087	.1187	.1284	9
10		.0184	.0225	.0272	.0325	.0385	.0450	.0522	.0600	.0683	.0771	8
11		.0060	.0077	.0097	.0122	.0151	.0184	.0223	.0267	.0318	.0374	7
12		.0016	.0021	.0028	.0037	.0047	.0063	.0076	.0096	.0118	.0145	6
13		.0003	.0005	.0006	.0009	.0012	.0016	.0021	.0027	.0035	.0045	5
14		.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0006	.0008	.0011	4
15		.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	3
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

Table 5 Binomial Distribution *(Continued)*

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
0	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	18
1	.0009	.0007	.0005	.0004	.0003	.0002	.0002	.0001	.0001	.0001	17
2	.0055	.0044	.0035	.0028	.0022	.0017	.0013	.0010	.0008	.0006	16
3	.0206	.0171	.0141	.0116	.0095	.0077	.0062	.0050	.0039	.0031	15
4	.0536	.0464	.0400	.0342	.0291	.0246	.0206	.0172	.0142	.0117	14
5	.1042	.0941	.0844	.0753	.0666	.0586	.0512	.0444	.0382	.0327	13
6	.1569	.1477	.1380	.1281	.1181	.1081	.0983	.0887	.0796	.0708	12
7	.1869	.1833	.1785	.1726	.1657	.1579	.1494	.1404	.1310	.1214	11
8	.1786	.1825	.1852	.1864	.1864	.1850	.1822	.1782	.1731	.1669	10
9	.1379	.1469	.1552	.1628	.1694	.1751	.1795	.1828	.1848	.1855	9
10	.0862	.0957	.1054	.1151	.1248	.1342	.1433	.1519	.1598	.1669	8
11	.0436	.0504	.0578	.0658	.0742	.0831	.0924	.1020	.1117	.1214	7
12	.0177	.0213	.0254	.0301	.0354	.0413	.0478	.0549	.0626	.0708	6
13	.0057	.0071	.0089	.0109	.0134	.0162	.0196	.0234	.0278	.0327	5
14	.0014	.0018	.0024	.0031	.0039	.0049	.0062	.0077	.0095	.0117	4
15	.0003	.0004	.0005	.0006	.0009	.0011	.0015	.0019	.0024	.0031	3
16	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0006	2
17	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	1
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

N = 19

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.8262	.6812	.5606	.4604	.3774	.3086	.2519	.2051	.1666	.1351	19
1	.1586	.2642	.3294	.3645	.3774	.3743	.3602	.3389	.3131	.2852	18
2	.0144	.0485	.0917	.1367	.1787	.2150	.2440	.2652	.2787	.2852	17
3	.0008	.0056	.0161	.0323	.0533	.0778	.1041	.1307	.1562	.1796	16
4	.0000	.0005	.0020	.0054	.0112	.0199	.0313	.0455	.0618	.0798	15
5	.0000	.0000	.0002	.0007	.0018	.0038	.0071	.0119	.0183	.0266	14
6	.0000	.0000	.0000	.0001	.0002	.0006	.0012	.0024	.0042	.0069	13
7	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0014	12
8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	11
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.1092	.0881	.0709	.0569	.0455	.0364	.0290	.0230	.0182	.0144	19
1	.2565	.2284	.2014	.1761	.1529	.1318	.1129	.0961	.0813	.0685	18
2	.2854	.2803	.2708	.2581	.2428	.2259	.2081	.1898	.1717	.1540	17
3	.1999	.2166	.2293	.2381	.2428	.2439	.2415	.2361	.2282	.2182	16
4	.0988	.1181	.1371	.1550	.1714	.1858	.1979	.2073	.2141	.2182	15
5	.0366	.0483	.0614	.0757	.0907	.1062	.1216	.1365	.1507	.1636	14
6	.0106	.0154	.0214	.0288	.0374	.0472	.0581	.0699	.0825	.0955	13
7	.0024	.0039	.0059	.0087	.0122	.0167	.0221	.0285	.0359	.0443	12
8	.0004	.0008	.0013	.0021	.0032	.0048	.0068	.0094	.0126	.0166	11
9	.0001	.0001	.0002	.0004	.0007	.0011	.0017	.0025	.0036	.0051	10
10	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0006	.0009	.0013	9
11	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	8
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.0113	.0089	.0070	.0054	.0042	.0033	.0025	.0019	.0015	.0011	19
1	.0573	.0477	.0396	.0326	.0268	.0219	.0178	.0144	.0116	.0093	18
2	.1371	.1212	.1064	.0927	.0803	.0692	.0592	.0503	.0426	.0358	17
3	.2065	.1937	.1800	.1659	.1517	.1377	.1240	.1109	.0985	.0869	16
4	.2196	.2185	.2151	.2096	.2023	.1935	.1835	.1726	.1610	.1491	15
5	.1751	.1849	.1928	.1986	.2023	.2040	.2036	.2013	.1973	.1916	14
6	.1086	.1217	.1343	.1463	.1574	.1672	.1757	.1827	.1880	.1916	13
7	.0536	.0637	.0745	.0858	.0974	.1091	.1207	.1320	.1426	.1525	12
8	.0214	.0270	.0334	.0406	.0487	.0575	.0670	.0770	.0874	.0981	11
9	.0069	.0093	.0122	.0157	.0198	.0247	.0303	.0366	.0436	.0514	10
10	.0018	.0026	.0036	.0050	.0066	.0087	.0112	.0142	.0178	.0220	9
11	.0004	.0006	.0009	.0013	.0018	.0025	.0034	.0045	.0060	.0077	8
12	.0001	.0001	.0002	.0003	.0004	.0006	.0008	.0012	.0016	.0022	7
13	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0004	.0005	6
14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	5
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

Table 5 Binomial Distribution *(Continued)*

R	P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40		
0		.0009	.0007	.0005	.0004	.0003	.0002	.0002	.0001	.0001	.0001		19
1		.0074	.0059	.0046	.0036	.0029	.0022	.0017	.0013	.0010	.0008		18
2		.0299	.0249	.0206	.0169	.0138	.0112	.0091	.0073	.0058	.0046		17
3		.0762	.0664	.0574	.0494	.0422	.0358	.0302	.0253	.0211	.0175		16
4		.1370	.1249	.1131	.1017	.0909	.0806	.0710	.0621	.0540	.0467		15
5		.1846	.1764	.1672	.1572	.1468	.1360	.1251	.1143	.1036	.0933		14
6		.1935	.1936	.1921	.1890	.1844	.1785	.1714	.1634	.1546	.1451		13
7		.1615	.1692	.1757	.1808	.1844	.1865	.1870	.1860	.1835	.1797		12
8		.1088	.1195	.1298	.1397	.1489	.1573	.1647	.1710	.1760	.1797		11
9		.0597	.0687	.0782	.0880	.0980	.1082	.1182	.1281	.1375	.1464		10
10		.0268	.0323	.0385	.0453	.0528	.0608	.0694	.0785	.0879	.0976		9
11		.0099	.0124	.0155	.0191	.0233	.0280	.0334	.0394	.0460	.0532		8
12		.0030	.0039	.0051	.0066	.0083	.0105	.0131	.0161	.0196	.0237		7
13		.0007	.0010	.0014	.0018	.0024	.0032	.0041	.0053	.0068	.0085		6
14		.0001	.0002	.0003	.0004	.0005	.0008	.0010	.0014	.0018	.0024		5
15		.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0004	.0005		4
16		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001		3
		.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P	R

R	P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50		
1		.0006	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0000		18
2		.0037	.0029	.0022	.0017	.0013	.0010	.0008	.0006	.0004	.0003		17
3		.0144	.0118	.0096	.0077	.0062	.0049	.0039	.0031	.0024	.0018		16
4		.0400	.0341	.0289	.0243	.0203	.0169	.0138	.0112	.0092	.0074		15
5		.0834	.0741	.0653	.0572	.0497	.0429	.0368	.0313	.0265	.0222		14
6		.1353	.1252	.1150	.1049	.0949	.0853	.0761	.0674	.0593	.0518		13
7		.1746	.1683	.1611	.1530	.1443	.1350	.1254	.1156	.1058	.0961		12
8		.1820	.1829	.1823	.1803	.1771	.1725	.1668	.1601	.1525	.1442		11
9		.1546	.1618	.1681	.1732	.1771	.1795	.1808	.1806	.1791	.1762		10
10		.1074	.1172	.1268	.1361	.1449	.1530	.1603	.1667	.1721	.1762		9
11		.0611	.0694	.0783	.0875	.0970	.1066	.1163	.1259	.1352	.1442		8
12		.0283	.0335	.0394	.0458	.0529	.0606	.0688	.0775	.0866	.0961		7
13		.0106	.0131	.0160	.0194	.0233	.0279	.0328	.0385	.0448	.0518		6
14		.0032	.0041	.0052	.0065	.0082	.0101	.0125	.0152	.0185	.0222		5
15		.0007	.0010	.0013	.0017	.0022	.0029	.0037	.0047	.0059	.0074		4
16		.0001	.0002	.0002	.0003	.0005	.0006	.0008	.0011	.0014	.0018		3
17		.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003		2
		.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P	R

N = 20

R	P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10		
0		.8179	.6676	.5438	.4420	.3585	.2901	.2342	.1887	.1516	.1216		20
1		.1652	.2725	.3364	.3683	.3774	.3703	.3526	.3282	.3000	.2702		19
2		.0159	.0528	.0988	.1458	.1887	.2246	.2521	.2711	.2818	.2852		18
3		.0010	.0065	.0183	.0364	.0596	.0860	.1139	.1414	.1672	.1901		17
4		.0000	.0006	.0024	.0065	.0133	.0233	.0364	.0523	.0703	.0898		16
5		.0000	.0000	.0002	.0009	.0022	.0048	.0088	.0145	.0222	.0319		15
6		.0000	.0000	.0000	.0001	.0003	.0008	.0017	.0032	.0055	.0089		14
7		.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0006	.0011	.0020		13
8		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004		12
9		.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001		11
		.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P	R

R	P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20		
0		.0972	.0776	.0617	.0490	.0388	.0306	.0241	.0189	.0148	.0115		20
1		.2403	.2115	.1844	.1595	.1368	.1165	.0986	.0829	.0693	.0576		19
2		.2822	.2740	.2618	.2466	.2293	.2109	.1919	.1730	.1545	.1369		18
3		.2093	.2242	.2347	.2409	.2428	.2410	.2358	.2278	.2175	.2054		17
4		.1099	.1299	.1491	.1666	.1821	.1951	.2053	.2125	.2168	.2182		16
5		.0435	.0567	.0713	.0868	.1028	.1189	.1345	.1493	.1627	.1746		15
6		.0134	.0193	.0266	.0353	.0454	.0566	.0689	.0819	.0954	.1091		14
7		.0033	.0053	.0080	.0115	.0160	.0216	.0252	.0360	.0448	.0545		13
8		.0007	.0012	.0019	.0030	.0046	.0067	.0094	.0128	.0171	.0222		12
9		.0001	.0002	.0004	.0007	.0011	.0017	.0026	.0038	.0053	.0074		11
10		.0000	.0000	.0001	.0001	.0002	.0004	.0006	.0009	.0014	.0020		10

Table 5 Binomial Distribution *(Continued)*

11	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	9
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	8
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
0	.0090	.0069	.0054	.0041	.0032	.0024	.0018	.0014	.0011	.0008	20
1	.0477	.0392	.0321	.0261	.0211	.0170	.0137	.0109	.0087	.0068	19
2	.1204	.1050	.0910	.0783	.0669	.0569	.0480	.0403	.0336	.0278	18
3	.1920	.1777	.1631	.1484	.1339	.1199	.1065	.0940	.0823	.0716	17
4	.2169	.2131	.2070	.1991	.1897	.1790	.1675	.1553	.1429	.1304	16
5	.1845	.1923	.1979	.2012	.2023	.2013	.1982	.1933	.1868	.1789	15
6	.1226	.1356	.1478	.1589	.1686	.1768	.1833	.1879	.1907	.1916	14
7	.0652	.0765	.0883	.1003	.1124	.1242	.1356	.1462	.1558	.1643	13
8	.0282	.0351	.0429	.0515	.0609	.0709	.0815	.0924	.1034	.1144	12
9	.0100	.0132	.0171	.0217	.0271	.0332	.0402	.0479	.0563	.0654	11
10	.0029	.0041	.0056	.0075	.0099	.0128	.0163	.0205	.0253	.0308	10
11	.0007	.0010	.0015	.0022	.0030	.0041	.0055	.0072	.0094	.0120	9
12	.0001	.0002	.0003	.0005	.0008	.0011	.0015	.0021	.0029	.0039	8
13	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0005	.0007	.0010	7
14	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	6
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
0	.0006	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0000	20
1	.0054	.0042	.0033	.0025	.0020	.0015	.0011	.0009	.0007	.0005	19
2	.0229	.0188	.0153	.0124	.0100	.0080	.0064	.0050	.0040	.0031	18
3	.0619	.0531	.0453	.0383	.0323	.0270	.0224	.0185	.0152	.0123	17
4	.1181	.1062	.0947	.0839	.0738	.0645	.0559	.0482	.0412	.0350	16
5	.1698	.1599	.1493	.1384	.1272	.1161	.1051	.0945	.0843	.0746	15
6	.1907	.1881	.1839	.1782	.1712	.1632	.1543	.1447	.1347	.1244	14
7	.1714	.1770	.1811	.1836	.1844	.1836	.1812	.1774	.1722	.1659	13
8	.1251	.1354	.1450	.1537	.1614	.1678	.1730	.1767	.1790	.1797	12
9	.0750	.0849	.0952	.1056	.1158	.1259	.1354	.1444	.1526	.1597	11
10	.0370	.0440	.0516	.0598	.0686	.0779	.0875	.0974	.1073	.1171	10
11	.0151	.0188	.0231	.0280	.0336	.0398	.0467	.0542	.0624	.0710	9
12	.0051	.0066	.0085	.0108	.0136	.0168	.0206	.0249	.0299	.0355	8
13	.0014	.0019	.0026	.0034	.0045	.0058	.0074	.0094	.0118	.0146	7
14	.0003	.0005	.0006	.0009	.0012	.0016	.0022	.0029	.0038	.0049	6
15	.0001	.0001	.0001	.0002	.0003	.0004	.0005	.0007	.0010	.0013	5
16	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	4
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
1	.0004	.0003	.0002	.0001	.0001	.0001	.0001	.0000	.0000	.0000	19
2	.0024	.0018	.0014	.0011	.0008	.0006	.0005	.0003	.0002	.0002	18
3	.0100	.0080	.0064	.0051	.0040	.0031	.0024	.0019	.0014	.0011	17
4	.0295	.0247	.0206	.0170	.0139	.0113	.0092	.0074	.0059	.0046	16
5	.0656	.0573	.0496	.0427	.0365	.0309	.0260	.0217	.0180	.0148	15
6	.1140	.1037	.0936	.0839	.0746	.0658	.0577	.0501	.0432	.0370	14
7	.1585	.1502	.1413	.1318	.1221	.1122	.1023	.0925	.0830	.0739	13
8	.1790	.1768	.1732	.1683	.1623	.1553	.1474	.1388	.1296	.1201	12
9	.1658	.1707	.1742	.1763	.1771	.1763	.1742	.1706	.1661	.1602	11
10	.1268	.1359	.1446	.1524	.1593	.1652	.1700	.1734	.1755	.1762	10
11	.0801	.0895	.0991	.1089	.1185	.1280	.1370	.1455	.1533	.1602	9
12	.0417	.0486	.0561	.0642	.0727	.0818	.0911	.1007	.1105	.1201	8
13	.0178	.0217	.0260	.0310	.0366	.0429	.0497	.0572	.0653	.0739	7
14	.0062	.0078	.0098	.0122	.0150	.0183	.0221	.0264	.0314	.0370	6
15	.0017	.0023	.0030	.0038	.0049	.0062	.0078	.0098	.0121	.0148	5
16	.0004	.0005	.0007	.0009	.0013	.0017	.0022	.0028	.0036	.0046	4
17	.0001	.0001	.0001	.0002	.0002	.0003	.0005	.0006	.0008	.0011	3
18	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	2
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

Table 5 Binomial Distribution *(Continued)*

N = 50

R	P .01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.6050	.3642	.2181	.1299	.0769	.0453	.0266	.0155	.0090	.0052	50
1	.3056	.3716	.3372	.2706	.2025	.1447	.0999	.0672	.0443	.0286	49
2	.0756	.1858	.2555	.2762	.2611	.2262	.1843	.1433	.1073	.0779	48
3	.0122	.0607	.1264	.1842	.2199	.2311	.2219	.1993	.1698	.1386	47
4	.0015	.0145	.0459	.0902	.1360	.1733	.1963	.2037	.1973	.1809	46
5	.0001	.0027	.0131	.0346	.0658	.1018	.1359	.1629	.1795	.1849	45
6	.0000	.0004	.0030	.0108	.0260	.0487	.0767	.1063	.1332	.1541	44
7	.0000	.0001	.0006	.0029	.0085	.0195	.0363	.0581	.0828	.1076	43
8	.0000	.0000	.0001	.0006	.0024	.0067	.0147	.0271	.0440	.0643	42
9	.0000	.0000	.0000	.0001	.0006	.0020	.0052	.0110	.0203	.0333	41
10	.0000	.0000	.0000	.0000	.0001	.0005	.0016	.0039	.0082	.0152	40
11	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0012	.0030	.0061	39
12	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0010	.0022	38
13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0007	37
14	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	36
15	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	35
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90 P	R

R	P .11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
0	.0029	.0017	.0009	.0005	.0003	.0002	.0001	.0000	.0000	.0000	50
1	.0182	.0114	.0071	.0043	.0025	.0016	.0009	.0005	.0003	.0002	49
2	.0552	.0382	.0259	.0172	.0113	.0073	.0046	.0029	.0018	.0011	48
3	.1091	.0833	.0619	.0449	.0319	.0222	.0151	.0102	.0067	.0044	47
4	.1584	.1334	.1086	.0858	.0661	.0496	.0364	.0262	.0185	.0128	46
5	.1801	.1674	.1493	.1286	.1072	.0869	.0687	.0530	.0400	.0295	45
6	.1670	.1712	.1674	.1570	.1419	.1242	.1055	.0872	.0703	.0554	44
7	.1297	.1467	.1572	.1606	.1575	.1487	.1358	.1203	.1037	.0870	43
8	.0862	.1075	.1263	.1406	.1493	.1523	.1495	.1420	.1307	.1169	42
9	.0497	.0684	.0880	.1068	.1230	.1353	.1429	.1454	.1431	.1364	41
10	.0252	.0383	.0539	.0713	.0890	.1057	.1200	.1309	.1376	.1398	40
11	.0113	.0190	.0293	.0422	.0571	.0732	.0894	.1045	.1174	.1271	39
12	.0045	.0084	.0142	.0223	.0328	.0453	.0595	.0745	.0895	.1033	38
13	.0016	.0034	.0062	.0106	.0169	.0252	.0356	.0478	.0613	.0755	37
14	.0005	.0012	.0025	.0046	.0079	.0127	.0193	.0277	.0380	.0499	36
15	.0002	.0004	.0009	.0018	.0033	.0058	.0095	.0146	.0214	.0299	35
16	.0000	.0001	.0003	.0006	.0013	.0024	.0042	.0070	.0110	.0164	34
17	.0000	.0000	.0001	.0002	.0005	.0009	.0017	.0031	.0052	.0082	33
18	.0000	.0000	.0000	.0001	.0001	.0003	.0007	.0012	.0022	.0037	32
19	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0009	.0016	31
20	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0006	30
21	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	29
22	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	28
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80 P	R

R	P .21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
1	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	49
2	.0007	.0004	.0002	.0001	.0001	.0000	.0000	.0000	.0000	.0000	48
3	.0029	.0018	.0011	.0007	.0004	.0002	.0001	.0001	.0000	.0000	47
4	.0088	.0059	.0039	.0025	.0015	.0010	.0006	.0004	.0002	.0001	46
5	.0214	.0152	.0106	.0073	.0049	.0033	.0021	.0014	.0009	.0006	45
6	.0427	.0322	.0238	.0173	.0123	.0087	.0060	.0040	.0027	.0018	44
7	.0713	.0571	.0447	.0344	.0259	.0191	.0139	.0099	.0069	.0048	43
8	.1019	.0865	.0718	.0583	.0463	.0361	.0276	.0207	.0152	.0110	42
9	.1263	.1139	.1001	.0859	.0721	.0592	.0476	.0375	.0290	.0220	41
10	.1377	.1317	.1226	.1113	.0985	.0852	.0721	.0598	.0485	.0386	40
11	.1331	.1351	.1332	.1278	.1194	.1089	.0970	.0845	.0721	.0602	39
12	.1150	.1238	.1293	.1311	.1294	.1244	.1166	.1068	.0957	.0838	38
13	.0894	.1021	.1129	.1210	.1261	.1277	.1261	.1215	.1142	.1050	37
14	.0628	.0761	.0891	.1010	.1110	.1186	.1233	.1248	.1233	.1189	36
15	.0400	.0515	.0639	.0766	.0888	.1000	.1094	.1165	.1209	.1223	35
16	.0233	.0318	.0417	.0529	.0648	.0769	.0885	.0991	.1080	.1147	34
17	.0124	.0179	.0249	.0334	.0432	.0540	.0655	.0771	.0882	.0983	33
18	.0060	.0093	.0137	.0193	.0264	.0348	.0444	.0550	.0661	.0772	32
19	.0027	.0044	.0069	.0103	.0148	.0206	.0277	.0360	.0454	.0558	31

Table 5 Binomial Distribution *(Continued)*

20	.0011	.0019	.0032	.0050	.0077	.0112	.0159	.0217	.0288	.0370	30
21	.0004	.0008	.0014	.0023	.0035	.0056	.0084	.0121	.0168	.0227	29
22	.0001	.0003	.0005	.0009	.0016	.0026	.0041	.0062	.0090	.0128	28
23	.0000	.0001	.0002	.0004	.0006	.0011	.0018	.0029	.0045	.0067	27
24	.0000	.0000	.0001	.0001	.0002	.0004	.0008	.0013	.0021	.0032	26
25	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005	.0009	.0014	25
26	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0006	24
27	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	23
28	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	22
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
4	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	46
5	.0003	.0002	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	45
6	.0011	.0007	.0005	.0003	.0002	.0001	.0001	.0000	.0000	.0000	44
7	.0032	.0022	.0014	.0009	.0005	.0004	.0002	.0001	.0001	.0000	43
8	.0078	.0055	.0037	.0025	.0017	.0011	.0007	.0004	.0003	.0002	42
9	.0164	.0120	.0086	.0061	.0042	.0029	.0019	.0013	.0008	.0005	41
10	.0301	.0231	.0174	.0128	.0093	.0066	.0046	.0032	.0022	.0014	40
11	.0493	.0395	.0311	.0240	.0182	.0136	.0099	.0071	.0050	.0035	39
12	.0719	.0604	.0498	.0402	.0319	.0248	.0189	.0142	.0105	.0076	38
13	.0944	.0831	.0717	.0606	.0502	.0408	.0325	.0255	.0195	.0147	37
14	.1121	.1034	.0933	.0825	.0714	.0607	.0505	.0412	.0330	.0260	36
15	.1209	.1168	.1103	.1020	.0923	.0819	.0712	.0606	.0507	.0415	35
16	.1188	.1202	.1189	.1149	.1088	.1008	.0914	.0813	.0709	.0606	34
17	.1068	.1132	.1171	.1184	.1171	.1133	.1074	.0997	.0906	.0808	33
18	.0880	.0976	.1057	.1118	.1156	.1169	.1156	.1120	.1062	.0987	32
19	.0666	.0774	.0877	.0970	.1048	.1107	.1144	.1156	.1144	.1109	31
20	.0463	.0564	.0670	.0775	.0875	.0965	.1041	.1098	.1134	.1146	30
21	.0297	.0379	.0471	.0570	.0673	.0776	.0874	.0962	.1035	.1091	29
22	.0176	.0235	.0306	.0387	.0478	.0575	.0676	.0777	.0873	.0959	28
23	.0096	.0135	.0183	.0243	.0313	.0394	.0484	.0580	.0679	.0778	27
24	.0049	.0071	.0102	.0141	.0190	.0249	.0319	.0400	.0489	.0584	26
25	.0023	.0035	.0052	.0075	.0106	.0146	.0195	.0255	.0325	.0405	25
26	.0010	.0016	.0025	.0037	.0055	.0079	.0110	.0151	.0200	.0259	24
27	.0004	.0007	.0011	.0017	.0026	.0039	.0058	.0082	.0113	.0154	23
28	.0001	.0003	.0004	.0007	.0012	.0018	.0028	.0041	.0060	.0084	22
29	.0000	.0001	.0002	.0003	.0005	.0008	.0012	.0019	.0029	.0043	21
30	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0008	.0013	.0020	20
31	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0009	19
32	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	18
33	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	17
	.69	.68	.67	.66	.65	.64	.63	.62	.61	.60	P R

R P	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	
8	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	42
9	.0003	.0002	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	41
10	.0009	.0006	.0004	.0002	.0001	.0001	.0001	.0000	.0000	.0000	40
11	.0024	.0016	.0010	.0007	.0004	.0003	.0002	.0001	.0001	.0000	39
12	.0054	.0037	.0026	.0017	.0011	.0007	.0005	.0003	.0002	.0001	38
13	.0109	.0079	.0057	.0040	.0027	.0018	.0012	.0008	.0005	.0003	37
14	.0200	.0152	.0113	.0082	.0059	.0041	.0029	.0019	.0013	.0008	36
15	.0334	.0264	.0204	.0155	.0115	.0085	.0061	.0043	.0030	.0020	35
16	.0508	.0418	.0337	.0267	.0207	.0159	.0119	.0086	.0062	.0044	34
17	.0706	.0605	.0508	.0419	.0339	.0269	.0219	.0159	.0119	.0087	33
18	.0899	.0803	.0703	.0604	.0508	.0420	.0340	.0270	.0210	.0160	32
19	.1053	.0979	.0893	.0799	.0700	.0602	.0507	.0419	.0340	.0270	31
20	.1134	.1099	.1044	.0973	.0883	.0795	.0697	.0600	.0506	.0419	30
21	.1126	.1137	.1126	.1092	.1039	.0967	.0884	.0791	.0695	.0598	29
22	.1031	.1086	.1119	.1131	.1113	.1086	.1033	.0963	.0880	.0788	28
23	.0872	.0957	.1028	.1082	.1115	.1126	.1115	.1082	.1029	.0960	27
24	.0682	.0780	.0872	.0956	.1025	.1079	.1112	.1124	.1112	.1080	26
25	.0493	.0587	.0684	.0781	.0873	.0956	.1026	.1079	.1112	.1123	25
26	.0329	.0409	.0497	.0590	.0697	.0783	.0875	.0957	.1027	.1080	24
27	.0203	.0263	.0333	.0412	.0500	.0593	.0690	.0786	.0877	.0960	23
28	.0116	.0157	.0206	.0266	.0335	.0415	.0502	.0596	.0692	.0788	22
29	.0061	.0086	.0118	.0159	.0209	.0268	.0338	.0417	.0504	.0598	21

Table 5 Binomial Distribution *(Continued)*

30	.0030	.0044	.0062	.0087	.0119	.0160	.0210	.0270	.0339	.0419	20
31	.0013	.0020	.0030	.0044	.0063	.0088	.0120	.0161	.0210	.0270	19
32	.0006	.0009	.0014	.0021	.0031	.0044	.0063	.0088	.0120	.0160	18
33	.0002	.0003	.0006	.0009	.0014	.0021	.0031	.0044	.0063	.0087	17
34	.0001	.0001	.0002	.0003	.0005	.0009	.0014	.0020	.0030	.0044	16
35	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0009	.0013	.0020	15
36	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0008	14
37	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	13
38	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	12
	.59	.58	.57	.56	.55	.54	.53	.52	.51	.50	P R

N = 100

R P	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	
0	.3660	.1326	.0476	.0169	.0059	.0021	.0007	.0002	.0001	.0000	100
1	.3697	.2707	.1471	.0703	.0312	.0131	.0053	.0021	.0008	.0003	99
2	.1849	.2734	.2252	.1450	.0812	.0414	.0198	.0090	.0039	.0016	98
3	.0610	.1823	.2275	.1973	.1396	.0864	.0486	.0254	.0125	.0059	97
4	.0149	.0902	.1706	.1994	.1781	.1338	.0888	.0536	.0301	.0159	96
5	.0029	.0353	.1013	.1595	.1800	.1639	.1283	.0895	.0571	.0339	95
6	.0005	.0114	.0496	.1052	.1500	.1657	.1529	.1233	.0895	.0596	94
7	.0001	.0031	.0206	.0589	.1060	.1420	.1545	.1440	.1188	.0889	93
8	.0000	.0007	.0074	.0285	.0649	.1054	.1352	.1455	.1366	.1148	92
9	.0000	.0002	.0023	.0121	.0349	.0687	.1040	.1293	.1381	.1304	91
10	.0000	.0000	.0007	.0046	.0167	.0399	.0712	.1024	.1243	.1319	90
11	.0000	.0000	.0002	.0016	.0072	.0209	.0439	.0728	.1006	.1199	89
12	.0000	.0000	.0000	.0005	.0028	.0099	.0245	.0470	.0738	.0988	88
13	.0000	.0000	.0000	.0001	.0010	.0043	.0125	.0276	.0494	.0743	87
14	.0000	.0000	.0000	.0000	.0003	.0017	.0058	.0149	.0304	.0513	86
15	.0000	.0000	.0000	.0000	.0001	.0006	.0025	.0074	.0172	.0327	85
16	.0000	.0000	.0000	.0000	.0000	.0002	.0010	.0034	.0090	.0193	84
17	0	.0000	.0000	.0000	.0000	.0001	.0004	.0015	.0044	.0106	83
18	0	.0000	.0000	.0000	.0000	.0000	.0001	.0006	.0020	.0054	82
19	0	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0009	.0026	81
20	0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0012	80
21	0	0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005	79
22	0	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002	78
23	0	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	77
	.99	.98	.97	.96	.95	.94	.93	.92	.91	.90	P R

R P	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	
1	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	99
2	.0007	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	98
3	.0027	.0012	.0005	.0002	.0001	.0000	.0000	.0000	.0000	.0000	97
4	.0080	.0038	.0018	.0008	.0003	.0001	.0001	.0000	.0000	.0000	96
5	.0189	.0100	.0050	.0024	.0011	.0005	.0002	.0001	.0000	.0000	95
6	.0369	.0215	.0119	.0063	.0031	.0015	.0007	.0003	.0001	.0001	94
7	.0613	.0394	.0238	.0137	.0075	.0039	.0020	.0009	.0004	.0002	93
8	.0881	.0625	.0414	.0259	.0153	.0086	.0047	.0024	.0012	.0006	92
9	.1112	.0871	.0632	.0430	.0276	.0168	.0098	.0054	.0029	.0015	91
10	.1251	.1090	.0860	.0637	.0444	.0292	.0182	.0108	.0062	.0034	90
11	.1265	.1205	.1051	.0849	.0640	.0454	.0305	.0194	.0118	.0069	89
12	.1160	.1219	.1165	.1025	.0838	.0642	.0463	.0316	.0206	.0128	88
13	.0970	.1125	.1179	.1130	.1001	.0827	.0642	.0470	.0327	.0216	87
14	.0745	.0954	.1094	.1143	.1098	.0979	.0817	.0641	.0476	.0335	86
15	.0528	.0745	.0938	.1067	.1111	.1070	.0960	.0807	.0640	.0481	85
16	.0347	.0540	.0744	.0922	.1041	.1082	.1044	.0941	.0798	.0638	84
17	.0212	.0364	.0549	.0742	.0908	.1019	.1057	.1021	.0924	.0789	83
18	.0121	.0229	.0379	.0557	.0739	.0895	.0998	.1033	.1000	.0909	82
19	.0064	.0135	.0244	.0391	.0563	.0736	.0882	.0979	.1012	.0981	81
20	.0032	.0074	.0148	.0258	.0402	.0567	.0732	.0870	.0962	.0993	80
21	.0015	.0039	.0084	.0160	.0270	.0412	.0571	.0728	.0859	.0946	79
22	.0007	.0019	.0045	.0094	.0171	.0282	.0420	.0574	.0724	.0849	78
23	.0003	.0009	.0023	.0052	.0103	.0182	.0292	.0427	.0576	.0720	77
24	.0001	.0004	.0011	.0027	.0058	.0111	.0192	.0301	.0433	.0577	76
25	.0000	.0002	.0005	.0013	.0031	.0064	.0119	.0201	.0309	.0439	75
26	.0000	.0001	.0002	.0006	.0015	.0035	.0071	.0127	.0209	.0316	74

Table 5 Binomial Distribution *(Continued)*

27	.0000	.0000	.0001	.0003	.0008	.0018	.0040	.0076	.0134	.0217	73
28	.0000	.0000	.0000	.0001	.0004	.0009	.0021	.0044	.0082	.0141	72
29	.0000	.0000	.0000	.0000	.0002	.0004	.0011	.0024	.0048	.0088	71
30	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0012	.0027	.0052	70
31	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0006	.0014	.0029	69
32	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0007	.0016	68
33	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0008	67
34	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	66
35	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	65
36	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	64
	.89	.88	.87	.86	.85	.84	.83	.82	.81	.80	P R

R P	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	
7	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	93
8	.0003	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	92
9	.0007	.0003	.0002	.0001	.0000	.0000	.0000	.0000	.0000	.0000	91
10	.0018	.0009	.0004	.0002	.0001	.0000	.0000	.0000	.0000	.0000	90
11	.0038	.0021	.0011	.0005	.0003	.0001	.0001	.0000	.0000	.0000	89
12	.0076	.0043	.0024	.0012	.0005	.0003	.0001	.0001	.0000	.0000	88
13	.0136	.0082	.0048	.0027	.0014	.0007	.0004	.0002	.0001	.0000	87
14	.0225	.0144	.0089	.0052	.0030	.0016	.0009	.0004	.0002	.0001	86
15	.0343	.0233	.0152	.0095	.0057	.0033	.0018	.0010	.0005	.0002	85
16	.0484	.0350	.0241	.0159	.0100	.0061	.0035	.0020	.0011	.0006	84
17	.0636	.0487	.0356	.0248	.0165	.0106	.0065	.0038	.0022	.0012	83
18	.0780	.0634	.0490	.0361	.0254	.0171	.0111	.0069	.0041	.0024	82
19	.0895	.0772	.0631	.0492	.0365	.0259	.0177	.0115	.0072	.0044	81
20	.0963	.0881	.0764	.0629	.0493	.0369	.0264	.0182	.0120	.0076	80
21	.0975	.0947	.0869	.0756	.0625	.0494	.0373	.0269	.0186	.0124	79
22	.0931	.0959	.0932	.0858	.0749	.0623	.0495	.0376	.0273	.0190	78
23	.0839	.0917	.0944	.0919	.0847	.0743	.0621	.0495	.0378	.0277	77
24	.0716	.0830	.0905	.0931	.0905	.0837	.0736	.0618	.0496	.0380	76
25	.0578	.0712	.0822	.0893	.0918	.0894	.0828	.0731	.0615	.0496	75
26	.0444	.0579	.0708	.0814	.0883	.0906	.0883	.0819	.0725	.0613	74
27	.0323	.0448	.0580	.0704	.0805	.0873	.0896	.0873	.0812	.0720	73
28	.0224	.0329	.0451	.0580	.0701	.0799	.0864	.0886	.0864	.0804	72
29	.0148	.0231	.0335	.0455	.0580	.0697	.0793	.0855	.0876	.0856	71
30	.0093	.0154	.0237	.0340	.0453	.0580	.0694	.0787	.0847	.0868	70
31	.0056	.0098	.0160	.0242	.0344	.0460	.0580	.0691	.0781	.0840	69
32	.0032	.0060	.0103	.0165	.0248	.0349	.0462	.0579	.0688	.0776	68
33	.0018	.0035	.0063	.0107	.0170	.0252	.0352	.0464	.0579	.0685	67
34	.0009	.0019	.0037	.0067	.0112	.0175	.0257	.0356	.0466	.0573	66
35	.0005	.0010	.0021	.0040	.0070	.0116	.0179	.0261	.0359	.0468	65
36	.0002	.0005	.0011	.0023	.0042	.0073	.0120	.0183	.0265	.0362	64
37	.0001	.0003	.0006	.0012	.0024	.0045	.0077	.0123	.0187	.0263	63
38	.0000	.0001	.0003	.0006	.0013	.0025	.0047	.0079	.0127	.0191	62
39	.0000	.0001	.0001	.0003	.0007	.0015	.0028	.0049	.0082	.0130	61
40	.0000	.0000	.0001	.0002	.0004	.0008	.0016	.0029	.0051	.0085	60
41	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0017	.0031	.0053	59
42	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0009	.0018	.0032	58
43	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0010	.0019	57
44	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0010	56
45	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0005	55
46	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	54
47	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	53
48	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	52
	.79	.78	.77	.76	.75	.74	.73	.72	.71	.70	P R

R P	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	
15	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	85
16	.0003	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	84
17	.0006	.0003	.0002	.0001	.0000	.0000	.0000	.0000	.0000	.0000	83
18	.0013	.0007	.0004	.0002	.0001	.0000	.0000	.0000	.0000	.0000	82
19	.0025	.0014	.0008	.0004	.0002	.0001	.0000	.0000	.0000	.0000	81
20	.0046	.0027	.0015	.0008	.0004	.0002	.0001	.0001	.0000	.0000	80
21	.0079	.0049	.0029	.0016	.0009	.0005	.0002	.0001	.0001	.0000	79
22	.0127	.0082	.0051	.0030	.0017	.0010	.0005	.0003	.0001	.0001	78
23	.0194	.0131	.0085	.0053	.0032	.0019	.0010	.0006	.0003	.0001	77

Table 5 Binomial Distribution *(Continued)*

24	.0280	.0198	.0134	.0088	.0055	.0033	.0019	.0011	.0006	.0003	76
25	.0382	.0283	.0201	.0137	.0090	.0057	.0035	.0020	.0012	.0006	75
26	.0496	.0384	.0286	.0204	.0140	.0092	.0059	.0036	.0021	.0012	74
27	.0610	.0495	.0386	.0288	.0207	.0143	.0095	.0060	.0037	.0022	73
28	.0715	.0608	.0495	.0387	.0290	.0209	.0145	.0097	.0062	.0038	72
29	.0797	.0710	.0605	.0495	.0388	.0292	.0211	.0147	.0098	.0063	71
30	.0848	.0791	.0706	.0603	.0494	.0389	.0294	.0213	.0149	.0100	70
31	.0860	.0840	.0785	.0702	.0601	.0494	.0389	.0295	.0215	.0151	69
32	.0833	.0853	.0834	.0779	.0698	.0599	.0493	.0390	.0296	.0217	68
33	.0771	.0827	.0846	.0827	.0774	.0694	.0597	.0493	.0390	.0297	67
34	.0683	.0767	.0821	.0840	.0821	.0769	.0691	.0595	.0492	.0391	66
35	.0578	.0680	.0763	.0816	.0834	.0816	.0765	.0688	.0593	.0491	65
36	.0469	.0578	.0678	.0759	.0811	.0829	.0811	.0761	.0685	.0591	64
37	.0365	.0471	.0578	.0676	.0755	.0806	.0824	.0807	.0757	.0682	63
38	.0272	.0367	.0472	.0577	.0674	.0752	.0802	.0820	.0803	.0754	62
39	.0194	.0275	.0369	.0473	.0577	.0672	.0749	.0799	.0816	.0799	61
40	.0133	.0197	.0277	.0372	.0474	.0577	.0671	.0746	.0795	.0812	60
41	.0087	.0136	.0200	.0280	.0373	.0475	.0577	.0670	.0744	.0792	59
42	.0055	.0090	.0138	.0203	.0282	.0375	.0476	.0576	.0668	.0742	58
43	.0033	.0057	.0092	.0141	.0205	.0285	.0377	.0477	.0576	.0667	57
44	.0019	.0035	.0059	.0094	.0143	.0207	.0287	.0378	.0477	.0576	56
45	.0011	.0020	.0036	.0060	.0095	.0145	.0210	.0289	.0380	.0478	55
46	.0006	.0011	.0021	.0037	.0062	.0098	.0147	.0212	.0290	.0381	54
47	.0003	.0006	.0012	.0022	.0038	.0063	.0099	.0149	.0213	.0292	53
48	.0001	.0003	.0007	.0012	.0023	.0039	.0064	.0101	.0151	.0215	52
49	.0001	.0002	.0003	.0007	.0013	.0023	.0040	.0066	.0102	.0152	51
50	.0000	.0001	.0002	.0004	.0007	.0013	.0024	.0041	.0067	.0103	50
51	.0000	.0000	.0001	.0002	.0004	.0007	.0014	.0025	.0042	.0068	49
52	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0014	.0025	.0042	48
53	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0015	.0026	47
54	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0008	.0015	46
55	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0008	45
56	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	44
57	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	43
58	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	42
59	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	41

Index